MATH 0200 – Prep for Scientific Calculus
SAMPLE FINAL EXAM 3

Exam length: 1 hour 50 minutes

INSTRUCTIONS:

1. NO TABLES, BOOKS, NOTES, HEADPHONES, CALCULATORS, OR COMPUTERS MAY BE USED.
2. Show ALL of your calculations and display answers clearly. You may leave your final answers in exact form. Unjustified answers will receive no credit.
3. WRITE YOUR SOLUTIONS in the space provided. EXTRA SPACE is available on the BACKS of the pages. When using these back pages, clearly LABEL the problem, and also clearly indicate on the appropriate front page where your back-page solution (or continuation of a solution) is located.
4. Write neatly. Cross out any work that you do not wish to be considered for grading.
5. Academic Integrity Strictly Applies. Looking at another person's paper is reason to assume cheating and your paper will be taken.
6. All cell phones and electronic devices must be OFF and put away, and hats removed.

1. (10 points)
Solve the following equation.

$$|-2x+1|-5=10$$

Solution.

$$|-2x+1|=15.$$

So either

$$-2x+1=15 \qquad \text{or} \qquad -2x+1=-15.$$

These give

$$x=-7 \qquad \text{or} \qquad x=8.$$

2. (10 points)
Find the center, the foci, the length of the major axis, and the length of the minor axis for the ellipse defined by the equation

$$x^2+4x+25y^2-50y=-4.$$

Solution. Complete the square:

$$(x^2+4x)+25(y^2-2y)=-4,$$

$$(x+2)^2-4+25\big((y-1)^2-1\big)=-4.$$

So

$$(x+2)^2+25(y-1)^2=25,$$

and dividing by 25 gives

$$\frac{(x+2)^2}{25}+\frac{(y-1)^2}{1}=1.$$

Hence the center is $(-2,1)$, with

$$a^2=25, \qquad b^2=1, \qquad c^2=a^2-b^2=24.$$

Therefore

$$a=5, \qquad b=1, \qquad c=2\sqrt{6}.$$

The major axis is horizontal, so the foci are

$$(-2\pm 2\sqrt{6},\,1).$$

The length of the major axis is $2a=10$, and the length of the minor axis is $2b=2$.

3. (10 points)
Find an equation for the line passing through the point $(3, 2)$ and parallel to the line $y = -x - 16$.

Solution. A line parallel to $y = -x - 16$ has slope -1. Using point-slope form,

$$y - 2 = -(x - 3).$$

Thus

$$y = -x + 5.$$

4. (10 points)
Find the vertex of the parabola $y = -3x^2 - 6x + 9$ and determine whether it is a minimum or maximum.

Solution. Complete the square:

$$y = -3(x^2 + 2x) + 9 = -3\big((x+1)^2 - 1\big) + 9.$$

So

$$y = -3(x+1)^2 + 12.$$

Therefore the vertex is

$$(-1, 12).$$

Since the coefficient of $(x+1)^2$ is negative, the parabola opens downward, so the vertex is a maximum.

5. (10 points)
Find all numbers x such that the following equation holds.

$$\log_3(8x - 7) = 2$$

Solution. Rewrite the logarithmic equation in exponential form:

$$8x - 7 = 3^2 = 9.$$

Thus

$$8x = 16,$$

so

$$x = 2.$$

Check the domain: $8x - 7 > 0$ gives $x > 7/8$, so $x = 2$ is valid.

6. (40 points)
Solve these equations for all possible real values of x.

(a) $e^{2x} + 2 = 8 - e^x$

Solution. Let $y = e^x$, so $y > 0$. Then

$$y^2 + 2 = 8 - y,$$

which becomes

$$y^2 + y - 6 = 0.$$

Factor:

$$(y + 3)(y - 2) = 0.$$

Since $y > 0$, we must have $y = 2$. Therefore

$$e^x = 2 \implies x = \ln 2.$$

(b) $3^x = 27$

Solution. Since $27 = 3^3$, we get

$$3^x = 3^3,$$

so

$$x = 3.$$

(c) $4\log_3(5x - 3) = 12$

Solution. Divide by 4:

$$\log_3(5x - 3) = 3.$$

Hence

$$5x - 3 = 3^3 = 27.$$

So

$$5x = 30 \implies x = 6.$$

Domain check: $5x - 3 > 0$ gives $x > 3/5$, so $x = 6$ is valid.

