MATH 0200 – Prep for Scientific Calculus
SAMPLE FINAL EXAM 1

Exam length: 1 hour 50 minutes

INSTRUCTIONS:

1. NO TABLES, BOOKS, NOTES, HEADPHONES, CALCULATORS, OR COMPUTERS MAY BE USED.
2. Show ALL of your calculations and display answers clearly. You may leave your final answers in exact form. Unjustified answers will receive no credit.
3. WRITE YOUR SOLUTIONS in the space provided. EXTRA SPACE is available on the BACKS of the pages. When using these back pages, clearly LABEL the problem, and also clearly indicate on the appropriate front page where your back-page solution (or continuation of a solution) is located.
4. Write neatly. Cross out any work that you do not wish to be considered for grading.
5. Academic Integrity Strictly Applies. Looking at another person's paper is reason to assume cheating and your paper will be taken.
6. All cell phones and electronic devices must be OFF and put away, and hats removed.

1. (30 points)

Short answer. Fill in the blank.

(a) Function $f(x) = 6x^3 + 7x - 3$ has degree ____________.

Solution. The degree is 3.

(b) State the Law of Cosines and sketch the corresponding triangle.

Solution. For a triangle with side lengths a, b, c opposite angles A, B, C,

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Equivalent formulas are obtained by symmetry:

$$a^2 = b^2 + c^2 - 2bc\cos A, \qquad b^2 = a^2 + c^2 - 2ac\cos B.$$

(c) A function with vertical asymptotes at $x = 2$ and $x = 3$ and horizontal asymptote $y = 7$ is ______________________.

Solution. One example is

$$f(x) = 7 + \frac{1}{(x-2)(x-3)}.$$

(d) $\tan\left(\frac{\pi}{2} - \theta\right) =$ ____________(θ).

Solution.

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta.$$

(e) The conic section

$$x^2 + 4x + 25y^2 - 50y = -4$$

is a _______________.

Solution. Complete the square:

$$(x+2)^2 - 4 + 25\big((y-1)^2 - 1\big) = -4,$$

so

$$(x+2)^2 + 25(y-1)^2 = 25.$$

This is an ellipse.

(f) The equation of the line parallel to $y = -x - 16$ and through the point $(3, 2)$ is __________________________.

> *Solution.* A parallel line has slope -1. Using point-slope form,
> $$y - 2 = -(x - 3),$$
> so
> $$y = -x + 5.$$

2. (30 points)
Short answer. Fill in the blank.

(a) $27° =$ _____________ rad.

> *Solution.*
> $$27° = 27 \cdot \frac{\pi}{180} = \frac{3\pi}{20}.$$

(b) The graph of $2f(x-1)$ can be obtained from the graph of f by _____________________ in the ____________ direction and _____________________ in the ____________ direction. Answers should be shifting/scaling/flipping and vertical/horizontal.

> *Solution.* Shift 1 unit to the right in the horizontal direction, and then stretch by a factor of 2 in the vertical direction.

(c) The period of $f(x) = \sin x$ is ___________. The period of $\cos(2x)$ is ___________.

> *Solution.*
> $$\text{Period of } \sin x = 2\pi, \qquad \text{period of } \cos(2x) = \pi.$$

(d) The range of $\cos^{-1}(x)$ is ______________. The range of $\sin^{-1}(x)$ is ______________.

> *Solution.*
> $$\cos^{-1}(x) \in [0, \pi], \qquad \sin^{-1}(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

(e) The polar equation for the circle centered at the origin with radius 3 is _____________.

> *Solution.*
> $$r = 3.$$

(f) Write the equation for the balance in a bank account after t years with annual interest rate of 7% compounded continuously and initial deposit \$15,000. ____________

Solution.

$$A(t) = 15000e^{0.07t}.$$

3. (30 points)
Solve:

(a) $|-2x+1| - 5 = 10$

Solution.

$$|-2x+1| = 15.$$

So either

$$-2x+1 = 15 \quad \text{or} \quad -2x+1 = -15.$$

These give

$$x = -7 \quad \text{or} \quad x = 8.$$

(b) $e^{2x} + 2 = 8 - e^x$

Solution. Let $y = e^x$, so $y > 0$. Then

$$y^2 + 2 = 8 - y,$$

which becomes

$$y^2 + y - 6 = 0.$$

Factor:

$$(y+3)(y-2) = 0.$$

Since $y > 0$, we must have $y = 2$. Therefore

$$e^x = 2 \implies x = \ln 2.$$

(c) $\tan\left(\frac{7\pi}{6}\right)$

Solution.

$$\tan\left(\frac{7\pi}{6}\right) = \tan\left(\pi + \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

