MATH 0200 – Prep for Scientific Calculus
SAMPLE FINAL EXAM 2

Exam length: 1 hour 50 minutes

INSTRUCTIONS:

1. NO TABLES, BOOKS, NOTES, HEADPHONES, CALCULATORS, OR COMPUTERS MAY BE USED.
2. Show ALL of your calculations and display answers clearly. You may leave your final answers in exact form. Unjustified answers will receive no credit.
3. WRITE YOUR SOLUTIONS in the space provided. EXTRA SPACE is available on the BACKS of the pages. When using these back pages, clearly LABEL the problem, and also clearly indicate on the appropriate front page where your back-page solution (or continuation of a solution) is located.
4. Write neatly. Cross out any work that you do not wish to be considered for grading.
5. Academic Integrity Strictly Applies. Looking at another person's paper is reason to assume cheating and your paper will be taken.
6. All cell phones and electronic devices must be OFF and put away, and hats removed.

1. (30 points)
Simplify or Solve.

(a) Find all x such that $\dfrac{x+2}{x-1} < 3$.

Solution. Move everything to one side:

$$\frac{x+2}{x-1} - 3 < 0.$$

Combine into a single fraction:

$$\frac{x+2-3(x-1)}{x-1} < 0 \quad \Longrightarrow \quad \frac{-2x+5}{x-1} < 0.$$

Multiply numerator and denominator by -1:

$$\frac{2x-5}{x-1} > 0.$$

Critical values are $x = 1$ and $x = \frac{5}{2}$. A sign chart shows the expression is positive on

$$(-\infty, 1) \cup \left(\frac{5}{2}, \infty\right).$$

Therefore,

$$\boxed{x < 1 \text{ or } x > \frac{5}{2}}.$$

(b) Give exact value for $\cos(\pi/12)$.

Solution. Use the difference formula:

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}.$$

Hence

$$\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

(c) Give exact value for $\sin(7\pi/12)$.

Solution. Write

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right).$$

Then

$$\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}.$$

So

$$\boxed{\sin\left(\frac{7\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}}.$$

(d) Evaluate $\cos^{-1}(\cos(10\pi/9))$.

Solution. The range of $\cos^{-1}$ is $[0, \pi]$. We need the angle in this interval having the same cosine as $10\pi/9$. Since

$$\cos\left(\frac{10\pi}{9}\right) = \cos\left(2\pi - \frac{10\pi}{9}\right) = \cos\left(\frac{8\pi}{9}\right),$$

we get

$$\boxed{\cos^{-1}(\cos(10\pi/9)) = \frac{8\pi}{9}}.$$

(e) Evaluate $\cos(\tan^{-1}(-4))$.

Solution. Let $\theta = \tan^{-1}(-4)$. Then $\tan\theta = -4$, and since $\theta \in (-\pi/2, \pi/2)$, θ is in quadrant IV. Draw a right triangle with adjacent side 1, opposite side -4, and hypotenuse

$$\sqrt{1^2 + (-4)^2} = \sqrt{17}.$$

Therefore,

$$\cos\theta = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}.$$

So

$$\boxed{\cos(\tan^{-1}(-4)) = \frac{\sqrt{17}}{17}}.$$

(f) Find all real and/or complex solutions to $z^2 + 4z + 6 = 0$.

Solution. Use the quadratic formula:

$$z = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 6}}{2} = \frac{-4 \pm \sqrt{16 - 24}}{2} = \frac{-4 \pm \sqrt{-8}}{2}.$$

Since $\sqrt{-8} = 2i\sqrt{2}$,

$$z = \frac{-4 \pm 2i\sqrt{2}}{2} = -2 \pm i\sqrt{2}.$$

Thus the solutions are

$$\boxed{z = -2 \pm i\sqrt{2}}.$$

2. (30 points)
Short answer.

(a) The graph of $g(x)$ is obtained from the graph of $f(x)$ by first stretching the graph of f horizontally by a factor of 2, then flipping it vertically (across the x-axis), then shifting it up 2 units. Give a formula for $g(x)$ in terms of $f(x)$.

> *Solution.* A horizontal stretch by factor 2 gives $f(x/2)$. Flipping across the x-axis gives $-f(x/2)$. Shifting up 2 units gives
>
> $$\boxed{g(x) = -f\left(\frac{x}{2}\right) + 2}.$$

(b) Sketch the graph of $f(x) = \sqrt{x}$ and its inverse. Clearly mark the domains and ranges of each.