(d) $\log_3(4x) - \log_3(x - 5) = 2$

Solution. First, the domain requires

$$4x > 0 \quad \text{and} \quad x - 5 > 0,$$

so $x > 5$.

Combine the logs:

$$\log_3\left(\frac{4x}{x-5}\right) = 2.$$

Therefore

$$\frac{4x}{x-5} = 3^2 = 9.$$

Solve:

$$4x = 9x - 45 \implies 45 = 5x \implies x = 9.$$

Since $9 > 5$, the solution is valid.

7. (10 points)
Construct an example of a rational function whose graph in the xy-plane has as vertical asymptotes the lines $x = 3$ and $x = 5$ and as an oblique asymptote the line $y = 2x - 3$.

Solution. One example is

$$f(x) = 2x - 3 + \frac{1}{(x-3)(x-5)}.$$

This is a rational function. The denominator $(x-3)(x-5)$ gives vertical asymptotes at $x = 3$ and $x = 5$, and the fraction tends to 0 as $|x| \to \infty$, so the oblique asymptote is

$$y = 2x - 3.$$

8. (10 points)
If \$15,000 is deposited in a saving account with an annual interest rate of 7% compounded continuously, how long will it take for this amount to double?

Solution. For continuous compounding,

$$A(t) = Pe^{rt}.$$

Here $P = 15000$, $r = 0.07$, and we want $A(t) = 30000$. So

$$30000 = 15000e^{0.07t}.$$

Divide by 15000:

$$2 = e^{0.07t}.$$

Taking natural logarithms,

$$\ln 2 = 0.07t.$$

Thus

$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}.$$

9. (10 points)
On a twelve hour clock, find the angle in degrees between the hour hand and the minute hand at 2:27. Your answer should be between 0° and 180°.

Solution. At 27 minutes, the minute hand is at

$$27 \cdot 6^\circ = 162^\circ$$

from the 12.

At 2:27, the hour hand is at

$$2 \cdot 30^\circ + 27 \cdot 0.5^\circ = 60^\circ + 13.5^\circ = 73.5^\circ.$$

Therefore the angle between the hands is

$$162^\circ - 73.5^\circ = 88.5^\circ.$$

So the required angle is

$$88.5^\circ.$$

10. (10 points)
Suppose

$$\frac{3\pi}{2} < \theta < 2\pi \qquad \text{and} \qquad \sin\theta = -0.6.$$

Evaluate the values of the other five trigonometric functions at θ.

Solution. Since θ is in quadrant IV, $\sin\theta < 0$ and $\cos\theta > 0$. Given

$$\sin\theta = -0.6 = -\frac{3}{5},$$

we can use a right triangle with hypotenuse 5 and opposite side -3. Then the adjacent side is

$$\sqrt{5^2 - 3^2} = 4.$$

Therefore

$$\cos\theta = \frac{4}{5}, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{4},$$
$$\csc\theta = -\frac{5}{3}, \qquad \sec\theta = \frac{5}{4}, \qquad \cot\theta = -\frac{4}{3}.$$

11. (10 points)
Suppose a 40-foot ladder is leaning against a wall, making a 45° angle with the ground.

(a) How high up the wall is the end of the ladder?

Solution. The ladder is the hypotenuse of a right triangle of length 40. The height is the side opposite the 45° angle:

$$h = 40 \sin 45^\circ = 40 \cdot \frac{\sqrt{2}}{2} = 20\sqrt{2}.$$

So the end of the ladder is

$$20\sqrt{2} \text{ feet}$$

up the wall.

(b) How far from the wall is the base of the ladder?

Solution. The distance from the wall is the side adjacent to the 45° angle:

$$d = 40 \cos 45^\circ = 40 \cdot \frac{\sqrt{2}}{2} = 20\sqrt{2}.$$

So the base is

$$20\sqrt{2} \text{ feet}$$

from the wall.

