Solutions

1. (5 pts) For a differentiable function, $f(x)$ at $x = a$, state the limit definition of $f'(a)$.

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

$$\left(\text{Or, } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ so } f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right)$$

2. (5 pts) Evaluate and simplify cosh(ln(10)).

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}, \text{ so } \cosh(\ln(10)) = \frac{e^{\ln(10)} - e^{-\ln(10)}}{2} = \frac{10 + e^{\ln(\frac{1}{10})}}{2} = \frac{10 + \frac{1}{10}}{2}$$

$$\boxed{= 5.05}$$

3. (10 pts) Consider the function $f(x) = 10 - 3e^{-0.5x}$

(a) Determine the inverse function $f^{-1}(x)$.

$$x = 10 - 3e^{-0.5y}$$

$$x - 10 = -3e^{-0.5y}$$

$$\frac{x-10}{-3} = e^{-0.5y}$$

$$\ln\left(\frac{x-10}{-3}\right) = -0.5y$$

$$\boxed{f^{-1}(x) = y = \frac{\ln\left(\frac{x-10}{-3}\right)}{-0.5}}$$

(b) Determine $(f^{-1})'(7)$. (Note that $f(0) = 7$.)

Inverse function theorem: ~~$(f^{-1})'(x)$~~

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(0)} = \frac{1}{3/2} = \boxed{\frac{2}{3}}$$

$$\left(\text{Note } f'(x) = \frac{3}{2}e^{-0.5x}, \text{ so } f'(0) = \frac{3}{2}e^{-0.5(0)} = \boxed{\frac{3}{2}} \right)$$

$$y = f^{-1}(x)$$

$$f(y) = x$$

$$f'(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

So, $\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x)}}$

4. (5 pts each) Differentiate the following functions. You do not need to simplify your answer.

(a) $f(x) = \sqrt{3x + 2e^{-2x}}$

$$\boxed{f'(x) = \frac{1}{2\sqrt{3x+2e^{-2x}}} \cdot \left(3 + 2\left(e^{-2x} \cdot -2\right)\right)}$$

(b) $y = \dfrac{4 + \sin(3x)}{x\cos(x)}$ ✓ Logarithmic differentiation

$$\ln(y) = \ln(4+\sin(3x)) - \ln(x) - \ln(\cos(x))$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{\cos(3x) \cdot 3}{4+\sin(3x)} - \frac{1}{x} - \frac{-\sin(x)}{\cos(x)}, \text{ so } \boxed{\frac{dy}{dx} = \left[\frac{3\cos(3x)}{4+\sin(3x)} - \frac{1}{x} + \tan(x)\right]\left(\frac{4+\sin(3x)}{x\cos(x)}\right)}$$

(c) $f(x) = \left(\ln(x^3 + 1)\right)^2$

$$\boxed{f'(x) = 2\ln(x^3+1) \cdot \frac{1}{x^3+1} \cdot 3x^2}$$

(d) $y = \dfrac{10}{(8x + 6e^{-0.5x})^2} = 10\left(8x + 6e^{-0.5x}\right)^{-2}$

$$\boxed{y' = -20\left(8x + 6e^{-0.5x}\right)^{-3} \cdot \left(8 + 6e^{-0.5x}(-0.5)\right)}$$

Product!

(e) $y = (3x+1)^2 \cos(3x)$

$$\boxed{y' = \{(3x+1)^2\} \cdot \left[-\sin(3x) \cdot 3\right] + \left[2(3x+1) \cdot 3\right]\{\cos(3x)\}}$$

(f) $f(x) = 4\arctan(2x) - 3\arcsin(3x)$

$$\boxed{f'(x) = 4 \cdot \frac{1}{1+(2x)^2} \cdot 2 - 3 \cdot \frac{1}{\sqrt{1-(3x)^2}} \cdot 3}$$

(g) $y = (2 + \sin(x))^x$ super-exponential

$$\ln(y) = \ln\left((2+\sin(x))^x\right) = x \cdot \ln(2+\sin(x))$$

$$\frac{1}{y} \cdot y' = x \cdot \frac{\cos(x)}{2+\sin(x)} + \ln(2+\sin(x))$$

$$\boxed{y' = \left[x + \frac{\cos(x)}{2+\sin(x)} + \ln(2+\sin(x))\right](2+\sin(x))^x}$$

