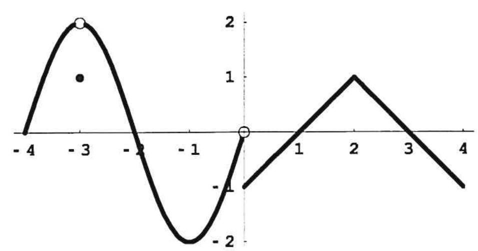
Solutions

1. (12 pts) For the function $f(x)$, graphed here,



a) (4 pts) Find the following limits. If the limit does not exist, please explain why not.

(i) $\lim_{x\to -3} f(x)$

$\boxed{2}$

(note $f(3)$ is irrelevant)

(ii) $\lim_{x\to 0} f(x)$

$\boxed{\text{DNE}}$, since $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = -1$. Left and right limits disagree

b) (2 pts) At what value(s) of x is $f(x)$ discontinuous?

$x = -3$ (jump)

$x = 0$ (jump)

c) (3 pts) At what value(s) of x does $f(x)$ have a removable discontinuity? Please list the value of y that you could use to redefine $f(x)$ so that $f(x)$ is continuous at the removable discontinuity.

$x = -3$ has a removable discontinuity. If $f(-3) = 2 = \lim_{x\to -3} f(x)$, $f(x)$ would be continuous at $x = -3$.

d) (3 pts) At what value(s) of x is $f(x)$ not differentiable?

At $\boxed{x=-3}$ and $\boxed{x=0}$ (not continuous)

and at $\boxed{x=2}$ (sharp corner)

2. (16 pts) Find the following limits, if the limit exists. If the limit does not exist, explain why.

(a) $\lim_{x\to 1} \frac{|2x^2+x-3|}{x-1}$

Consider left and right limits:

When $x>1$, $2x^2+x-3>0$ (try $x=10$)

Thus, $\lim_{x\to 1^+} \frac{|2x^2+x-3|}{x-1} = \lim_{x\to 1^+} \frac{2x^2+x-3}{x-1}$

$= \lim_{x\to 1^+} \frac{(2x+3)(x-1)}{x-1} = \lim_{x\to 1^+}(2x+3) = \boxed{5}$

When $x<1$, $2x^2+x-3<0$ (try $x=0$)

Thus, $\lim_{x\to 1^-} \frac{|2x^2+x-3|}{x-1} = \lim_{x\to 1^-} \frac{-(2x^2+x-3)}{x-1} = \boxed{-5}$

Since left and right limits disagree, $\lim_{x\to 1} \frac{|2x^2+x-3|}{x-1}$ $\boxed{DNE}$.

(b) $\lim_{x\to\infty} \frac{\sin^2 x}{x^2}$

Note: $-1 \le \sin(x) \le 1$, hence $0 \le \sin^2(x) \le 1$. Therefore,

$\lim_{x\to\infty} \frac{0}{x^2} \le \lim_{x\to\infty} \frac{\sin^2(x)}{x^2} \le \lim_{x\to\infty} \frac{1}{x^2}$, so

$0 \le \lim_{x\to\infty} \frac{\sin^2(x)}{x^2} \le 0.$

Conclude $\boxed{\lim_{x\to\infty} \frac{\sin^2(x)}{x^2} = 0}$ by Squeeze Theorem

$\lim_{x\to 0} \frac{e^{-2x}-1+2x}{1-\cos(3x)}$

Direct substitution:

$\frac{e^{-2(0)}-1+2(0)}{1-\cos(3(0))} \to \frac{0}{0}$ INDETERMINATE.

Use L'Hospital's Rule:

$= \lim_{x\to 0} \frac{-2e^{-2x}+2}{3\sin(3x)} \to \frac{0}{0}$ INDETERMINATE.

..., and again!