(d) $\tan\left(\cos^{-1}\left(\frac{1}{3}\right)\right)$

Solution. Let $\theta = \cos^{-1}(1/3)$. Then $\cos\theta = 1/3$ and $\theta \in [0, \pi]$. Since $1/3 > 0$, actually $\theta \in [0, \pi/2)$. Draw a right triangle with adjacent side 1 and hypotenuse 3. Then the opposite side is

$$\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}.$$

Hence

$$\tan\theta = \frac{2\sqrt{2}}{1} = 2\sqrt{2}.$$

(e) $\sin\left(-\sin^{-1}\left(\frac{3}{13}\right)\right)$

Solution. Using oddness of sine,

$$\sin\left(-\sin^{-1}\left(\frac{3}{13}\right)\right) = -\sin\left(\sin^{-1}\left(\frac{3}{13}\right)\right) = -\frac{3}{13}.$$

(f) $\log_2(x+3) + \log_2(x-2) = 3$

Solution. First require $x + 3 > 0$ and $x - 2 > 0$, so $x > 2$. Using log rules,

$$\log_2\big((x+3)(x-2)\big) = 3.$$

Therefore

$$(x+3)(x-2) = 2^3 = 8.$$

Expand:

$$x^2 + x - 6 = 8,$$

so

$$x^2 + x - 14 = 0.$$

Thus

$$x = \frac{-1 \pm \sqrt{57}}{2}.$$

Because $x > 2$, the only valid solution is

$$\boxed{x = \frac{-1 + \sqrt{57}}{2}}.$$

4. (30 points)
Simplify:

(a) $27^{-4/3}$

Solution.

$$27^{-4/3} = \left(27^{1/3}\right)^{-4} = 3^{-4} = \frac{1}{81}.$$

(b) $\left(\dfrac{x^3y^2 + xy^{-2}}{x^{-3}y^{-4}}\right)^{-1}$

Solution. First simplify the fraction inside:

$$\frac{x^3y^2 + xy^{-2}}{x^{-3}y^{-4}} = (x^3y^2 + xy^{-2})x^3y^4 = x^6y^6 + x^4y^2.$$

Now raise to the power -1:

$$\left(x^6y^6 + x^4y^2\right)^{-1} = \frac{1}{x^6y^6 + x^4y^2}.$$

A factored form is

$$\frac{1}{x^4y^2(x^2y^4 + 1)}.$$

(c) $\ln\left(\dfrac{x^3}{y^2}\right)$ if $\ln x = 2$ and $\ln y = 6.1$

Solution.

$$\ln\left(\frac{x^3}{y^2}\right) = \ln(x^3) - \ln(y^2) = 3\ln x - 2\ln y = 3(2) - 2(6.1) = 6 - 12.2 = -6.2.$$

(d) $\dfrac{3-4i}{6-5i}$

Solution. Multiply numerator and denominator by the conjugate $6 + 5i$:

$$\frac{3-4i}{6-5i} \cdot \frac{6+5i}{6+5i} = \frac{(3-4i)(6+5i)}{6^2+5^2}.$$

The numerator is

$$18 + 15i - 24i - 20i^2 = 18 - 9i + 20 = 38 - 9i.$$

So

$$\frac{3-4i}{6-5i} = \frac{38-9i}{61} = \frac{38}{61} - \frac{9}{61}i.$$

(e) $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{300}$

Solution. Note that

$$\frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right).$$

By De Moivre's Theorem,

$$\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)^{300} = \cos(-100\pi) + i\sin(-100\pi) = 1.$$

5. (15 points)
Let

$$f(x) = \frac{x}{x-2}, \qquad g(x) = \sqrt{x+1}.$$

(a) What is the domain of $g \circ f(x)$?

Solution. We need

$$g(f(x)) = \sqrt{\frac{x}{x-2} + 1}$$

to be defined. So we require $x \neq 2$ and

$$\frac{x}{x-2} + 1 \geq 0.$$

Combine terms:

$$\frac{x}{x-2} + 1 = \frac{2x-2}{x-2} = \frac{2(x-1)}{x-2}.$$

Thus we solve

$$\frac{x-1}{x-2} \geq 0.$$

Critical values are $x = 1$ and $x = 2$. A sign chart gives

$$(-\infty, 1] \cup (2, \infty).$$

So the domain is

$$\boxed{(-\infty, 1] \cup (2, \infty)}.$$

(b) Evaluate $g \circ f(3)$.

Solution.

$$f(3) = \frac{3}{3-2} = 3,$$

so

$$g(f(3)) = g(3) = \sqrt{3+1} = 2.$$

(c) Write an expression for $f^{-1}(x)$.

Solution. Let

$$y = \frac{x}{x-2}.$$

Solve for x:

$$y(x-2) = x \implies yx - 2y = x \implies x(y-1) = 2y \implies x = \frac{2y}{y-1}.$$

Therefore

$$f^{-1}(x) = \frac{2x}{x-1}.$$

6. (10 points)
Find a number t such that the vectors $u = \langle 2\cos t, 4\rangle$ and $v = \langle 10, 3\rangle$ are perpendicular.

Solution. Perpendicular vectors have dot product zero:

$$\langle 2\cos t, 4\rangle \cdot \langle 10, 3\rangle = 0.$$

Thus

$$20\cos t + 12 = 0,$$

so

$$\cos t = -\frac{3}{5}.$$

One possible value is

$$\boxed{t = \arccos\left(-\frac{3}{5}\right)}.$$

More generally,

$$t = \arccos\left(-\frac{3}{5}\right) + 2\pi k \quad \text{or} \quad t = 2\pi - \arccos\left(-\frac{3}{5}\right) + 2\pi k,$$

where $k \in \mathbb{Z}$.

7. (10 points)
Suppose a 40-foot ladder is leaning against a wall, making a $45°$ angle with the ground.

(a) How high up the wall is the end of the ladder?

Solution. The ladder is the hypotenuse of a right triangle. The height is

$$40 \sin 45^\circ = 40 \cdot \frac{\sqrt{2}}{2} = 20\sqrt{2} \text{ ft.}$$

(b) How far from the wall is the base of the ladder?

Solution. The horizontal distance is

$$40 \cos 45^\circ = 40 \cdot \frac{\sqrt{2}}{2} = 20\sqrt{2} \text{ ft.}$$

8. (15 points)
Suppose u and v are angles in the interval $(0, \pi/2)$ with $\cos u = 1/3$ and $\sin v = 1/4$. Evaluate:

(a) $\sin u$

Solution. Since $u \in (0, \pi/2)$, $\sin u > 0$. Thus

$$\sin u = \sqrt{1 - \cos^2 u} = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}.$$

(b) $\cos v$

Solution. Since $v \in (0, \pi/2)$, $\cos v > 0$. Hence

$$\cos v = \sqrt{1 - \sin^2 v} = \sqrt{1 - \frac{1}{16}} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}.$$

(c) $\sin(2u)$

Solution. Use $\sin(2u) = 2 \sin u \cos u$:

$$\sin(2u) = 2 \cdot \frac{2\sqrt{2}}{3} \cdot \frac{1}{3} = \frac{4\sqrt{2}}{9}.$$

(d) $\cos(2v)$

Solution. Use $\cos(2v) = 1 - 2\sin^2 v$:

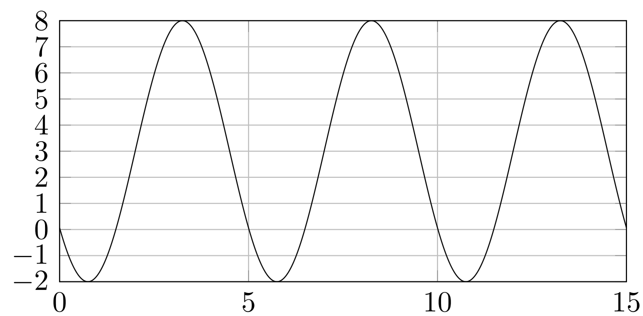
$$\cos(2v) = 1 - 2\left(\frac{1}{4}\right)^2 = 1 - \frac{2}{16} = 1 - \frac{1}{8} = \frac{7}{8}.$$

(e) $\cos(u/2)$

Solution. Since $u/2 \in (0, \pi/4)$, cosine is positive. Use the half-angle formula:

$$\cos\left(\frac{u}{2}\right) = \sqrt{\frac{1+\cos u}{2}} = \sqrt{\frac{1+1/3}{2}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}.$$

9. (15 points)
Find a function which models the periodic behavior. The first minimum occurs at $x = 3/4$.



Solution. From the graph, the maximum is 8 and the minimum is -2, so the amplitude is

$$\frac{8-(-2)}{2} = 5$$

and the midline is

$$\frac{8+(-2)}{2} = 3.$$

The distance between consecutive minima is 5, so the period is 5. Hence

$$B = \frac{2\pi}{5}.$$

Because the first minimum occurs at $x = \frac{3}{4}$, one convenient model is

$$\boxed{y = 3 - 5\cos\left(\frac{2\pi}{5}\left(x - \frac{3}{4}\right)\right)}.$$

There are other equivalent correct models.

10. (15 points)
Determine the area of a regular octagon with vertices on the unit circle.

Solution. A regular octagon inscribed in the unit circle can be divided into 8 congruent isosceles triangles with central angle

$$\frac{2\pi}{8} = \frac{\pi}{4}.$$

Each triangle has side lengths 1 and 1, so its area is

$$\frac{1}{2}(1)(1)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}.$$

Therefore the total area is

$$8 \cdot \frac{\sqrt{2}}{4} = 2\sqrt{2}.$$

So the area is

$$\boxed{2\sqrt{2}}.$$