> *Solution.* The inverse of $f(x) = \sqrt{x}$ is
>
> $$f^{-1}(x) = x^2 \qquad (x \geq 0).$$
>
> The graph of $y = \sqrt{x}$ has domain $[0, \infty)$ and range $[0, \infty)$. The graph of its inverse $y = x^2$ with $x \geq 0$ also has domain $[0, \infty)$ and range $[0, \infty)$. The two graphs are reflections of each other across the line $y = x$.

(c) Give the equation of the line perpendicular to $y = 3x + 5$ and through the point $(1, 2)$.

> *Solution.* The given line has slope 3, so a perpendicular line has slope $-\frac{1}{3}$. Using point-slope form,
>
> $$y - 2 = -\frac{1}{3}(x - 1).$$
>
> Simplifying,
>
> $$y = -\frac{1}{3}x + \frac{1}{3} + 2 = -\frac{1}{3}x + \frac{7}{3}.$$
>
> Thus
>
> $$\boxed{y = -\frac{1}{3}x + \frac{7}{3}}.$$

(d) What is the minimum value of $x^2 - 6x + 2$?

> *Solution.* Complete the square:
>
> $$x^2 - 6x + 2 = (x - 3)^2 - 7.$$

Since $(x-3)^2 \geq 0$, the minimum value is

$$\boxed{-7}.$$

(e) You have a calculator which can only compute square roots and multiply numbers. Explain how you can use it to compute $7^{3/4}$.

Solution. Write

$$7^{3/4} = \sqrt{7^{3/2}} = \sqrt{7\sqrt{7}}.$$

So first compute $\sqrt{7}$, then multiply by 7, and finally take the square root of the result. That uses only square roots and multiplication.

(f) Find a number b such that 3 is a zero of $p(x) = 1 - 4x + bx^2 + 2x^3$.

Solution. If 3 is a zero, then $p(3) = 0$:

$$1 - 4(3) + b(3)^2 + 2(3)^3 = 0.$$

Thus

$$1 - 12 + 9b + 54 = 0 \implies 9b + 43 = 0.$$

Hence

$$\boxed{b = -\frac{43}{9}}.$$

3. (30 points)
Short answer.

(a) List all asymptotes of $f(x) = \dfrac{x^2+1}{(x-1)(x+2)}$.

Solution. Vertical asymptotes occur where the denominator is zero and the numerator is nonzero. Since

$$(x-1)(x+2) = 0$$

at $x = 1$ and $x = -2$, and $x^2 + 1 \neq 0$ for real x, the vertical asymptotes are

$$x = 1 \quad \text{and} \quad x = -2.$$

Since the degrees of numerator and denominator are equal, the horizontal asymptote is the ratio of leading coefficients, namely

$$y = 1.$$

Therefore the asymptotes are

$$\boxed{x = 1, \quad x = -2, \quad y = 1}.$$

(b) Find a number n such that $\log_3(\log_2 n) = 2$.

Solution. From

$$\log_3(\log_2 n) = 2,$$

we get

$$\log_2 n = 3^2 = 9.$$

Hence

$$n = 2^9 = 512.$$

So

$$\boxed{n = 512}.$$

(c) Write as a single logarithm: $\log x + 2\log y - \frac{1}{2}\log z^4$.

Solution. Use log rules:

$$2\log y = \log(y^2), \qquad \frac{1}{2}\log z^4 = \log\big((z^4)^{1/2}\big) = \log(z^2).$$

Therefore

$$\log x + 2\log y - \frac{1}{2}\log z^4 = \log x + \log(y^2) - \log(z^2) = \log\left(\frac{xy^2}{z^2}\right).$$

So the single logarithm is

$$\boxed{\log\left(\frac{xy^2}{z^2}\right)}.$$

(d) Write an expression for the amount of money you will have after t years if you invest \$1000 at a rate of 2% compounded quarterly.

Solution. Use the compound-interest formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Here $P = 1000$, $r = 0.02$, and $n = 4$. Thus

$$\boxed{A(t) = 1000\left(1 + \frac{0.02}{4}\right)^{4t} = 1000(1.005)^{4t}}.$$

(e) What is the slope of the line that makes an angle of $30°$ with the positive x-axis?