$= \lim_{x\to 0} \frac{4e^{-2x}}{9\cos(3x)} = \boxed{\frac{4}{9}}$

(d) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

Say $L = \lim_{x\to 0} (\cos(x))^{\frac{1}{x^2}}$. Then $\ln(L) = \lim_{x\to 0} \ln\left((\cos(x))^{\frac{1}{x^2}}\right)$

$= \lim_{x\to 0} \frac{\ln(\cos(x))}{x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{\left(\frac{-\sin(x)}{\cos(x)}\right)}{2x} = \lim_{x\to 0} \frac{-\sin(x)}{2x\cos(x)}$

$= \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right) \cdot \left(\lim_{x\to 0} \frac{-1}{2\cos(x)}\right) = 1 \cdot -\frac{1}{2} = \boxed{-\frac{1}{2}}$

Thus, $\boxed{L = e^{-1/2}}$.

3. (10 pts) Find the derivative of the function $f(x) = \sqrt{1-2x}$ using the limit definition of the derivative. NO CREDIT will be given if Limit Definition is not used.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{1-2(x+h)} - \sqrt{1-2x}\right)\left(\sqrt{1-2(x+h)}+\sqrt{1-2x}\right)}{h\left(\sqrt{1-2(x+h)}+\sqrt{1-2x}\right)}$$

$$= \lim_{h\to 0} \frac{1-2(x+h)-(1-2x)}{h\left(\sqrt{1-2(x+h)}+\sqrt{1-2x}\right)} = \lim_{h\to 0} \frac{1-2x-2h-1+2x}{h\left(\sqrt{1-2(x+h)}+\sqrt{1-2x}\right)} = \lim_{h\to 0} \frac{-2h}{h\left(\sqrt{1-2(x+h)}+\sqrt{1-2x}\right)}$$

$$= \lim_{h\to 0} \frac{-2}{\sqrt{1-2(x+h)}+\sqrt{1-2x}} = \frac{-2}{2\sqrt{1-2x}} = \boxed{\frac{-1}{\sqrt{1-2x}}}$$

4. (10 pts) Let $f(x) = 2x - 2 - \cos x$. Use Intermediate Value Theorem to show that $f(x) = 0$ has at least one root in the interval $[0, \pi]$.

When $x = 0$, $f(x) = 2(0) - 2 - \cos(0) = -2 - 1 = \boxed{-3} < 0$

When $x = \pi$, $f(x) = 2\pi - 2 - \cos(\pi) = 2\pi - 2 + 1 = \boxed{2\pi - 1} > 0$

Also, $f(x)$ is continuous on $[0, \pi]$.

We've satisfied the hypotheses for IVT; hence there is a root in the interval. ☒

5. (10 pts) Use implicit differentiation to find the equation of the tangent line to the curve

$$xy^2 + \ln(2x+1) = y$$

at the point (0,0).

Differentiate w.r.t x:

$$\{x\}\cdot\{2y\frac{dy}{dx}\} + \{1\}\{2y\} + \frac{2}{2x+1} = \frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$: $2yx\frac{dy}{dx} - \frac{dy}{dx} = \frac{-2}{2x+1} - 2y$

$$\frac{dy}{dx}(2yx-1) = \frac{-2}{2x+1} - 2y$$

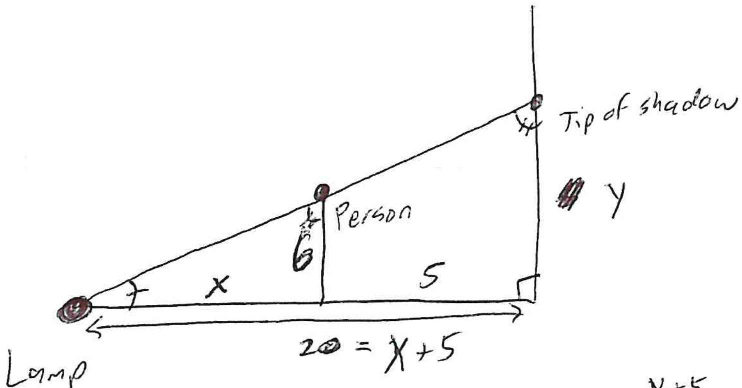
$$\frac{dy}{dx} = \frac{\frac{-2}{2x+1} - 2y}{2yx-1}.$$

At $(0,0)$, $\frac{dy}{dx} = \frac{\frac{-2}{0+1} - 0}{0-1} = \frac{-2}{-1} = \boxed{2}.$

Equation of tangent line: $y - y_1 = m(x - x_1)$.