(h) $f(x) = \int_4^{2x^3} e^u \tan(u)\, du$

$$\boxed{f'(x) = e^{2x^3} \tan(2x^3) \cdot 6x^2 - e^4 \tan(4) \cdot 0}$$

5. (15 pts) Determine the limit (show all work)

(a) $\lim_{x\to\infty} (\ln(2x+1) - \ln(x+3))$ $\to \infty - \infty$ (indeterminate)

$= \lim_{x\to\infty} \left\{ \ln\left(\frac{2x+1}{x+3}\right)\right\} = \ln\left(\lim_{x\to\infty}\frac{2x+1}{x+3}\right)$ [redacted]

$= \ln\left(\lim_{x\to\infty}\left(\frac{2+\frac{1}{x}}{1+\frac{3}{x}}\right)\right) = \ln\left(\frac{2}{1}\right) = \boxed{\ln(2)}$

(b) $\lim_{x\to 0^+} x^{x/2}$ $\to 0^0$ INDETERMINATE

$L = \lim_{x\to0^+} x^{x/2}$, so $\ln(L) = \lim_{x\to0^+}\ln\left(x^{x/2}\right) = \lim_{x\to0^+}\frac{x}{2}\cdot\ln(x) = \lim_{x\to0}\frac{\frac{1}{2}\ln(x)}{\frac{1}{x}}$

L'H
↓
$= \lim_{x\to0^+}\frac{\frac{1}{2}\cdot\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to0^+} -\frac{1}{2}x = -\frac{1}{2}(0) = \boxed{0}$. Hence, $\boxed{L = e^0 = 1}$

(c) $\lim_{x\to 0}\frac{3\tan(x)}{x+\sin(2x)}$ $\to \frac{0}{0}$ [redacted] INDETERMINATE

L'H
↓
$= \lim_{x\to0}\frac{3\sec^2(x)}{1+2\cos(2x)} = \frac{3\sec^2(0)}{1+2\cos(0)} = \frac{3\cdot1}{1+2\cdot1} = \frac{3}{1+2} = \boxed{3}$

6. (10 pts) Oil is dripping from a tank forming a circular puddle whose circumference is changing at a rate of 6 inches per second. Determine the rate at which the area of the circle is changing when the radius is 40 inches.

$A = \pi r^2$, and $C = 2\pi r$, so $A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{\pi C^2}{4\pi^2} = \frac{C^2}{4\pi}$.

Differentiate with respect to time: $\frac{dA}{dt} = \frac{1}{4\pi} \cdot 2C\,\frac{dC}{dt}$.

When $r = 40$, $C = 80\pi$. We know $\frac{dC}{dt} = 6$. Get $\frac{dA}{dt} = \frac{1}{4\pi} \cdot 2(80\pi) \cdot 6 = \boxed{240 \text{ in}^2/\text{sec}}$

7. (10 pts) Determine the range (interval from the minimum value to the maximum value) on $0 \leq x \leq \pi$ of the function $f(x) = \cos(x) + \cos^2(x)$.

To find the min & max values, find where $f'(x) = 0$ and test endpoints:

$f'(x) = -\sin(x) + 2\cos(x)(-\sin(x)) = -\sin(x)(1 + 2\cos(x)) = 0$

$\Rightarrow -\sin(x) = 0$ or $1 + 2\cos(x) = 0$

$\Rightarrow \boxed{x = 0,\ x = \pi}$ or $\cos(x) = \frac{-1}{2} \Rightarrow \boxed{x = \frac{2\pi}{3}}$.

Now, $f(0) = \cos(0) + \cos^2(0) = 1 + 1 = 2$

$f(\pi) = \cos(\pi) + \cos^2(\pi) = -1 + 1 = 0$

$f\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + \cos^2\left(\frac{2\pi}{3}\right) = \frac{-1}{2} + \frac{1}{4} = \frac{-1}{4}$.

The range is thus $\boxed{\frac{-1}{4} \leq f(x) \leq 2}$ on the given interval.

