MATH 0120 - BUSINESS CALCULUS

SAMPLE FINAL EXAM

1. Evaluate the following limits

(a) (5 points) $\lim_{x\to 4} \frac{\sqrt{x}-1}{x+2}$ $\to \boxed{\frac{1}{6}}$

In more detail: $\frac{\sqrt{4}-1}{4+2} = \frac{2-1}{4+2} = \frac{1}{6}$.

(b) (5 points) $\lim_{x\to 1} \frac{3x^3+3x^2-6x}{x^2-x}$ $\to \frac{0}{0}$

$= \lim_{x\to 1} \frac{3x(x^2+x-2)}{x(x-1)} = \lim_{x\to 1} \frac{3x(x-1)(x+2)}{x(x-1)} = \lim_{x\to 1} 3(x+2) = 3((1)+2) = \boxed{9}$.

(c) (5 points) $\lim_{x\to 0^-} \frac{|x|}{x}$

As $x \to 0^-$, $x < 0$, so treat $|x|$ as $-x$.

thus, get $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = \boxed{-1}$.

(d) (5 points) $\lim_{x\to -5^+} \frac{3}{x+5}$ $\to \frac{3}{0}$

As $x \to -5^+$, $x+5 > 0$. Hence, get $\frac{+}{+}$, and so

$\lim_{x\to -5^+} \frac{3}{x+5}$ $\boxed{\text{DNE } (+\infty)}$.

2. (10 points) Find the derivative of $f(x) = x^2 - 3x + 5$ using **the limit definition** of the derivative. NO CREDIT will be given if the limit definition is not used.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) + 5 - (x^2 - 3x + 5)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 5 - x^2 + 3x - 5}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h} = \lim_{h \to 0} \frac{h(2x + h - 3)}{h}$$

$$= \lim_{h \to 0} 2x + h - 3 = \boxed{2x - 3}.$$

3. Find the derivatives. **You need not simplify.**

(a) (6 points) $f(x) = \frac{x}{\ln(2-x)}$

$$f'(x) = \frac{[\ln(2-x)]\cdot[1] - [x]\cdot\left[\frac{1}{2-x}\cdot(-1)\right]}{[\ln(2-x)]^2}$$

(b) (6 points) $g(x) = 2xe^{(3x+7)} + \pi$

$$g'(x) = [2x][e^{3x+7}\cdot(3)] + [2][e^{3x+7}] + 0$$

(c) (6 points) $h(x) = (5x^2 - x)(x^4 + 4x)^{-2}$

$$h'(x) = [5x^2 - x][-2(x^4+4x)^{-3}\cdot(4x^3+4)] + [10x-1][(x^4-4x)^{-2}]$$

4. (3 points) Let $R(x)$ represent the revenue, in dollars, generated by producing and selling x bicycles per day. Suppose it is known that $\boxed{R(100) = -2.}$ If the company is currently producing 100 bicycles, would you recommend increasing production? Explain your answer.

** $R(100) = -2$ makes no sense; the question should read $R'(100) = -2$. **

If $R'(100) = -2$, then an increase in x results in a decrease to revenue. Thus, we do <u>not</u> recommend increasing production.

5. (8 points) Suppose for a particular person that the time T (in minutes) required to learn a list of length n is

$$T = f(n) = 2n\sqrt{n-2}.$$

Find $f'(11)$ and interpret your answer.

$$f'(n) = [2n]\left[\tfrac{1}{2}(n-2)^{-1/2}\cdot(1)\right] + [2][\sqrt{n-2}], \text{ so}$$

$$f'(11) = 2(11)\left(\tfrac{1}{2}((11)-2)^{-1/2}\cdot(1)\right) + 2\sqrt{(11)-2} = 22\cdot\tfrac{1}{2}\cdot\tfrac{1}{3}\cdot 1 + 2\cdot 3$$

$$= \frac{11}{3} + 6 = \boxed{\frac{29}{3}}.$$

We could say $f'(11)$ is the "marginal learning time" at $n = 11$, so it would take about $\frac{29}{3}$ more minutes to learn the 12th item in the list.