$x_1 = 0$, $y_1 = 0$, $m = 2$, so $y - 0 = 2(x-0) \Rightarrow \boxed{y = 2x}$.

6. (12 pt) A flood lamp is installed on the ground 20 feet from a vertical wall. A 6-foot-tall man is walking towards the wall at the rate of 5 ft/s. How fast is the tip of his shadow moving down the wall when he is 5 feet from the wall?



By similar triangles, $\frac{x}{6} = \frac{x+5}{y}$. Since $x = 15$, get $\frac{15}{6} = \frac{20}{y} \Rightarrow y = 20 \cdot \frac{6}{15}$
$= 8$

Our equation is $\frac{x}{6} = \frac{x+5}{y} \Rightarrow xy = 6x + 30$.

Differentiate w.r.t. time: $x \cdot \frac{dy}{dt} + \frac{dx}{dt} \cdot y = 6 \frac{dx}{dt}$.

Since $\frac{dx}{dt} = 5$, we get $\frac{dy}{dt} = \frac{6 \cdot 5 - 5 \cdot 8}{15} = \frac{30 - 40}{15} = \frac{-10}{15} = \boxed{\frac{-2}{3} \text{ ft/s}}$

7. (20 pts) Differentiate the following functions. You don't have to simplify your answers.

(a) (5 pts) Find y'' for $y = \sec^3(2x)$.

$$y' = 3\sec^2(2x)\cdot(\sec(2x)\tan(2x))\cdot 2$$

$$= 6\underbrace{\sec^3(2x)\tan(2x)}_{\text{product Rule}}$$

$$y'' = 6\left([3\sec^2(2x)\cdot(\sec(2x)\tan(2x))\cdot 2][\tan(2x)] + [\sec^3(2x)][\sec^2(2x)\cdot 2]\right)$$

(b) (5 pts) $y = \dfrac{\cos^{-1}\sqrt{-x}}{4^x+1} + \tan^{-1}x\sin^{-1}x$.

Quotient Product

$$y' = \frac{[(4^x+1)]\left[\frac{-1}{\sqrt{1-(\sqrt{-x})^2}}\cdot\frac{-1}{2\sqrt{-x}}\right] - [\arccos(\sqrt{-x})][4^x\ln(4)]}{(4^x+1)^2}$$

$$+ \frac{1}{1+x^2}\arcsin(x) + \arctan(x)\frac{1}{\sqrt{1-x^2}}$$

(c) (5 pts) $y = \dfrac{x^2e^{-2x}}{\sqrt{(1+3x)(x-4)}}$.

Logarithmic differentiation

$$\ln(y) = \ln(x^2) + \ln(e^{-2x}) - \ln\left(\sqrt{(1+3x)(x-4)}\right)$$

$$= \ln(x^2) + \ln(e^{-2x}) - \frac{1}{2}\ln(1+3x) - \frac{1}{2}\ln(x-4)$$

$$= \ln(x^2) - 2x - \frac{1}{2}\ln(1+3x) - \frac{1}{2}\ln(x-4)$$

$$\frac{1}{y}\cdot y' = \frac{2x}{x^2} - 2 - \frac{1}{2}\cdot\frac{3}{1+3x} - \frac{1}{2}\cdot\frac{1}{x-4}$$

$$y' = \left(\frac{2}{x} - 2 - \frac{3/2}{1+3x} - \frac{1/2}{x-4}\right)\left[\frac{x^2e^{-2x}}{\sqrt{(1+3x)(x-4)}}\right]$$

(d) (5 pts) $y = \displaystyle\int_{x^2}^{0} \sin(u)(2-4u)^8\,du$.

$$\boxed{y' = \sin(0)(2-4(0))^8\cdot 0 - \sin(x^2)(2-4(x^2))^8\cdot 2x}$$

8. a) (5 pts) Find the linear approximation of the function $f(x) = x^{\frac{2}{3}}$ at $a = 8$ and use the linearization to approximate $f(7.9)$.

