Solutions

1. (10 pts) The position of a body on $0 \leq t \leq 4$ can determined by the function:

$$s(t) = 8t^2 - t + 2$$

(a) (5 pts) Determine the average velocity on the interval $[2, 2+h]$.

$$\text{Average velocity} = \frac{\text{Distance}}{\text{Time}} = \frac{s(2+h)-s(2)}{2+h-2} = \frac{8(2+h)^2-(2+h)+2-\left(8(2)^2-2+2\right)}{h}$$

$$= \frac{32+32h+8h^2-2-h+2-32+2-2}{h} = \frac{8h^2+31h}{h} = \boxed{8h+31}.$$

(b) (3 pts) Determine the average velocity on the interval $[2, 2.21]$

Average velocity $= 8h+31$, where $h = 0.21$. Thus, $AV = 8(0.21)+31 = 1.68+31 = \boxed{32.68}$

(c) (2 pts) Deternine the instantaneous velocity at time $t = 2$.

$$\text{Velocity} = \lim_{h \to 0}\left(\text{Average velocity}\right) = \lim_{h\to 0}(8h+31) = \boxed{31}$$

2. (15 pts) If $f(x) = e^{-x/2}\cos(x)$,

(a) (10 pts) Determine the linearization of the function at $x = 0$

$f(x) \approx f(a) + f'(a)(x-a)$, where $a = 0$. $\boxed{\text{Get } f(x) \approx 1 + \frac{-1}{2}(x-0)}$

Note $f(0) = e^{-(0)/2}\cos(0) = 1 \cdot 1 = \boxed{1}$, $f'(x) = -e^{-x/2}\sin(x) - \frac{1}{2}e^{-x/2}\cos(x)$ so

$$f'(0) = -e^{-(0)/2}\sin(0) - \frac{1}{2}e^{-(0)/2}\cos(0) = \boxed{-\frac{1}{2}}.$$

(b) (5 pts) Use the linearization to approximate $f(0.2)$

$$f(0.2) \approx 1 - \frac{1}{2}(0.2-0) = 1 - 0.1 = \boxed{0.9}.$$

3. (40 pts) Differentiate the following functions. You do not need to simplify your answer.

(a) $f(x) = 3(6x+5)^3(x^2-1)^4$ Product Rule!

$$\boxed{f'(x) = 3\left((6x+5)^3 \cdot \left[4(x^2-1)^3 \cdot 2x\right] + \left\{3(6x+5)^2 \cdot 6\right\} \cdot (x^2-1)^4\right)}$$

(b) $y = \dfrac{3xe^{-x}}{5+9x^2}$

Solution 1: Quotient/Product Rule

$$\boxed{y' = \frac{\{5+9x^2\} \cdot \left[3\left(x(-e^{-x}) + e^{-x}\right)\right] - \left[3xe^{-x}\right] \cdot \left[18x\right]}{(5+9x^2)^2}}$$

Solution 2: Logarithmic Differentiation:

$$\ln|y| = \ln(3) + \ln(x) + \ln(e^{-x}) - \ln(5+9x^2)$$

$$= \ln(3) + \ln(x) - x - \ln(5+9x^2)$$

$$\frac{1}{y} \cdot y' = 0 + \frac{1}{x} - 1 - \frac{18x}{5+9x^2} \Rightarrow \boxed{y' = \left(\frac{1}{x} - 1 - \frac{18x}{5+9x^2}\right)\left[\frac{3xe^{-x}}{5+9x^2}\right]}$$

(c) $f(x) = \sin^3(4x)$

Chain Rule Note $f(x) = (\sin(4x))^3$

$$\boxed{f'(x) = 3\sin^2(4x) \cdot \cos(4x) \cdot 4}$$

(d) $y = \ln\left(\sqrt{\dfrac{x^2+7x+1}{12x-7}}\right)$ ↙ Logarithmic differentiation

$$y = \frac{1}{2}\ln\left(\frac{x^2+7x+1}{12x-7}\right) = \frac{1}{2}\left(\ln(x^2+7x+1) - \ln(12x-7)\right)$$

$$\boxed{y' = \frac{1}{2}\left(\frac{2x+7}{x^2+7x+1} - \frac{12}{12x-7}\right)}$$