6. Follow the steps to graph the stated function.

$$f(x) = -x^3 + 12x,\ f'(x) = -3x^2 + 12, \text{ and } f''(x) = -6x.$$

(a) (8 points) Make a sign diagram (or sign chart) for the first derivative of $f(x)$, and find all open intervals of increase and all open intervals of decrease.

$f'(x) = 0$ when $0 = -3x^2 + 12 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow \boxed{x = \pm 2}$.

$f'(x)$: $-$ 0 $+$ 0 $-$

x: -3, -2, 0, 2, 3

$f'(-3) = -3(-3)^2 + 12 = -15$

$f'(0) = -3(0)^2 + 12 = 12$

$f'(-3) = -3(3)^2 + 12 = -15$

$f(x)$ is increasing on $(-2, 2)$ and $f(x)$ is decreasing on $(-\infty, -2) \cup (2, \infty)$.

(b) (8 points) Make a sign diagram (or sign chart) for the second derivative of $f(x)$ and find all open intervals on which the graph is concave up and all open intervals on which the graph is concave down.

$f''(x) = 0$ when $-6x = 0 \Rightarrow \boxed{x = 0}$.

$f''(x)$: $+$ 0 $-$

x: -1, 0, 1

$f(-1) = -6(-1) = 6$

$f(1) = -6(1) = -6$

$f(x)$ is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.

(c) (6 points) Find the critical numbers and the inflection points of $f(x)$ and classify each critical point as a relative maximum, relative minimum, or inflection point.

Critical #s: $x = -2$, $x = 2$. $x = -2$ is a local min and $x = 2$ is a local max.

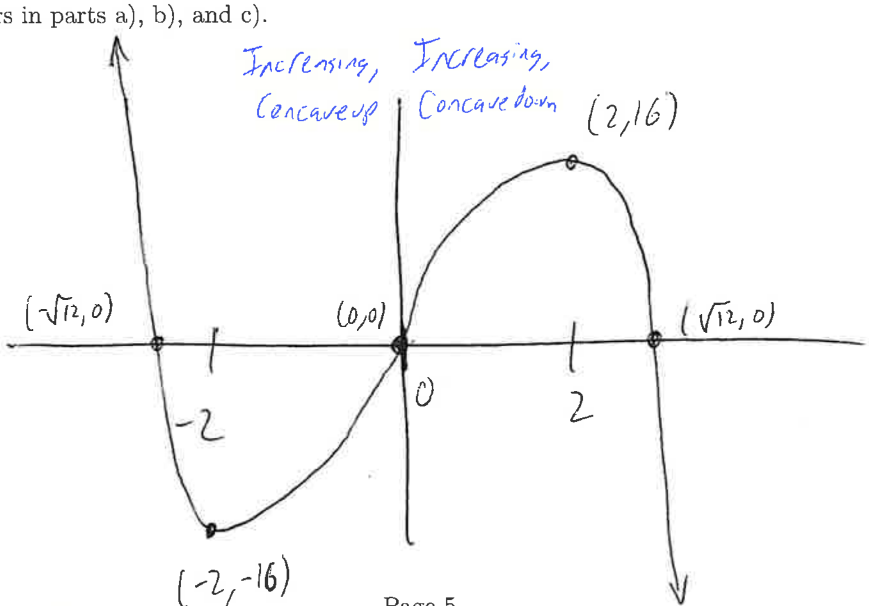
Inflection points: $x = 0$.

(d) (4 points) Sketch the graph of $y = f(x)$ by hand, plotting and labeling **only** the relative extreme points, inflection points and the y-intercept. To be considered correct, your graph must match your answers in parts a), b), and c).

$f(-2) = -(-2)^3 + 12(-2)$
$= 8 - 24 = -16$

$f(0) = 0$

$f(2) = -(2)^3 + 12(2)$
$= -8 + 24 = 16$.



7. (8 points) Find the equation of the line tangent to $4x^2y - xy^3 = 0$ at the point $(1, 2)$.