Solution. The slope of a line making angle θ with the positive x-axis is $m = \tan\theta$. Hence

$$m = \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

So the slope is

$$\boxed{\frac{\sqrt{3}}{3}}.$$

(f) Find two angles u and v so that $\cos u = \cos v$ but $\sin u \neq \sin v$.

Solution. One example is

$$\boxed{u = \frac{\pi}{3}, \qquad v = \frac{5\pi}{3}}.$$

Indeed,

$$\cos\frac{\pi}{3} = \frac{1}{2} = \cos\frac{5\pi}{3},$$

but

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}e - \frac{\sqrt{3}}{2} = \sin\frac{5\pi}{3}.$$

4. (10 points)
Show that if $f(x) = mx + b$ then for $m \neq 0$ its inverse is given by the equation

$$f^{-1}(y) = \frac{1}{m}y - \frac{b}{m}.$$

Solution. Let

$$y = mx + b.$$

Solve for x:

$$y - b = mx \quad \Longrightarrow \quad x = \frac{y-b}{m} = \frac{1}{m}y - \frac{b}{m}.$$

This expresses x in terms of y, so

$$\boxed{f^{-1}(y) = \frac{1}{m}y - \frac{b}{m}}.$$

Equivalently, writing the input variable as x,

$$f^{-1}(x) = \frac{x-b}{m}.$$

5. (15 points)
For the functions

$$r(x) = \frac{3x+4}{x^2+1}, \qquad s(x) = \frac{x^2+2}{2x-1}$$

find and SIMPLIFY the formula for the composition $r \circ s(x)$.

Solution. We compute

$$(r \circ s)(x) = r(s(x)) = \frac{3s(x)+4}{(s(x))^2+1}.$$

Since

$$s(x) = \frac{x^2+2}{2x-1},$$

we have

$$3s(x)+4 = \frac{3(x^2+2)}{2x-1} + 4 = \frac{3x^2+6+8x-4}{2x-1} = \frac{3x^2+8x+2}{2x-1}.$$

Also,

$$(s(x))^2+1 = \left(\frac{x^2+2}{2x-1}\right)^2 + 1 = \frac{(x^2+2)^2+(2x-1)^2}{(2x-1)^2}.$$

Expand the numerator:

$$(x^2+2)^2+(2x-1)^2 = x^4+4x^2+4+4x^2-4x+1 = x^4+8x^2-4x+5.$$

Therefore,

$$(r \circ s)(x) = \frac{\frac{3x^2+8x+2}{2x-1}}{\frac{x^4+8x^2-4x+5}{(2x-1)^2}} = \frac{(3x^2+8x+2)(2x-1)}{x^4+8x^2-4x+5}.$$

So

$$\boxed{(r \circ s)(x) = \frac{(2x-1)(3x^2+8x+2)}{x^4+8x^2-4x+5}}.$$

6. (15 points)
Solve

$$\frac{\ln(12x)}{\ln(5x)} = 2.$$

Solution. We need $x > 0$ and also $\ln(5x) \neq 0$, so $x \neq \frac{1}{5}$. Now

$$\frac{\ln(12x)}{\ln(5x)} = 2 \quad \Longrightarrow \quad \ln(12x) = 2\ln(5x).$$

Using log rules,

$$2\ln(5x) = \ln\big((5x)^2\big) = \ln(25x^2).$$

Hence

$$\ln(12x) = \ln(25x^2).$$

Since logarithms are one-to-one on positive numbers,

$$12x = 25x^2.$$

So

$$25x^2 - 12x = 0 \quad \Longrightarrow \quad x(25x - 12) = 0.$$

Because $x > 0$, we discard $x = 0$. Thus

$$\boxed{x = \frac{12}{25}}.$$

7. (15 points)
Simplify $\left(\frac{4}{5} - \frac{3}{5}i\right)^{75}$.