(e) $f(x) = \tan(4x) + \arctan(4x)$

$$\boxed{f'(x) = \sec^2(4x) \cdot 4 + \frac{1}{1+(4x)^2} \cdot 4}$$

(f) $y = \left(1 + 3e^{x/6}\right)^{1/x}$

$$\ln(y) = \ln\left(\left(1+3e^{x/6}\right)^{1/x}\right) = \frac{1}{x} \cdot \ln\left(1+3e^{x/6}\right)$$

$$\frac{1}{y} \cdot y' = \left[\frac{1}{x}\right] \cdot \left[\frac{3e^{x/6} \cdot \frac{1}{6}}{1+3e^{x/6}}\right] + \left[\frac{-1}{x^2}\right] \cdot \ln\left(1+3e^{x/6}\right)$$

$$\boxed{y' = \left(\frac{1}{x} \cdot \frac{3e^{x/6} \cdot \frac{1}{6}}{1+3e^{x/6}} + \frac{-1}{x^2} \cdot \ln\left(1+3e^{x/6}\right)\right) \cdot \left(1+3e^{x/6}\right)^{1/x}}$$

(g) $f(x) = 5x\sqrt{25 - x^2}$

$$\boxed{f'(x) = 5\left(x \cdot \left[\frac{-2x}{2\sqrt{25-x^2}}\right] + \sqrt{25-x^2}\right)}$$

(h) $y = \int_4^{\sqrt{x}} \cos^4(6u) \sin^4(8u)\, du$

$$\boxed{y' = \cos^4(6\sqrt{x}) \sin^4(8\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} - \cos^4(6(4)) \sin^4(8(4)) \cdot 0}$$

4. (15 pts) Determine the limit (show all work)

(a) $\lim_{x\to 0} \frac{\ln(e^x + 3\sin x)}{x + \sin(\pi x)}$ $\to \ln(1) = 0$ / $\to 0+0=0$ INDETERMINATE

$\overset{L'H}{=} \lim_{x\to 0} \frac{\left(\frac{e^x + 3\cos(x)}{e^x + 3\sin(x)}\right)}{1 + \cos(\pi x)\cdot\pi} = \frac{\left(\frac{1+3}{1+0}\right)}{1+1\cdot\pi} = \boxed{\frac{4}{1+\pi}}$

(b) $\lim_{x\to\infty} \left(1 - \frac{1}{2x}\right)^x = L$ $\to 1^\infty$ INDETERMINATE

$\ln(L) = \lim_{x\to\infty} \ln\left(\left(1-\frac{1}{2x}\right)^x\right) = \lim_{x\to\infty} x \cdot \ln\left(1 - \frac{1}{2x}\right) = \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{2x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{2x}} \cdot \frac{2}{(2x)^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{2x}} \cdot \frac{1}{2} \cdot \frac{1}{x^2}}{\frac{-1}{x^2}}$

$= \lim_{x\to\infty} \frac{1}{1-\frac{1}{2x}} \cdot \frac{-1}{2} = \boxed{\frac{-1}{2}}$ Thus, $\boxed{L = e^{-1/2}}$

(c) $\lim_{x\to 1^-} \frac{\arccos(x)}{\sqrt{1-x^2}}$ Since $\cos(0) = 1$, $\arccos(1) = 0$. Get $\frac{0}{0}$ INDETERMINATE

$\overset{L'H}{=} \lim_{x\to 1^-} \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{-2x}{2\sqrt{1-x^2}}} = \lim_{x\to 1^-} \frac{1}{x} = \frac{1}{1} = \boxed{1}$

5. (15 pts) A jogger runs along an elliptic track which has the path

$$80x^2 + 100y^2 = 10500$$

measured in meters. As he reaches the point $(10, 5)$, the x-coordinate of his path is changing at a rate of 3 m/sec.