$f(x) \approx f(a) + f'(a)(x-a)$.

$f(8) = 8^{2/3} = 2^2 = \boxed{4}$

$f'(x) = \frac{2}{3} x^{-1/3}$, so $f'(8) = \frac{2}{3(8)^{1/3}} = \boxed{\frac{1}{3}}$

Get $f(x) \approx 4 + \frac{1}{3}(x-8)$, so

$f(7.9) \approx 4 + \frac{1}{3}(7.9-8) = 4 + \frac{-1}{30} = \boxed{\frac{119}{30}}$

b) (5 pts) Use **Newton's Method** with the initial approximation $x_1 = 1$ to find the second approximation x_2 to the solution to the equation

$$e^{-2x} = \sqrt{x} + x$$

You don't have to simplify your result.

Formula: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(x) = e^{-2x} - \sqrt{x} - x = 0$ $\quad f(1) = e^{-2} - 1 - 1 = e^{-2} - 2$

$f'(x) = -2e^{-2x} - \frac{1}{2\sqrt{x}} - 1$, so $f'(1) = -2e^{-2} - \frac{1}{2} - 1$

Get $\boxed{x_2 = 1 - \frac{e^{-2} - 2}{-2e^{-2} - \frac{1}{2} - 1}}$.

9. (18 pts) Given the function $f(x) = \dfrac{x^2}{x^2+3}$,

(a)(3 pts) Find its $x-$ and $y-$ intercepts, identify all vertical and horizontal asymptotes of the graph of this function if there exists.

$f(x) = 0$ when $x^2 = 0 \Rightarrow \boxed{x=0}$

Vertical asymptotes when $x^2+3=0 \Rightarrow$ No solutions

Horizontal asymptotes: $\lim\limits_{x\to\infty} \dfrac{x^2}{x^2+3} = \lim\limits_{x\to\infty} \dfrac{x^2}{x^2} = \lim\limits_{x\to\infty} 1 = \boxed{1}$, $\lim\limits_{x\to-\infty} \dfrac{x^2}{x^2+3} = \lim\limits_{x\to-\infty} 1 = \boxed{1}$

(b) (6 pts) Find the critical numbers of $f(x)$ and give the increasing and decreasing intervals. Determine whether the critical numbers are local minimizers, local maximizers or neither.

$f'(x) = \dfrac{(x^2+3)2x - x^2(2x)}{(x^2+3)^2} = \dfrac{2x^3+6x-2x^3}{(x^2+3)^2} = \dfrac{+6x}{(x^2+3)^2}$, $f'(x)=0$ when $+6x=0 \Rightarrow \boxed{x=0}$

$f'(x)$ DNE when $(x^2+3)^2=0 \Rightarrow$ No solutions

Sign Chart: $f'(x)$: $\ominus$ 0 $\oplus$

x: -1, 0, 1

$x=0$ is a local [illegible].

$f(x)$ increases on $(0, \infty)$

$f(x)$ decreases on $(-\infty, 0)$

(c) (5 pts) Find the intervals where $f(x)$ are concave upward and concave downward .

$f''(x) = \dfrac{(x^2+3)^2 \cdot +6 - (+6x)2(x^2+3)\cdot 2x}{(x^2+3)^4} = \dfrac{+6(x^2+3) - 6x(4x)}{(x^2+3)^3} = \dfrac{+6x^2+18-24x^2}{(x^2+3)^3} = \dfrac{-18x^2+18}{(x^2+3)^3}$

$= \dfrac{-18(x-1)(x+1)}{(x^2+3)^3}$. $f''(x)=0$ when $-18(x-1)(x+1)=0 \Rightarrow \boxed{x=-1}$ or $\boxed{x=1}$

$f''(x)$ DNE when $(x^2+3)^3=0 \Rightarrow$ No solutions

Sign Chart: $f''(x)$: $\oplus$ 0 $\ominus$ 0 $\oplus$

x: -2, -1, 0, 1, 2

$f(x)$ is concave up on $(-\infty,-1)\cup(1,\infty)$.

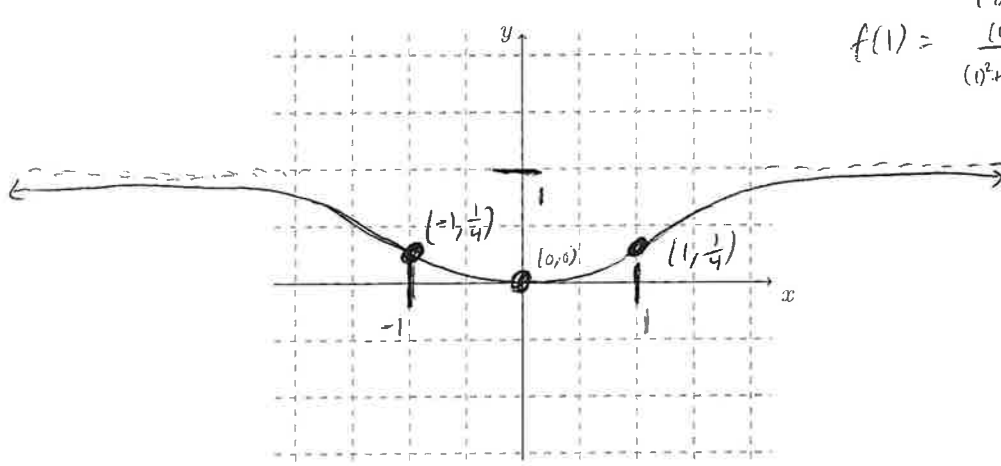
$f(x)$ is concave down on $(-1,1)$.

(Inflection pts @ $x=-1$, $x=1$)

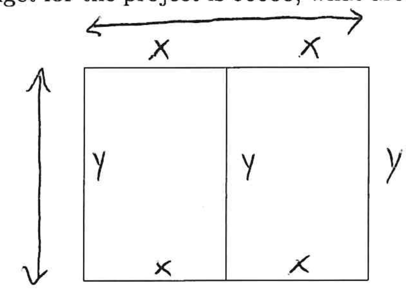
(d) (4 pts) Sketch the graph of the function $f(x)$ representing clearly the information gathered above using the axis provided below.

$f(-1) = \dfrac{(-1)^2}{(-1)^2+3} = \dfrac{1}{4}$

$f(1) = \dfrac{(1)^2}{(1)^2+3} = \dfrac{1}{4}$



10. (12 pts) A large rectangular area is to be fenced off as in the diagram below (a large rectangle divided into two smaller rectangles). The fence used to divide the space costs \$10 per foot and the fence used for the perimeter costs \$15 per foot. If the total budget for the project is \$6000, what are the dimensions which yield the largest area?



Objective: Max area $= (2x)(y)$

Constraint: Cost $= 6000 = (4x+2y)(15) + (y)(10) \Rightarrow 6000 = 60x + 40y$

$\Rightarrow y = \dfrac{6000-60x}{40} = 150 - 1.5x$

So, $A(x) = 2x(150-1.5x) = 300x - 3x^2$

$A'(x) = 300 - 6x = 0 \Rightarrow 300 = 6x \Rightarrow \boxed{x=50}$ (so $2x = 100$)

$y = 150 - 1.5(50) = 150 - 75 = \boxed{75}$.

So the dimensions are $\boxed{100 \text{ by } 75}$.

11. (10 pts) A particle moves along a straight line with the velocity $v(t) = t - \frac{16}{t^3}$ ft/s.

(a) (4 pts) Find the total displacement of the particle over the time interval $[1, 4]$.

Total Displacement = Net change in distance = $\int_1^4 v(t)\,dt = \int_1^4 \left(t - \frac{16}{t^3}\right) dt$

$$= \left[\frac{t^2}{2} - \frac{16}{-2t^2}\right]_1^4 = \left[\frac{(4)^2}{2} + \frac{16}{2(4)^2}\right] - \left[\frac{(1)^2}{2} + \frac{16}{2(1)^2}\right] = \left(8+\frac{1}{2}\right) - \left(\frac{1}{2}+8\right) = \boxed{0}$$

Note this is the final position − the initial position

(b) (6 pts) Find the total distance traveled by the particle over the interval $[1, 4]$.

Total Distance ⇔ All movement is treated as positive, so

Total distance = $\int_1^4 |v(t)|\,dt = \int_1^4 \left|t - \frac{16}{t^3}\right| dt$.

Note $t - \frac{16}{t^3} = 0 \Rightarrow t^4 = 16 \Rightarrow \boxed{t=2}$

Sign chart: $v(t)$ ⊖ 0 ⊕

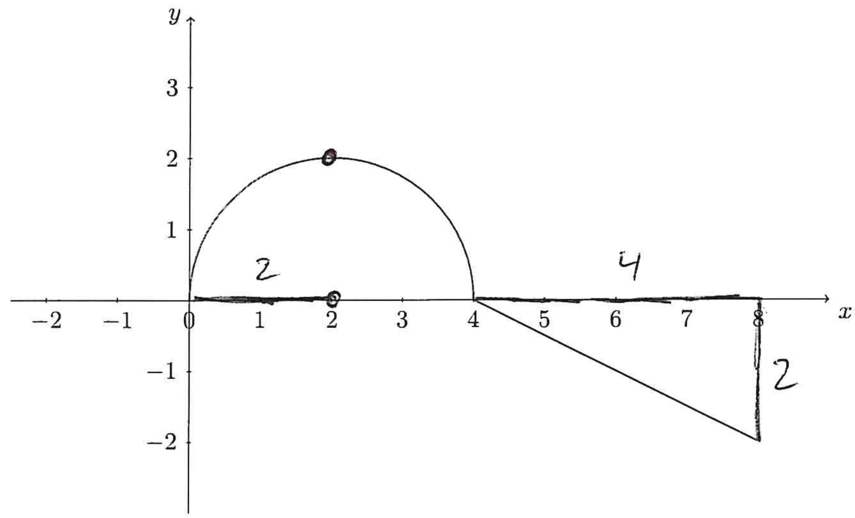
* 1 2 4

$$= \int_1^2 \left(\frac{16}{t^3} - t\right) dt + \int_2^4 \left(t - \frac{16}{t^3}\right) dt$$

$$= \left[\frac{-16}{2t^2} - \frac{t^2}{2}\right]_1^2 + \left[\frac{t^2}{2} + \frac{16}{2t^2}\right]_2^4 = \left(\frac{-16}{2(2)^2} - \frac{(2)^2}{2}\right) - \left(\frac{-16}{2(1)^2} - \frac{(1)^2}{2}\right) + \left(\frac{(4)^2}{2} + \frac{16}{2(4)^2}\right) - \left(\frac{(2)^2}{2} + \frac{16}{2(2)^2}\right)$$

$$= -2 - 2 + 8 + \frac{1}{2} + 8 + \frac{1}{2} - 2 - 2 = 17 - 8 = \boxed{9}.$$

12. (12 pts) Let $A(x) = \int_0^x f(t)dt$ where $f(t)$ is shown in the graph below. Note that the graph consists of a semicircle and a straight line segment.



(a) (4 pts) Find $A(4)$ and $A(8)$.

$$A(4) = \int_0^4 f(t)\,dt = \text{Area(semicircle)} = \tfrac{1}{2}\pi(2)^2 = \boxed{2\pi}$$

$$A(8) = \int_0^8 f(t)\,dt = \text{Area(semicircle)} - \text{Area(triangle)} = 2\pi - \tfrac{1}{2}(4)(2) = \boxed{2\pi - 4}$$

(b) (2 pts) Find $A'(2)$.

$$A'(x) = \frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x), \text{ so } A'(2) = f(2) = \boxed{2}$$

(c) (3 pts) Find all critical points of $A(x)$ in the interval $(0, 8)$ and classify each of them as a local maximum, local minimum, or neither.

Critical points occur when $A'(x)=0$. Since $A'(x)=f(x)$, this is when $\boxed{x=0}$ or $\boxed{x=4}$.

Sign chart: $A'(x)=f(x)$ + 0 −

x 0 4 8

$\boxed{x=4 \text{ is a local max}}$

Disregard, not in interval.

(d) (3 pts) Determine the interval(s) on which $A(x)$ is concave up.

$A(x)$ is concave up when $A''(x)>0$. Since $A'(x)=f'(x)$, find when $f(x)$ is increasing. This occurs from $x=0$ to $x=2$, so $\boxed{A(x) \text{ is concave up on } (0,2).}$

13. (48 pts, 8 pts each) Evaluate the following integrals.

(a) $\displaystyle\int \sqrt[3]{x^2} + \frac{1}{1+x^2} + e^2\,dx$

$= \displaystyle\int \left(x^{2/3} + \frac{1}{1+x^2} + e^2\right) dx$

$= \boxed{\dfrac{3x^{5/3}}{5} + \arctan(x) + e^2 \cdot x + C}$

(b) $\displaystyle\int_{-4}^{4} x\sqrt[3]{x+4}\,dx$ ← u-sub

$U = x+4 \qquad x = u-4$

$du = dx$

lower limit $= (-4)+4 = 0$

upper limit $= (4)+4 = 8$

$= \displaystyle\int_0^8 (u-4)u^{1/3}\,du = \int_0^8 (u^{4/3} - 4u^{1/3})\,du = \left[\frac{3u^{7/3}}{7} - 4\cdot\frac{3u^{4/3}}{4}\right]_0^8$

$= \boxed{\left(\dfrac{3(8)^{7/3}}{7} - 3(8)^{4/3}\right) - \left(\dfrac{3(0)^{7/3}}{7} - 3(0)^{4/3}\right)}$

(c) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ ← u-sub

$U = e^x + e^{-x}$

$du = (e^x - e^{-x})dx$

$= \displaystyle\int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|e^x + e^{-x}| + C}$

(d) $\displaystyle\int_0^{\pi/4} 4\cos^2(2x)\,dx$ ← Double-angle formula

$$\left\{ \cos^2(\theta) = \frac{1}{2}\left(1+\cos(2\theta)\right) \right\}$$

$$= 4\int_0^{\pi/4} \frac{1}{2}\left(1+\cos(4x)\right)dx = 2\left(\int_0^{\pi/4} 1\,d\cancel{x} + \int_0^{\pi/4}\cos(4x)\,dx\right)$$

$$= 2\left(\left(\frac{\pi}{4}\right)-(0) + \left[\frac{1}{4}\sin(4x)\right]_0^{\pi/4}\right) = 2\left(\frac{\pi}{4} + \left(\frac{1}{4}\sin(\pi)\right) - \left(\frac{1}{4}\sin(0)\right)\right)$$

u-sub

$u=4x$

$$= 2\left(\frac{\pi}{4}+0-0\right) = \boxed{\frac{\pi}{2}}$$

(e) $\displaystyle\int \sin^{2014}x\cdot\cos^3 x\,dx$ ← u-sub

$u=\sin(x)$

$du=\cos(x)dx$

$$= \int u^{2014}\cos^2(x)\,du = \int u^{2014}(1-\sin^2(x))\,du = \int u^{2014}(1-u^2)\,du$$

$$= \int\left(u^{2014}-u^{2016}\right)du = \frac{u^{2015}}{2015} - \frac{u^{2017}}{2017} + C = \boxed{\frac{\sin^{2015}(x)}{2015} - \frac{\sin^{2017}(x)}{2017} + C}$$

(f) $\displaystyle\int \frac{x}{e^{2x}}\,dx = \int x e^{-2x}\,dx$ ← Parts

$$\boxed{\int f\cdot dg = f\cdot g - \int g\cdot df}$$

$f=x$ $\quad dg = e^{-2x}dx$

$df = dx$ $\quad g = -\frac{1}{2}e^{-2x}$

u-sub

$u=-2x$

$$= -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$$

$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$$

$$= \boxed{-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C}$$