12. (20 points)
Find the exact values for the indicated quantities if we are given that

$$\cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

(a) $\sin\left(\frac{\pi}{12}\right)$

Solution. Since $\frac{\pi}{12}$ is in quadrant I,

$$\sin \frac{\pi}{12} = \sqrt{1 - \cos^2 \frac{\pi}{12}} = \sqrt{1 - \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

(b) $\sin\left(\frac{5\pi}{12}\right)$

Solution. Use the sum formula with $\frac{5\pi}{12} = \frac{\pi}{3} + \frac{\pi}{12}$:

$$\sin\left(\frac{5\pi}{12}\right) = \sin \frac{\pi}{3} \cos \frac{\pi}{12} + \cos \frac{\pi}{3} \sin \frac{\pi}{12}.$$

Therefore

$$\sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{6}+\sqrt{2}}{4} + \frac{1}{2}\cdot\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

(c) $\cos\left(\frac{5\pi}{12}\right)$

Solution. Again using $\frac{5\pi}{12} = \frac{\pi}{3} + \frac{\pi}{12}$,

$$\cos\left(\frac{5\pi}{12}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{12} - \sin\frac{\pi}{3}\sin\frac{\pi}{12}.$$

Hence

$$\cos\left(\frac{5\pi}{12}\right) = \frac{1}{2}\cdot\frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

(d) $\tan\left(\frac{5\pi}{12}\right)$

Solution. Using the previous two parts,

$$\tan\left(\frac{5\pi}{12}\right) = \frac{\sin(5\pi/12)}{\cos(5\pi/12)} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}.$$

Rationalizing gives

$$\tan\left(\frac{5\pi}{12}\right) = 2+\sqrt{3}.$$

(e) $\sin\left(\frac{\pi}{24}\right)$

Solution. Since $\frac{\pi}{24}$ is in quadrant I, the half-angle formula gives

$$\sin\frac{\pi}{24} = \sqrt{\frac{1-\cos(\pi/12)}{2}} = \sqrt{\frac{1-\frac{\sqrt{6}+\sqrt{2}}{4}}{2}} = \sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}}.$$

(f) $\cos\left(\frac{\pi}{24}\right)$

Solution. Again using the half-angle formula and choosing the positive sign,

$$\cos\frac{\pi}{24} = \sqrt{\frac{1+\cos(\pi/12)}{2}} = \sqrt{\frac{1+\frac{\sqrt{6}+\sqrt{2}}{4}}{2}} = \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}}.$$

13. (20 points)
Evaluate the following expressions.

(a) $\cos^{-1}\big(\cos(3\pi)\big)$

Solution. Since $\cos(3\pi) = -1$ and the range of $\cos^{-1}$ is $[0, \pi]$,

$$\cos^{-1}(\cos(3\pi)) = \cos^{-1}(-1) = \pi.$$

(b) $\sin^{-1}\left(\sin\left(\frac{6\pi}{7}\right)\right)$

Solution. Since

$$\sin\left(\frac{6\pi}{7}\right) = \sin\left(\pi - \frac{6\pi}{7}\right) = \sin\frac{\pi}{7},$$

and $\pi/7 \in [-\pi/2, \pi/2]$, we get

$$\sin^{-1}\left(\sin\left(\frac{6\pi}{7}\right)\right) = \frac{\pi}{7}.$$

(c) $\cos^{-1}\big(\cos(40°)\big)$

Solution. Since $40° \in [0°, 180°]$, it lies in the range of $\cos^{-1}$. Therefore

$$\cos^{-1}(\cos 40°) = 40°.$$

(d) $\sin^{-1}\big(\sin(70°)\big)$

Solution. Since $70° \in [-90°, 90°]$, it lies in the range of $\sin^{-1}$. Hence

$$\sin^{-1}(\sin 70°) = 70°.$$

(e) $\sin\left(\cos^{-1}\left(\frac{1}{3}\right)\right)$

Solution. Let $\theta = \cos^{-1}(1/3)$. Then $\cos\theta = 1/3$ with $\theta \in [0, \pi]$. Since $1/3 > 0$, actually $\theta \in [0, \pi/2)$. In a right triangle, take adjacent side 1 and hypotenuse 3. Then the opposite side is