8. (20 pts) Answer the following for the function $f(x) = \frac{x-1}{x^2}$ $\left(= \frac{1}{x} - \frac{1}{x^2}\right)$

(a) (5 pts) State any vertical asymptotes. List the critical values of $f(x)$ and state whether there is a local maximum, local minimum, inflection point, or the derivative does not exist.

excluded, $f'(x)$ DNE there

Vertical asymptotes when $x^2=0 \Rightarrow \boxed{x=0}$

$f'(x) = \frac{-1}{x^2} + \frac{2}{x^3} = 0 \Rightarrow \frac{2}{x^3} = \frac{1}{x^2} \Rightarrow x^3 = 2x^2 \Rightarrow x^3 - 2x^2 = 0 \Rightarrow x^2(x-2) = 0 \Rightarrow \cancel{x=0}, \boxed{x=2}$

Sign chart:

$f'(x)$	⊖	DNE	⊕	0	⊖
x	-1	0	1	2	1000

So, $x=2$ is a local max.
$f'(x)$ DNE at $x=0$.

(b) (5 pts) Determine the x value of any points of inflection.

$f''(x) = \frac{2}{x^3} - \frac{6}{x^4} = 0 \Rightarrow \frac{2}{x^3} = \frac{6}{x^4} \Rightarrow 2x^4 = 6x^3 \Rightarrow 2x^4 - 6x^3 = 0 \Rightarrow 2x^3(x-3) = 0 \Rightarrow \cancel{x=0}, \boxed{x=3}$

Sign chart:

$f''(x)$	⊖	DNE	⊖	0	⊕
x	-1	0	1	3	1000

So, $x=3$ is an inflection point.

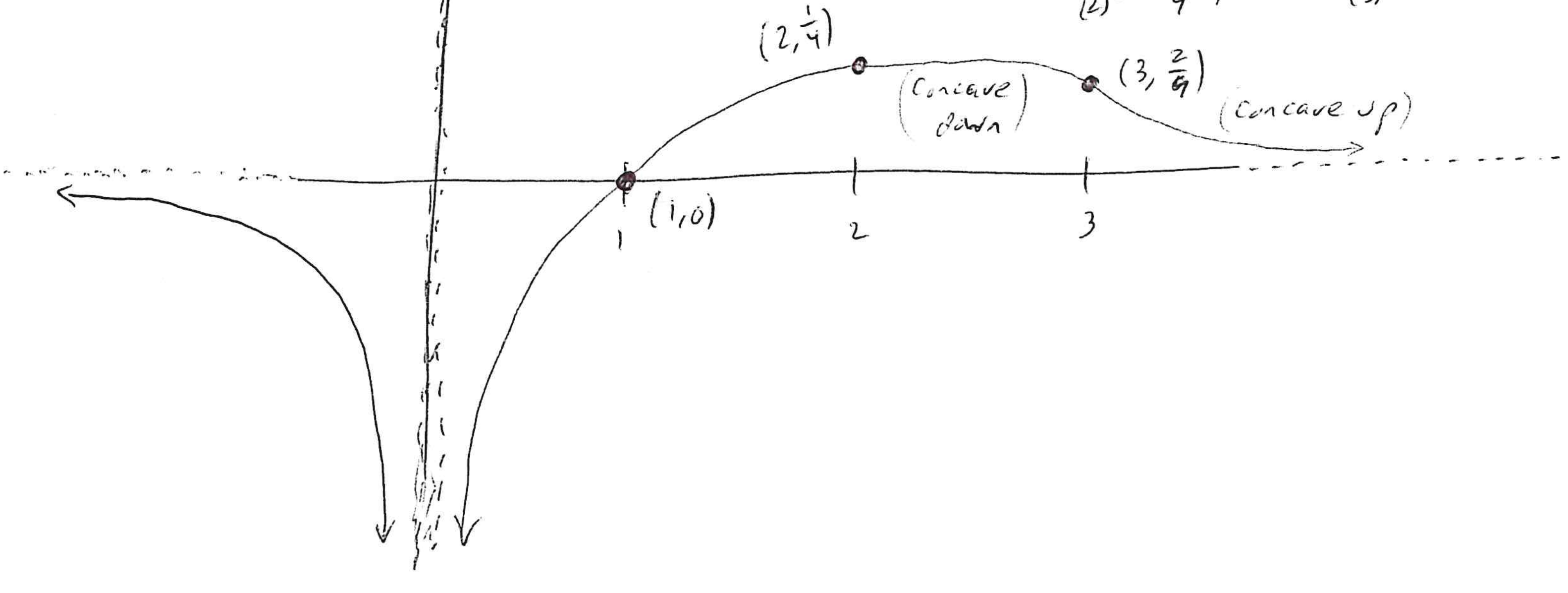
(c) (5 pts) $\lim_{x\to-\infty} f(x) =$ 0 $\qquad \lim_{x\to\infty} f(x) =$ 0