(a) (5 pts) At what rate is the y coordinate changing?

Differentiate with respect to time: $160x \cdot \frac{dx}{dt} + 200y \cdot \frac{dy}{dt} = 0$

Now, $x = 10$, $y = 5$, $\frac{dx}{dt} = 3$. So, $\frac{dy}{dt} = \frac{-160(10)(3)}{200(5)} = \frac{-4800}{1000} = \boxed{-4.8 \text{ m/sec}}$

(b) (10 pts) A spectator stands at the point $(22, 10)$. At what rate is the distance from runner to the spectator changing at this time?

Distance $D = \sqrt{(x-22)^2 + (y-10)^2}$, So $D^2 = (x-22)^2 + (y-10)^2$.

Differentiate with respect to time: $2D \cdot \frac{dD}{dt} = 2(x-22) \cdot \frac{dx}{dt} + 2(y-10)\frac{dy}{dt}$.

When $x = 10$, $y = 5$, $D = \sqrt{144 + 25} = \sqrt{169} = \boxed{13}$. We have $\frac{dx}{dt}$, $\frac{dy}{dt}$, x, y, D.

So, $\frac{dD}{dt} = \frac{2(10-22) \cdot 3 + 2(5-10) \cdot (-4.8)}{2 \cdot 13} = \frac{-72 - 48}{26} = \frac{-120}{26} = \boxed{\frac{-60}{13} \text{ m/sec}}$

6. (20 pts) For the function $f(x) = \dfrac{x^3}{x-2}$ answer the following:

$f'(x) = \dfrac{(x-2)3x^2 - x^3 \cdot 1}{(x-2)^2}$

$= \dfrac{3x^3 - 6x^2 - x^3}{(x-2)^2} = \dfrac{2x^3 - 6x^2}{(x-2)^2}$

$\boxed{= \dfrac{2x^2(x-3)}{(x-2)^2}}$

(a) (5 pts) List the critical values of $f(x)$.

$f'(x) = 0$ when $2x^2 = 0$ or $x - 3 = 0$

$\Rightarrow \boxed{x = 0}$ or $\boxed{x = 3}$.

$f'(x)$ DNE when $(x-2)^2 = 0 \Rightarrow \boxed{x = 2}$

(b) (5 pts) For each listed x above, state whether there is a local maximum, local minimum, inflection point, or vertical asymptote.