$$\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}.$$

Therefore

$$\sin\theta = \frac{2\sqrt{2}}{3}.$$

So

$$\sin\left(\cos^{-1}\left(\frac{1}{3}\right)\right) = \frac{2\sqrt{2}}{3}.$$

(f) $\cos\left(\sin^{-1}\left(\frac{1}{4}\right)\right)$

Solution. Let $\theta = \sin^{-1}(1/4)$. Then $\sin\theta = 1/4$ with $\theta \in [-\pi/2, \pi/2]$, so $\cos\theta \geq 0$. In a right triangle, take opposite side 1 and hypotenuse 4. Then the adjacent side is

$$\sqrt{4^2 - 1^2} = \sqrt{15}.$$

Hence

$$\cos\theta = \frac{\sqrt{15}}{4}.$$

Therefore

$$\cos\left(\sin^{-1}\left(\frac{1}{4}\right)\right) = \frac{\sqrt{15}}{4}.$$

(g) $\cos\big(\tan^{-1}(-4)\big)$

Solution. Let $\theta = \tan^{-1}(-4)$. Then $\tan\theta = -4 = -4/1$, and $\theta \in (-\pi/2, \pi/2)$, so $\cos\theta > 0$. In a right triangle, take opposite side -4 and adjacent side 1. Then the hypotenuse is

$$\sqrt{1^2 + 4^2} = \sqrt{17}.$$

Hence

$$\cos\theta = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}.$$

Therefore

$$\cos\big(\tan^{-1}(-4)\big) = \frac{\sqrt{17}}{17}.$$

(h) $\sin\left(-\sin^{-1}\left(\frac{3}{13}\right)\right)$

Solution. Using the oddness of sine,

$$\sin\left(-\sin^{-1}\left(\frac{3}{13}\right)\right) = -\sin\left(\sin^{-1}\left(\frac{3}{13}\right)\right) = -\frac{3}{13}.$$

14. (6 points)
Suppose t is such that $\cos^{-1} t = \dfrac{\pi}{17}$. Evaluate the following expressions:

(a) $\cos^{-1}(-t)$

Solution. Since

$$t = \cos\frac{\pi}{17},$$

we have

$$-t = -\cos\frac{\pi}{17} = \cos\left(\pi - \frac{\pi}{17}\right) = \cos\frac{16\pi}{17}.$$

Because $16\pi/17 \in [0, \pi]$, it follows that

$$\cos^{-1}(-t) = \frac{16\pi}{17}.$$

(b) $\sin^{-1}(t)$

Solution. Since

$$t = \cos\frac{\pi}{17} = \sin\left(\frac{\pi}{2} - \frac{\pi}{17}\right) = \sin\frac{15\pi}{34},$$

and $15\pi/34 \in [-\pi/2, \pi/2]$, we get

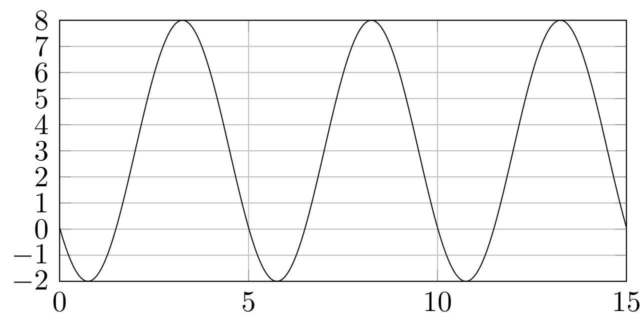
$$\sin^{-1}(t) = \frac{15\pi}{34}.$$

(c) $\sin^{-1}(-t)$

Solution. Since $\sin^{-1}$ is odd,

$$\sin^{-1}(-t) = -\sin^{-1}(t) = -\frac{15\pi}{34}.$$

15. (10 points)
Find a function that models the periodic behavior described by the graph below. You need to use the fact that the first minimum occurs at $x = \dfrac{3}{4}$.



Solution. From the graph, the maximum is 8 and the minimum is -2, so the midline is
$$\frac{8+(-2)}{2} = 3$$
and the amplitude is
$$\frac{8-(-2)}{2} = 5.$$
Successive minima occur 5 units apart, so the period is 5. Thus
$$B = \frac{2\pi}{5}.$$
Because the first minimum occurs at $x = 3/4$, a convenient model is a negative cosine:
$$y = 3 - 5\cos\left(\frac{2\pi}{5}\left(x - \frac{3}{4}\right)\right).$$
Any equivalent sinusoidal model is acceptable.

16. (10 points)
Suppose the wind at airplane heights is 40 miles per hour (relative to the ground) moving 30° North of East. Relative to the air, an airplane is flying at 40 miles per hour 60° South of the wind direction. Find the speed and direction of the airplane relative to the ground.

Solution. The wind is 40 mph at angle 30° north of east, so its vector is
$$\vec{w} = \langle 40\cos 30^\circ,\, 40\sin 30^\circ\rangle = \langle 20\sqrt{3},\, 20\rangle.$$

The airplane is flying 60° south of the wind direction. Since the wind direction is 30° north of east, the airplane's airspeed direction is
$$30^\circ - 60^\circ = -30^\circ,$$

that is, 30° south of east. Hence the airplane's airspeed vector is

$$\vec{p} = \langle 40\cos 30°,\ -40\sin 30°\rangle = \langle 20\sqrt{3},\ -20\rangle.$$

Therefore the ground-velocity vector is

$$\vec{v} = \vec{p} + \vec{w} = \langle 20\sqrt{3},\ -20\rangle + \langle 20\sqrt{3},\ 20\rangle = \langle 40\sqrt{3},\ 0\rangle.$$

So the speed is

$$|\vec{v}| = 40\sqrt{3}\text{ mph},$$

and the direction is due east.

17. (10 points)
Find a number t such that $0 \le t < \pi$ and the vectors $\vec{u} = \langle 3\sin t, 4\rangle$ and $\vec{v} = \langle 6, 3\rangle$ are perpendicular.

Solution. Perpendicular vectors have dot product 0. So

$$\langle 3\sin t, 4\rangle \cdot \langle 6, 3\rangle = 0.$$

Thus

$$(3\sin t)(6) + (4)(3) = 0,$$

so

$$6\sin t + 4 = 0.$$

Hence

$$\sin t = -\frac{2}{3}.$$

$$t = \sin^{-1}\left(-\frac{2}{3}\right) + \pi = \pi - \sin^{-1}\left(\frac{2}{3}\right).$$

More generally,

$$t = 2\pi k \pm \cos^{-1}\left(-\frac{3}{5}\right), \qquad k \in \mathbb{Z}.$$

18. (20 points)
Perform the indicated operation and write each of the following expressions in the form $a + bi$, where a and b are real numbers.

(a) $(5 + 7i)(4 + 6i)$

Solution. Expand:

$$(5+7i)(4+6i) = 20+30i+28i+42i^2.$$

Since $i^2 = -1$,

$$20+58i-42 = -22+58i.$$

(b) $\dfrac{4+3i}{5-2i}$

Solution. Multiply numerator and denominator by the conjugate $5+2i$:

$$\frac{4+3i}{5-2i} \cdot \frac{5+2i}{5+2i} = \frac{(4+3i)(5+2i)}{(5-2i)(5+2i)}.$$

Compute the numerator:

$$(4+3i)(5+2i) = 20+8i+15i+6i^2 = 14+23i.$$

Compute the denominator:

$$(5-2i)(5+2i) = 25-(2i)^2 = 25+4 = 29.$$

Therefore

$$\frac{4+3i}{5-2i} = \frac{14}{29} + \frac{23}{29}i.$$

(c) $(\sqrt{11}+\sqrt{3}\,i)^2$

Solution. Use $(a+b)^2 = a^2+2ab+b^2$:

$$(\sqrt{11}+\sqrt{3}\,i)^2 = (\sqrt{11})^2 + 2(\sqrt{11})(\sqrt{3}\,i) + (\sqrt{3}\,i)^2.$$

So

$$11+2\sqrt{33}\,i+3i^2 = 11+2\sqrt{33}\,i-3 = 8+2\sqrt{33}\,i.$$

(d) $\overline{(\sqrt{7}+\sqrt{3}\,i)}$

Solution. The complex conjugate changes the sign of the imaginary part, so

$$\overline{(\sqrt{7}+\sqrt{3}\,i)} = \sqrt{7}-\sqrt{3}\,i.$$