$= \lim_{x\to-\infty} \frac{x-1}{x^2} \overset{L'H}{=} \lim_{x\to-\infty} \frac{1}{2x} = \boxed{0}$ $\qquad \lim_{x\to\infty} \frac{x-1}{x^2} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{2x} = \boxed{0}$

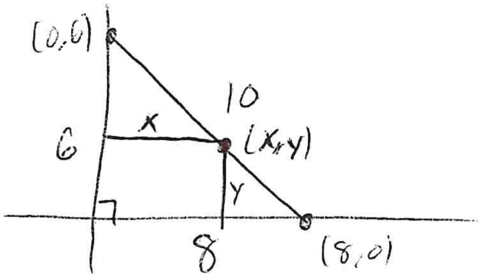
(d) (5pts) Graph the function. Show asymptote(s), intercept(s) and the exact points $(x,\ y)$ where the function has any local maximum or local minimum values. Also be specific as to where there is any change in concavity.

$f(x) = 0$ when $x - 1 = 0 \Rightarrow \boxed{x=1}$

$f(2) = \frac{2-1}{(2)^2} = \frac{1}{4}$, $f(3) = \frac{3-1}{(3)^2} = \frac{2}{9}$



9. (15 pts) A rectangle is to be inscribed in a right triangle having length of sides 6 and 8 and hypotenuse 10. Determine the dimensions of the rectangle (length and width) having the largest area. (It would help to put this on an xy-coordinate system with the right angle at the origin.)



Objective: Maximize $xy = A$

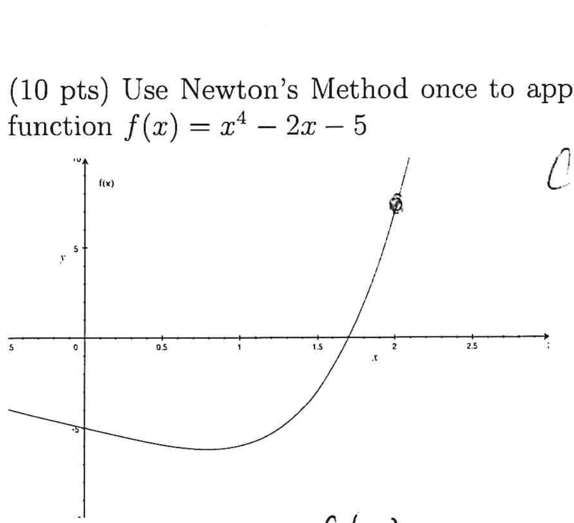
Constraint: (x,y) is on the hypotenuse. In other words, $y = \dfrac{\overset{\overset{\text{rise}}{\downarrow}}{-6}}{\underset{\underset{\text{run}}{\uparrow}}{8}}x + 6$, so

$$A(x) = x\left(\frac{-6}{8}x + 6\right) = \frac{-3}{4}x^2 + 6x$$

$$A'(x) = \frac{-6}{4}x + 6 = 0 \Rightarrow 6 = \frac{6}{4}x \Rightarrow \boxed{x = 4}$$

$$y = \frac{-6}{8}(4) + 6 = -3 + 6 = \boxed{3}.$$

10. (10 pts) Use Newton's Method once to approximate the first postitive root of the function $f(x) = x^4 - 2x - 5$



Choose $x_0 = 2$. Then $f(x_0) = (2)^4 - 2(2) - 5$

$= 16 - 4 - 5 = \boxed{7}$

$f'(x) = 4x^3 - 2$, so $f'(2) = 4(2)^3 - 2 = 32 - 2 = \boxed{30}$

Formula: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. So $x_1 = 2 - \frac{7}{30} = \frac{60}{30} - \frac{7}{30} = \boxed{\frac{53}{30}}$

11. (10 pts) At the point (3, 2), determine the equation of the tangent line to the curve

$$6\sqrt{x + 3y} + x^2 y - y^2 = 32.$$

Differentiate with respect to x:

$$6 \cdot \frac{1 + 3\frac{dy}{dx}}{2\sqrt{x+3y}} + 2xy [illegible] + x^2 \frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