Solution. First note that

$$\left|\frac{4}{5} - \frac{3}{5}i\right| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2} = 1.$$

So we may write

$$\frac{4}{5} - \frac{3}{5}i = \cos\theta + i\sin\theta$$

with

$$\cos\theta = \frac{4}{5}, \qquad \sin\theta = -\frac{3}{5}.$$

Thus $\theta = -\tan^{-1}(3/4)$. By De Moivre's Theorem,

$$\left(\frac{4}{5} - \frac{3}{5}i\right)^{75} = \cos(75\theta) + i\sin(75\theta).$$

Substituting $\theta = -\tan^{-1}(3/4)$,

$$\boxed{\left(\frac{4}{5} - \frac{3}{5}i\right)^{75} = \cos\left(-75\tan^{-1}\left(\frac{3}{4}\right)\right) + i\sin\left(-75\tan^{-1}\left(\frac{3}{4}\right)\right)}.$$

Equivalently,

$$\boxed{\left(\frac{4}{5} - \frac{3}{5}i\right)^{75} = \cos\big(75\tan^{-1}(3/4)\big) - i\sin\big(75\tan^{-1}(3/4)\big)}.$$

8. (25 points)

Suppose θ is in the interval $(\pi/2, \pi)$ with $\sin\theta = 1/3$. Compute:

(a) $\cos\theta$

Solution. Since θ is in quadrant II, cosine is negative. Thus

$$\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}.$$

(b) $\tan\theta$

Solution.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/3}{-2\sqrt{2}/3} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}.$$

(c) $\sec\theta$

Solution.

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-2\sqrt{2}/3} = -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4}.$$

(d) $\csc\theta$

Solution.

$$\csc\theta = \frac{1}{\sin\theta} = 3.$$

(e) $\cot\theta$

Solution.

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{-2\sqrt{2}/3}{1/3} = -2\sqrt{2}.$$

(f) $\sin(\theta+\pi)$

Solution.

$$\sin(\theta+\pi) = -\sin\theta = -\frac{1}{3}.$$

(g) $\cos(\theta+\pi)$

Solution.
$$\cos(\theta+\pi) = -\cos\theta = \frac{2\sqrt{2}}{3}.$$

(h) $\sin(\pi/2-\theta)$

Solution. Use the cofunction identity:
$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta = -\frac{2\sqrt{2}}{3}.$$

(i) $\cos(\pi/2-\theta)$

Solution.
$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta = \frac{1}{3}.$$

(j) $\sin(-\theta)$

Solution. Sine is odd, so
$$\sin(-\theta) = -\sin\theta = -\frac{1}{3}.$$

(k) $\cos(-\theta)$

Solution. Cosine is even, so
$$\cos(-\theta) = \cos\theta = -\frac{2\sqrt{2}}{3}.$$

(l) $\sin(\theta/2)$

Solution. Since $\theta \in (\pi/2, \pi)$, we have $\theta/2 \in (\pi/4, \pi/2)$, so sine is positive. Use the half-angle formula:
$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-(-2\sqrt{2}/3)}{2}} = \sqrt{\frac{1+2\sqrt{2}/3}{2}} = \sqrt{\frac{3+2\sqrt{2}}{6}}.$$
Therefore,
$$\boxed{\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{3+2\sqrt{2}}{6}}}.$$

(m) $\cos(2\theta)$

> *Solution.*
> $$\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2\left(\frac{1}{3}\right)^2 = 1 - \frac{2}{9} = \frac{7}{9}.$$

9. (15 points)
Find the area of a regular hexagon inscribed in the unit circle.

> *Solution.* A regular hexagon inscribed in the unit circle can be divided into six congruent central triangles, each with central angle
> $$\frac{2\pi}{6} = \frac{\pi}{3}.$$
> Each such triangle has side lengths $1, 1, 1$, so it is an equilateral triangle of side length 1. Hence each triangle has area
> $$\frac{\sqrt{3}}{4}.$$
> Therefore the total area is
> $$6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}.$$
> So the area is
> $$\boxed{\frac{3\sqrt{3}}{2}}.$$

10. (15 points)
Find all solutions to $z^3 = -8i$.

> *Solution.* Write $-8i$ in trigonometric form. Since $-8i$ has modulus 8 and argument $3\pi/2$,
> $$-8i = 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right).$$
> Its cube roots have modulus $\sqrt[3]{8} = 2$ and arguments
> $$\frac{\frac{3\pi}{2} + 2\pi k}{3} = \frac{\pi}{2} + \frac{2\pi k}{3}, \qquad k = 0, 1, 2.$$
> Thus the roots are:
> $$2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i,$$

$$2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\sqrt{3} - i,$$

$$2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3} - i.$$

Therefore,

$$\boxed{z = 2i, \quad z = -\sqrt{3} - i, \quad z = \sqrt{3} - i}.$$