Differentiate both sides with respect to x:

$$8xy + 4x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx} = 0.$$

Plug in (1,2):

$$8(1)(2) + 4(1)^2\frac{dy}{dx} - (2)^3 - 3(1)(2)^2\frac{dy}{dx} = 0$$

$$\Rightarrow 16 + 4\frac{dy}{dx} - 8 - 12\frac{dy}{dx} = 0$$

$$\Rightarrow 8 = 8\frac{dy}{dx} \Rightarrow \boxed{\frac{dy}{dx} = 1}.$$

8. (8 points) As a snowball is rolling down a hill its radius is increasing at a rate of 2 inches per second. Find the rate at which the volume is changing at the moment when the volume of the snowball is 36π cubic inches (NOTE: For a sphere, $V = \frac{4}{3}\pi r^3$).

Note $\frac{dr}{dt} = 2$ is given, and $V = 36\pi$ is given. Thus,

$$36\pi = \frac{4}{3}\pi r^3 \Rightarrow 27 = r^3 \Rightarrow \boxed{r = 3}.$$

From $V = \frac{4}{3}\pi r^3$, differentiate both sides with respect to time:

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi(3)^2(2) = \boxed{72\pi}.$$

9. A company's demand function for a luxury item is $D(p) = 180e^{-0.2p}$. The company is currently selling this luxury item for \$10 each.

(a) (8 points) Find the Elasticity of Demand at this price and determine if the price is elastic, inelastic or unit-elastic.

$$E(p) = \frac{-p\,D'(p)}{D(p)} = \frac{-p\left(180e^{-0.2p}\cdot(-0.2)\right)}{180e^{-0.2p}} = \boxed{0.2p}.$$

At $p=10$, $E(p)=2$, so the demand is elastic.

(b) (3 points) How much should the company charge for each luxury item it wants to maximize revenue?

To maximize revenue, set $E(p)=1 \Rightarrow 0.2p=1 \Rightarrow \boxed{p=5}$.

10. (6 points) State **but do not evaluate** the expression which gives the area bounded by the two curves.

$f(x) = 5 - x^2$ $\qquad\qquad$ $g(x) = x^2 - 3$

Solve $f(x)=g(x)$: $5-x^2 = x^2-3 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow \boxed{x=2,\ x=-2}$.

Now, determine the top function using a test point in the interval $[-2,2]$:

Test $x=0$, $f(0)=5-(0)^2=5$ and $g(0)=(0)^2-3=-3$.

Conclude $f(x)$ is the top function, so get

$$\boxed{A=\int_{-2}^{2}\left((5-x^2)-(x^2-3)\right)dx}$$

11. Evaluate.

(a) (8 points) $\int (\sqrt[3]{x^5} - e^{-2x} - \frac{3}{x^2} + 3)\, dx$

U-sub: $U = -2x$, $dU = -2dx$

$= \int x^{5/3} - \boxed{e^{-2x}} - 3x^{-2} + 3 \, dx$

$$\boxed{= \frac{x^{8/3}}{8/3} - \frac{e^{-2x}}{-2} - \frac{3x^{-1}}{-1} + 3x + C}$$

(b) (8 points) $\int_0^1 \frac{e^{2x} + 3}{e^{2x}} \, dx$

U-sub: $U = 2x$, $dU = -2dx$

$= \int_0^1 \frac{e^{2x}}{e^{2x}} + \frac{3}{e^{2x}} \, dx = \int_0^1 1 + 3\boxed{e^{-2x}} \, dx = \left[x + 3 \cdot \frac{e^{-2x}}{-2} \right]_0^1$

$= \left((1) - \frac{3}{2} e^{-2(1)} \right) - \left((0) - \frac{3}{2} e^{-2(0)} \right) = \boxed{1 - \frac{3}{2} e^{-2} + \frac{3}{2}}$

(c) (8 points) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$ U-sub: $U = \sqrt{x}$, $dU = \frac{1}{2\sqrt{x}} dx$, $dx = dU \cdot 2\sqrt{x} = dU \cdot 2U$

$= \int \frac{e^U}{U} \cdot dU \cdot 2U = \int 2e^U dU = 2e^U + C = \boxed{2e^{\sqrt{x}} + C}$

(d) (8 points) $\int x \ln x \, dx$ Integration by Parts: $\int f \cdot dg = f \cdot g - \int g \cdot df$

$f = \ln(x)$ $\quad dg = x\,dx$

$df = \frac{1}{x} dx$ $\quad g = \frac{1}{2} x^2$

$= \frac{1}{2} x^2 \ln(x) - \int \frac{1}{x} \cdot \frac{1}{2} x^2 dx$

$= \frac{1}{2} x^2 \ln(x) - \int \frac{1}{2} x \, dx = \boxed{\frac{1}{2} x^2 \ln(x) - \frac{1}{4} x^2 + C}$

12. (8 points) A company's marginal cost function is $MC(x) = 10e^{0.01x}$ where x is the number of units. Supposing that fixed costs are $3000, find the cost function.

$$TC(x) = \int MC(x)\,dx = \int 10e^{0.01x}\,dx = 10e^{0.01x} \cdot \frac{1}{0.01} + C$$

Also, $TC(0) = 3000$, so $3000 = 1000e^{0.01(0)} + C \Rightarrow 3000 = 1000 + C$

$\Rightarrow \boxed{C = 2000}$

Conclude $\boxed{TC(x) = 1000e^{0.01x} + 2000}$

13. Suppose for a certain product the demand function is $d(x) = 600 - x^2$ and the supply function is $s(x) = 50x$.

(a) (4 points) What is the market demand for x?

Set $d(x) = s(x)$, getting $600 - x^2 = 50x \Rightarrow x^2 + 50x - 600 = 0 \Rightarrow (x-10)(x+60) = 0$

$\Rightarrow \boxed{x^* = 10}$, $\cancel{x = -60}$ (negative demand does not make sense)

(b) (2 points) What is the market price for x?

$p^* = s(x^*) = 50(10) = \boxed{500}$.

(c) (6 points) State **but do not evaluate** the expression that gives the consumer's surplus at the market demand.

$$CS = \int_0^{x^*} (d(x) - p^*)\,dx = \boxed{\int_0^{10} ((600 - x^2) - (500))\,dx}$$

(d) (6 points) State **but do not evaluate** the expression that gives the producer's surplus at the market demand.

$$PS = \int_0^{x^*} (p^* - s(x))\,dx = \boxed{\int_0^{10} ((500) - (50x))\,dx}$$

14. (12 points) Find all critical points extreme of the function below and classify each as a relative maximum, relative minimum, or saddle point.

$$f(x, y) = y^3 - x^2 - 2x - 12y$$

Set $f_x = 0$ and $f_y = 0$:

Note $f_x = -2x - 2$ and $f_y = 3y^2 - 12$.

From f_x, $\boxed{x = -1}$ and from f_y, $\boxed{y = \pm 2}$

The critical points are $\boxed{(-1, -2) \text{ and } (-1, 2)}$

To test these, use $D = f_{xx} f_{yy} - (f_{xy})^2$.

Note $f_{xx} = -2$, $f_{yy} = 6y$, $f_{xy} = 0$, so $D = -12y - (0)^2 = \boxed{-12y}$.

At $(-1, -2)$, get $D = -12(-2) = 24 > 0$ and $f_{xx} = -2 < 0$, so $\underline{(-1, -2) \text{ is a local max}}$.

At $(-1, 2)$, get $D = -12(2) = -24 < 0$, so $\underline{(-1, 2) \text{ is a saddlepoint}}$.

15. (12 points) A company manufactures two products with $x =$ the number of units of product A produced and $y =$ the number of units of product B produced. Because of limited materials and capital, the quantities produced must satisfy the equation $4x + 2y = 80$ (this is called a *production possibilities curve*). Given the company's profit function is $P = 4x^2 + y^2$, use Lagrange Multipliers to find the production levels of products A and B that maximize the company's profit.

Solve $\nabla f(x,y) = \lambda \cdot \nabla g(x,y)$, where $f(x,y) = 4x^2 + y^2$ and

$g(x,y) = 4x + 2y - 80 = 0$;

$f_x = 8x$, $f_y = 2y$, $g_x = 4$, $g_y = 2$, so get

$8x = \lambda \cdot 4$ and $2y = \lambda \cdot 2$.

Conclude $x = \frac{1}{2}\lambda$ and $y = \lambda$, so

$4(\frac{1}{2}\lambda) + 2(\lambda) = 80 \Rightarrow 4\lambda = 80 \Rightarrow \lambda = 20$.

Hence, $\boxed{x = \frac{1}{2}(20) = 10}$ and $\boxed{y = 20}$ maximizes profit.