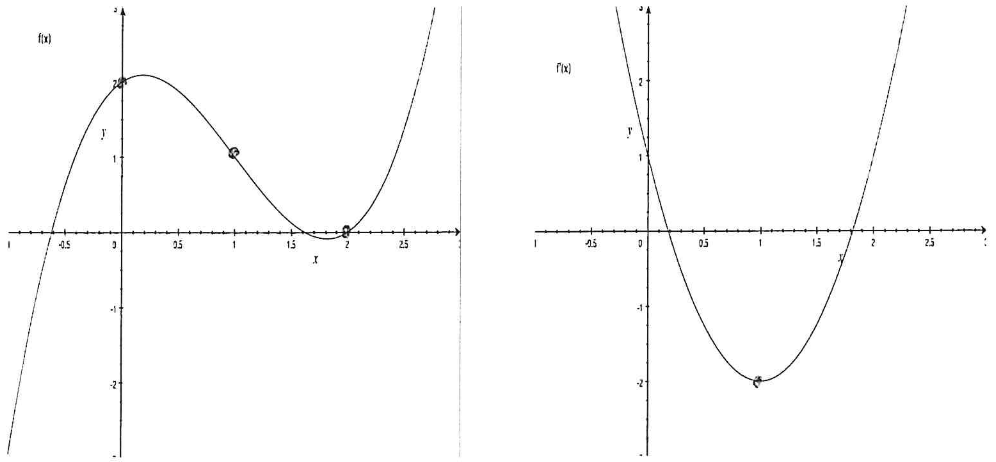
$$\frac{6}{2\sqrt{x+3y}} + \frac{18\frac{dy}{dx}}{2\sqrt{x+3y}} + 2xy + x^2\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(\frac{9}{\sqrt{x+3y}} + x^2 - 2y\right) = \frac{-6}{2\sqrt{x+3y}} - 2xy \Rightarrow \frac{dy}{dx} = \frac{\frac{-3}{\sqrt{x+3y}} - 2xy}{\frac{9}{\sqrt{x+3y}} + x^2 - 2y}$$

At (3,2), $\frac{dy}{dx} = \frac{\frac{-3}{\sqrt{3+6}} - 2(3)(2)}{\frac{9}{\sqrt{3+6}} + (3)^2 - 2(2)} = \frac{-1-12}{3+9-4} = \boxed{\frac{-13}{8}}$.

Equation of tangent line is $y - y_1 = m(x - x_1)$ where $x_1 = 3$, $y_1 = 2$, $m = \frac{-13}{8}$. Get $\boxed{y - 2 = \frac{-13}{8}(x-3)}$

12. (10 pts) Below is the plot of $f(x)$ on the left and $f'(x)$ on the right.



(a) Determine $\int_0^2 f'(x)\,dx$.

$= f(2) - f(0) = 0 - 2 = \boxed{-2}$

(b) Determine the equation of the tangent line to the curve $f(x)$ at $x = 1$.

Slope: $f'(1) = 1$. $x_1 = 1$, $y_1 = f(1) = -2$.

Get $\boxed{y - (-2) = 1(x - 1)}$

13. (10 pts) A particle moves along a line with velocity function $v(t) = 6t^2 - 6t$, where v is measured in meters per second.

(a) Find the displacement of the particle during the time interval $[0, 5]$.

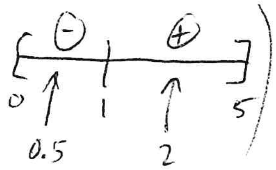
Displacement = Final position - Initial position $= \int_0^5 v(t)\,dt = \int_0^5 (6t^2 - 6t)\,dt = \left[2t^3 - 3t^2\right]_0^5$

$= \left(2(5)^3 - 3(5)^2\right) - \left(2(0)^3 - 3(0)^2\right) = 250 - 75 = \boxed{175}$

(b) Find the distance traveled by the particle during the time interval $[0, 5]$.