Sign Chart: $f'(x)$ ⊖ 0 ⊖ DNE ⊖ 0 ⊕

x ↑ 0 ↑ 2 ↑ 3 ↑

-1 1 2.5 100

Vertical asymptote at $\boxed{x = 2}$

Inflection point at $\boxed{x = 0}$

Local min at $\boxed{x = 3}$

(c) (5 pts) $\lim_{x \to -\infty} f(x) = \boxed{\text{DNE } (+\infty)}$

$= \lim_{x \to -\infty} \dfrac{x^3}{x} = \lim_{x \to -\infty} x^2 = \infty$

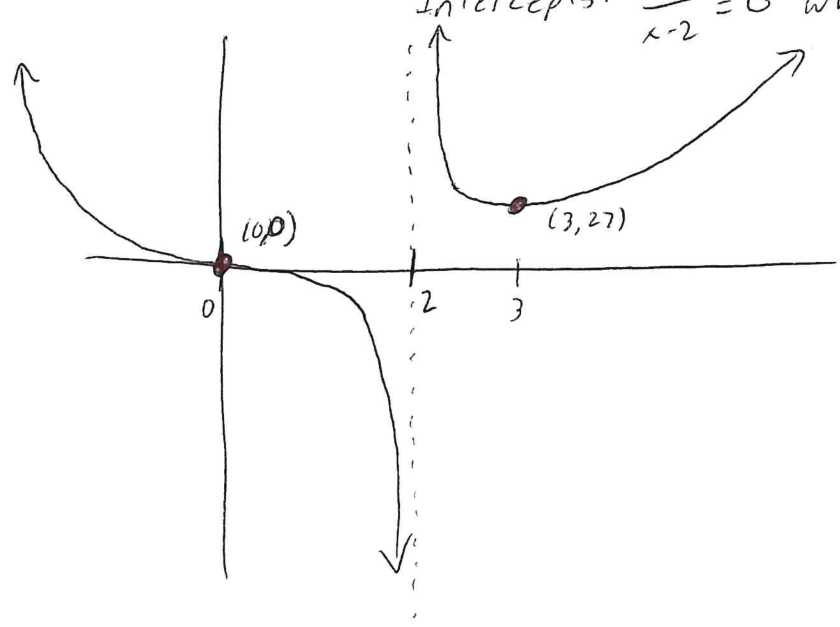
$\lim_{x \to \infty} f(x) = \boxed{\text{DNE } (+\infty)}$

$= \lim_{x \to \infty} \dfrac{x^3}{x} = \lim_{x \to \infty} x^2 = \infty$

(d) (5pts) Graph the function. Show asymptote(s), intercepts and the exact points (x and y coordinates) where the function has any or all local maximum, local minimum values.

Intercepts: $\dfrac{x^3}{x-2} = 0$ when $\boxed{x = 0}$.

When $x = 3$, $f(x) = \dfrac{(3)^3}{3-2} = \dfrac{27}{1} = 27$



7. (15 pts) A cylindrical can is to have volume 62.5π cm^3. If the cost to make the top and bottom lid is \$1.00/cm^2 and the cost to make the cylindrical side is \$2.00/cm^2, what are the dimensions of such a cylinder that would minimize cost?
Note: $v = \pi r^2 h$. Area of each lid is $aL = \pi r^2$. Area of cylindrical side is $aS = 2\pi r h$.

Objective: Minimize $\text{Cost} = 1 \cdot (2\pi r^2) + 2 \cdot (2\pi r h)$

Constraint: $V = \pi r^2 h \Rightarrow 62.5\pi = \pi r^2 h \Rightarrow \boxed{h = \dfrac{62.5}{r^2}}$

So, $\text{cost} = C(r) = 2\pi r^2 + 4\pi r\left(\dfrac{62.5}{r^2}\right) = 2\pi r^2 + \dfrac{250\pi}{r}$

$$C'(r) = 4\pi r - \frac{250\pi}{r^2} = 0 \Rightarrow 4\pi r = \frac{250\pi}{r^2}$$

$$\Rightarrow r^3 = \frac{250\pi}{4\pi} \Rightarrow r^3 = 62.5 \Rightarrow \boxed{r = \sqrt[3]{62.5}}$$

$$\boxed{h = \frac{62.5}{\left(\sqrt[3]{62.5}\right)^2}}$$

8. (10 pts) Use Newton's Method once starting with $x_0 = 3$ to approximate the solution to $x = \sqrt{25 - x^2}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. Now, $f(x) = \sqrt{25-x^2} - x = 0$, so $f(3) = \sqrt{25-(3)^2} - 3 = \sqrt{16} - 3$
$= 4 - 3 = \boxed{1}$

and $f'(x) = \frac{-2x}{2\sqrt{25-x^2}} - 1$, so $f'(3) = \frac{-6}{2(4)} - 1 = \frac{-3}{4} - 1 = \boxed{\frac{-7}{4}}$.

Get $x_1 = 3 - \frac{1}{\left(\frac{-7}{4}\right)} = 3 - \frac{4}{7} = \frac{21}{7} - \frac{4}{7} = \boxed{\frac{17}{7}}$.

9. (10 pts) At the point $(1, 0)$, determine the equation of the tangent line to the curve

$$(2x + 3y)^3 - 6x - 12y = 2.$$

Differentiate w.r.t. x: $3(2x+3y)^2 \cdot \left(2 + 3\frac{dy}{dx}\right) - 6 - 12\frac{dy}{dx} = 0$

$6(2x+3y)^2 + 9(2x+3y)^2 \cdot \frac{dy}{dx} - 6 - 12\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(9(2x+3y)^2 - 12\right) = 6 - 6(2x+3y)^2$

$\frac{dy}{dx} = \frac{6 - 6(2x+3y)^2}{9(2x+3y)^2 - 12}$. At $(1,0)$, $\frac{dy}{dx} = \frac{6 - 6(2)^2}{9(2)^2 - 12} = \frac{6-24}{36-12} = \frac{-18}{24} = \boxed{\frac{-3}{4}}$.

Equation of tangent line: $y - y_1 = m(x - x_1)$

$y_1 = 0$, $x_1 = 1$, $m = \frac{-3}{4}$, so get $\boxed{y - 0 = \frac{-3}{4}(x - 1)}$

10. (30 pts)Evaluate the following integrals.

(a) $\int_0^1 \left(\frac{4}{1+x^2} + \sqrt{8+x}\right)\, dx$ $\quad u = 8+x,\ du = dx$

$= \left[4\arctan(x) + \frac{2(8+x)^{3/2}}{3} \right]_0^1 = \left(4\arctan(1) + \frac{2(9)^{3/2}}{3}\right) - \left(4\arctan(0) + \frac{2(8)^{3/2}}{3}\right)$

$\boxed{= 4a\frac{\sqrt{2}}{2} + 18 - 0 - \frac{2\sqrt{512}}{3}}$

(b) $\int 3x^4 \ln(x)\, dx$ $\quad$ Parts: $\int f \cdot dg = f \cdot g - \int g \cdot df$

$\boxed{\begin{array}{ll} f = 3\ln(x) & dg = x^4\,dx \\ df = \frac{3}{x}dx & g = \frac{x^5}{5} \end{array}}$

$\int 3x^4 \ln(x) = \frac{3}{5}x^5 \ln(x) - \int \frac{x^5}{5} \cdot \frac{3}{x}\, dx$

$= \frac{3}{5}x^5 \ln(x) - \frac{3}{5}\int x^4\,dx = \boxed{\frac{3}{5}x^5 \ln(x) - \frac{3}{5} \cdot \frac{x^5}{5} + C}$

(c) $\int \frac{x+1}{x^2+2x+7}\, dx$ $\quad$ u-sub

$u = x^2 + 2x + 7$

$du = (2x+2)dx$

$dx = \frac{du}{2x+2}$

$= \int \frac{x+1}{u} \cdot \frac{du}{2x+2} = \int \frac{1}{2} \cdot \frac{1}{u}\, du = \frac{1}{2}\ln|u| + C$

$\boxed{= \frac{1}{2}\ln|x^2+2x+7| + C}$

(d) $\displaystyle\int \frac{5}{(4x+5)^3}\,dx$ ← u-sub

$$\boxed{\begin{array}{l} u = 4x+5 \\ du = 4\,dx \\ dx = \frac{du}{4} \end{array}}$$

$$= \int \frac{5}{u^3} \cdot \frac{du}{4} = \frac{5}{4}\int u^{-3}\,du = \frac{5}{4}\cdot\frac{u^{-2}}{-2} + C = \boxed{\frac{-5}{8}(4x+5)^{-2} + C}$$

(e) $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$

$$\boxed{\begin{array}{l} u = \sin(x) \\ du = \cos(x)\,dx \end{array}}$$

$$= \int u^2 \cos^2(x)\,du = \int u^2(1-\sin^2(x))\,du = \int u^2(1-u^2)\,du$$

$$= \int (u^2 - u^4)\,du = \frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$$

(f) $\displaystyle\int 7x\sin(3x)\,dx$ ← Parts

$$\boxed{\begin{array}{ll} f = 7x & dg = \sin(3x)\,dx \\ df = 7\,dx & g = -\frac{1}{3}\cos(3x) \end{array}}$$

↑ u-sub $u=3x$

$$= \frac{-7x}{3}\cos(3x) - \int \frac{-7}{3}\cos(3x)\,dx$$

$$= \frac{-7x}{3}\cos(3x) + \frac{7}{3}\int \cos(3x)\,dx$$

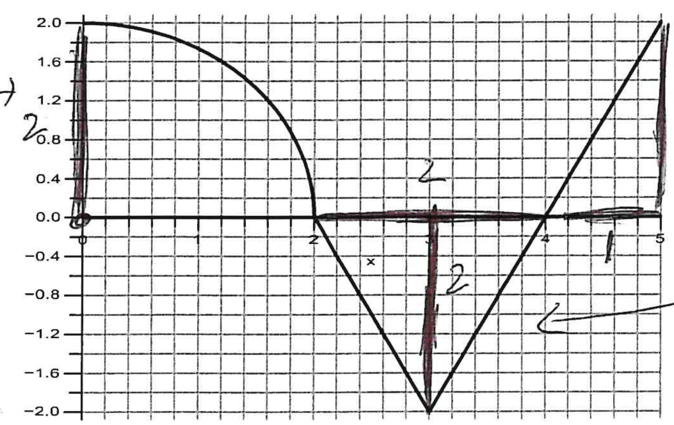
$$\boxed{= \frac{-7x}{3}\cos(3x) + \frac{7}{9}\sin(3x) + C}$$

↑ u-sub $u=3x$

11. (10 pts) Below is the plot of $f'(x) = \begin{cases} \sqrt{4-x^2} & 0 \le x \le 2 \\ -2+2|x-3| & 2 \le x \le 5 \end{cases}$

Use geometry to evaluate the integrals below:

Quarter-circle, radius = 2

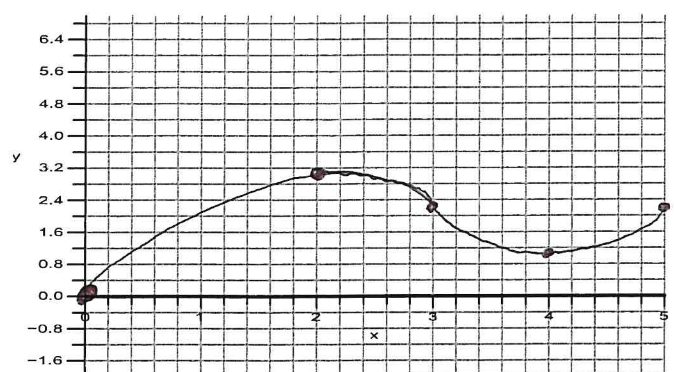


Triangle
Base = 1, Height = 2
Area = $\frac{1}{2}(1)(2) = \boxed{1}$

Triangle.
Base = 2, Height = 2
Area = $\frac{1}{2}(2)(2) = \boxed{2}$

$$\int_0^2 \sqrt{4-x^2}\,dx = \underline{\tfrac{1}{4}\pi(2)^2} \qquad \int_2^5 (-2+2|x-3|)\,dx = \underline{-1}$$

12. (10 pts) Sketch the position function in the grid below starting with $f(0) = 0$. Show proper concavity of the curve. Give the exact values of the function in the spaces to the right.



$f'(x) > 0$, $f''(x) < 0$	$f'(x) < 0$, $f''(x) < 0$	$f'(x) < 0$, $f''(x) > 0$	$f'(x) > 0$, $f''(x) > 0$
Concave down	Concave down	Concave up	Concave up

$f(2) = \underline{\tfrac{1}{4}\pi(2)^2 = \boxed{\pi}}$

$f(3) = \underline{\pi - 1}$

$f(4) = \underline{\pi - 2}$

$f(5) = \underline{\pi - 1}$