Distance ⇔ All movement is positive, so get $\int_0^5 |v(t)|\,dt = \int_0^5 |6t^2 - 6t|\,dt$ (Note $6t^2 - 6t = 0$ $\Rightarrow 6t(t-1) = 0$ $\Rightarrow \boxed{t = 0, t = 1}$)

(Sign chart:



)

$= \int_0^1 (6t - 6t^2)\,dt + \int_1^5 6t^2 - 6t\,dt$

$= \left[3t^2 - 2t^3\right]_0^1 + \left[2t^3 - 3t^2\right]_1^5 = 3 - 2 - (0 - 0) + 250 - 75 - (2 - 3) = 175 + 2 = \boxed{177}$

$\left(\text{Since } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}\right)$

14. (30 pts)Evaluate the following integrals.

(a) $\displaystyle\int_0^{1/2} \left(\frac{4}{\sqrt{1-x^2}} + \overset{\text{u-sub, } u=2x}{\sqrt{8+2x}} \right) dx$ (give a numerical evaluation))

$$= \left[4\arcsin(x) + \frac{2(8+2x)^{3/2}}{3\cdot 2} \right]_0^{1/2} = \left(4\arcsin\left(\tfrac{1}{2}\right) + \frac{\left(8+2\left(\frac{1}{2}\right)\right)^{3/2}}{3}\right) - \left(4\arcsin(0) + \frac{(8+2(0))^{3/2}}{3}\right)$$

$$\boxed{= 4\cdot\frac{\pi}{6} + \frac{27}{3} - 0 - \frac{8^{3/2}}{3}}$$

(b) $\displaystyle\int 2x^3 \ln(x)\, dx$ ↙ Parts

$$\boxed{\int f\cdot dg = f\cdot g - \int g\cdot df}$$

$f = \ln(x) \quad dg = 2x^3 dx$

$df = \frac{1}{x} dx \quad g = \frac{1}{2}x^4$

$$= \frac{1}{2}x^4 \ln(x) - \int \frac{1}{2}x^4 \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{2}x^4 \ln(x) - \frac{1}{2}\int x^3 dx = \boxed{\frac{1}{2}x^4 \ln(x) - \frac{1}{8}x^4 + C}$$

(c) $\displaystyle\int \frac{3x}{e^{2x}}\, dx = \int 3xe^{-2x}$ ↙ Parts

$f = 3x \quad dg = e^{-2x} dx$

$df = 3dx \quad g = -\frac{1}{2}e^{-2x}$

↑ u-sub $u = -2x$

$$= \frac{-3}{2}xe^{-2x} - \int \frac{-3}{2}e^{-2x} dx$$

$$= \frac{-3}{2}xe^{-2x} + \frac{3}{2}\int e^{-2x} dx$$

$$\boxed{= \frac{-3}{2}xe^{-2x} - \frac{3}{4}e^{-2x} + C}$$

(d) $\int 4\sin^3(\overset{\swarrow u\text{-sub}}{3x})\,dx$

$\boxed{\begin{array}{l} U=3x \\ du=3dx \\ dx=\frac{du}{3} \end{array}}$

$$= 4\int \sin^3(u)\frac{du}{3} = \frac{4}{3}\int \overset{w\text{-sub}\,\downarrow}{\sin^3(u)}du \qquad \boxed{\begin{array}{l} w=\cos(u) \\ dw=-\sin(u)du \end{array}}$$

$$= \frac{4}{3}\int \sin^2(u)\,dw = \frac{-4}{3}\int(1-\cos^2(u))dw = \frac{-4}{3}\int(1-w^2)dw$$

$$= \frac{-4}{3}\left(w - \frac{w^3}{3}\right)+C = \frac{-4}{3}\left(\cos(u) - \frac{\cos^3(u)}{3}\right)+C = \boxed{\frac{-4}{3}\left(\cos(3x) - \frac{\cos^3(3x)}{3}\right)+C}$$

(e) $\int \frac{3x}{(1+x^2)^4}\,dx$

$\uparrow$ u-sub

$\boxed{\begin{array}{l} U=1+x^2 \\ du=2x\,dx \\ dx=\frac{du}{2x} \end{array}}$

$$= \int \frac{3x}{u^4}\cdot\frac{du}{2x} = \frac{3}{2}\int\frac{1}{u^4}du = \frac{3}{2}\cdot\frac{1}{-3u^3}+C = \boxed{\frac{-1}{2(1+x^2)}+C}$$

(f) $\int_{-5}^{4} |x+3|\,dx$

$$= \frac{1}{2}(2)(2)+\frac{1}{2}(7)(7) = \frac{4}{2}+\frac{49}{2} = \boxed{\frac{53}{2}}$$

Use area: