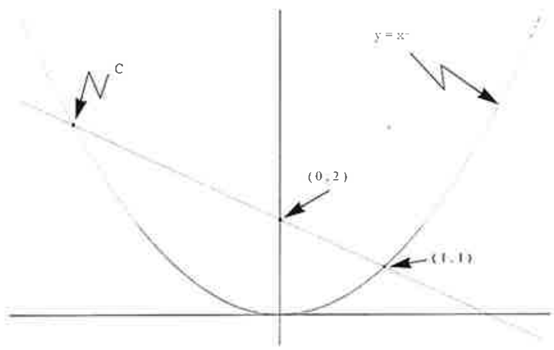
MATH 0120 - BUSINESS CALCULUS

SAMPLE FINAL EXAM

(1) (a) Using the graph below, find the coordinates of the point C. [10 points]



The equation of the line is $y = -x + 2$, since $m = \frac{1-2}{1-0} = \frac{-1}{1} = \boxed{-1}$ and $(0,2)$ is given.

C is located at the intersection of the line and parabola, so solve

$-x+2 = x^2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x-1)(x+2) = 0 \Rightarrow \boxed{x=1}$, $\boxed{x=-2}$.

Since $x=1$ is already marked, conclude the x-coordinate of C is -2. Thus, the y-coordinate is $(-2)^2 = 4$. So C is at $(-2,4)$.

(b) Simplify the following expression as much as possible [10 points]:

$$\frac{\left(\dfrac{y^3-1}{y^2-1}\right)}{\left(\dfrac{y^2+y+1}{y^2+2y+1}\right)}$$

where $y \neq \pm 1$

$$= \frac{\dfrac{(y-1)(y^2+y+1)}{(y-1)(y+1)}}{\dfrac{y^2+y+1}{(y+1)^2}} = \frac{y^2+y+1}{y+1} \cdot \frac{(y+1)^2}{y^2+y+1} = \boxed{y+1}.$$

(2) Find the following limits [12 points] :

(a) $\lim_{x\to -2}\left(\frac{x+2}{x^2-4}\right) \to \frac{0}{0}$

$= \lim_{x\to -2} \frac{x+2}{(x-2)(x+2)} = \lim_{x\to -2} \frac{1}{x-2} = \frac{1}{(-2)-2} = \boxed{\frac{-1}{4}}$.

(b) $\lim_{h\to 0}\left(\frac{(3+h)^2-9}{h}\right) \to \frac{0}{0}$

$= \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0} 6+h = \boxed{6}$.

(c) $\lim_{x\to 2^+}\left(\frac{(x+2)^2}{x^2-4}\right) \to \frac{16}{0}$

As $x\to 2^+$, $x^2-4>0$. Hence, get $\frac{+}{+}$, so $\lim_{x\to 2^+} \frac{(x+2)^2}{x^2-4}$ $\boxed{\text{DNE } (+\infty)}$.

(3) Using the **definition of the derivative only**, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$,[10 points]
(a) find the derivative function of

$$f(x) = \frac{1}{x}.$$

$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\left(\frac{x-(x+h)}{x(x+h)}\right)}{h}$

$= \lim_{h\to 0} \frac{x-x-h}{x(x+h)h} = \lim_{h\to 0} \frac{-h}{x(x+h)\cancel{h}} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \boxed{\frac{-1}{x^2}}$.

(b) Using the derivative function you found in (a) above, find the equation of the tangent line to $f(x)$ above at the point $(2, \frac{1}{2})$. [2 points]

Use point-slope form $y-y_1 = m(x-x_1)$. ~~[illegible]~~

$(2, \frac{1}{2}) = (x_1, y_1)$ is given, ~~and~~ get $m = f'(x_1) = \frac{-1}{(2)^2} = \frac{-1}{4}$.

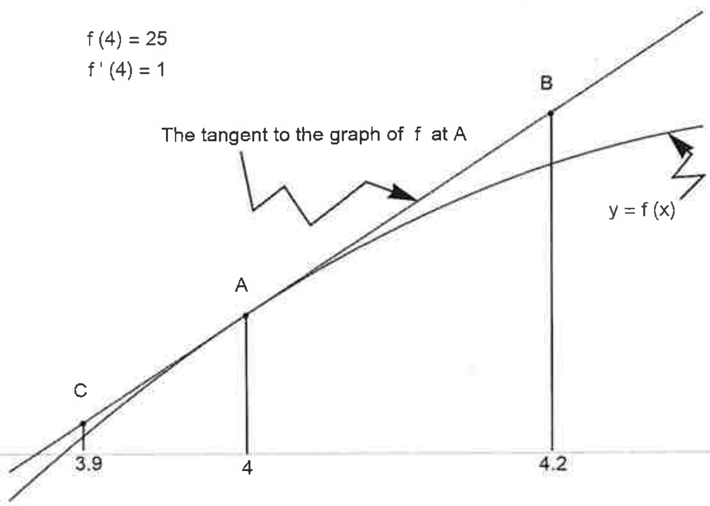
So, get $\boxed{y - \frac{1}{2} = \frac{-1}{4}(x-2)}$.

(4) (a) The quantity, q, of a certain skateboard sold depends on the selling price, p, so we write $q = f(p)$. You are given that $f(140) = 15,000$ and $f'(140) = -100$. What does $f(140) = 15,000$ and $f'(140) = -100$ tell you about the sale of the skateboards ? [8 points]

$f(140) = 15000$ means 15000 skateboards sell when the price is \$140.

$f'(140) = -100$ means ~~the~~ a price increase of some tiny amount h results in 100h fewer skateboards sold.

(b) State the coordinates of the point A. Calculate the coordinates of the points B and C using the figure below. Recall : $f'(x)$ can also be written as $\frac{dy}{dx}$. [12 points]



A is located at $(4, f(4)) = \boxed{(4, 25)}$.

B and C are on the line tangent to $f(x)$ at A. Since $f'(4) = 1$, the equation of this line is $y - 25 = 1(x-4) \Rightarrow \boxed{y = x + 21}$. Say $y = g(x)$.

Conclude B is located at $(4.2, g(4.2)) = \boxed{(4.2, 25.2)}$

and C is located at $(3.9, g(3.9)) = \boxed{(3.9, 24.9)}$.

(5) (a) Find $\frac{dy}{dx}$ **(Do not simplify your answer).** [7 points]

$$y = \frac{e^{2x}}{1+e^{-2x}}$$

$$\frac{dy}{dx} = \frac{[1+e^{-2x}][e^{2x}\cdot(2)] - [e^{2x}][e^{-2x}\cdot(-2)]}{[1+e^{-2x}]^2}$$

(b) Find $f'(x)$ **(Do not simplify your answer) :** [7 points]

$$f(x) = (-9x^2-2)\sqrt[3]{9x^2+2}$$

$$f'(x) = [-9x^2-2]\left[\tfrac{1}{3}(9x^2+2)^{-2/3}\cdot(18x)\right] + [-18x]\left[(9x^2+2)^{1/3}\right]$$

(Alternatively, write $f(x) = -(9x^2+2)(9x^2+2)^{1/3} = -(9x^2+2)^{4/3}$

and get $\boxed{f'(x) = -\frac{4}{3}(9x^2+2)^{1/3}\cdot(18x).}$

(c) Find $\dfrac{dy}{dx}$ where x and y are related by

$$x^3 + 2xy^2 + y^3 = 1$$

[8 points]

Differentiate both sides with respect to x:

$$3x^2 + 2y^2 + 4xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + 2y^2 = -4xy\frac{dy}{dx} - 3y^2\frac{dy}{dx}$$

$$\Rightarrow 3x^2 + 2y^2 = (-4xy - 3y^2)\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{3x^2+2y^2}{-4xy-3y^2}}$$

(6) Consider the function $f(x) = 3x^4 - 4x^3 + 1$. [For your information : $f(\frac{2}{3}) = \frac{33}{81} = 0.4074$.] [20 points]

(a) Find the critical number(s) of $f(x)$ and the open intervals on which $f(x)$ is increasing and decreasing.

$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$, so $f'(x) = 0$ when $\boxed{x=0,\ x=1}$.

These are the critical numbers.

$f'(x)$: $\ominus$ 0 $\ominus$ 0 $\oplus$

x: -1, 0, $\frac{1}{2}$, 1, 2

$f'(-1) = 12(-1)^2((-1)-1) = -24$

$f'(\frac{1}{2}) = 12(\frac{1}{2})^2((\frac{1}{2})-1) = \frac{-12}{8}$

$f'(2) = 12(2)^2((2)-1) = 48$

$f(x)$ is increasing on $(1, \infty)$ and decreasing on $(-\infty, 0) \cup (0, 1)$.

(b) Find the inflection point(s) of $f(x)$ and the open intervals on which $f(x)$ is concave up and concave down.

$f''(x) = 36x^2 - 24x = 12x(3x-2)$, so $f''(x) = 0$ when $\boxed{x=0,\ x=\frac{2}{3}}$

These are the inflection points.

$f''(x)$: $\oplus$ 0 $\ominus$ 0 $\oplus$

x: -1, 0, $\frac{1}{2}$, $\frac{2}{3}$, 1

$f''(-1) = 12(-1)(3(-1)-2) = 60$

$f''(\frac{1}{2}) = 12(\frac{1}{2})(3(\frac{1}{2})-2) = -3$

$f''(1) = 12(1)(3(1)-2) = 12$

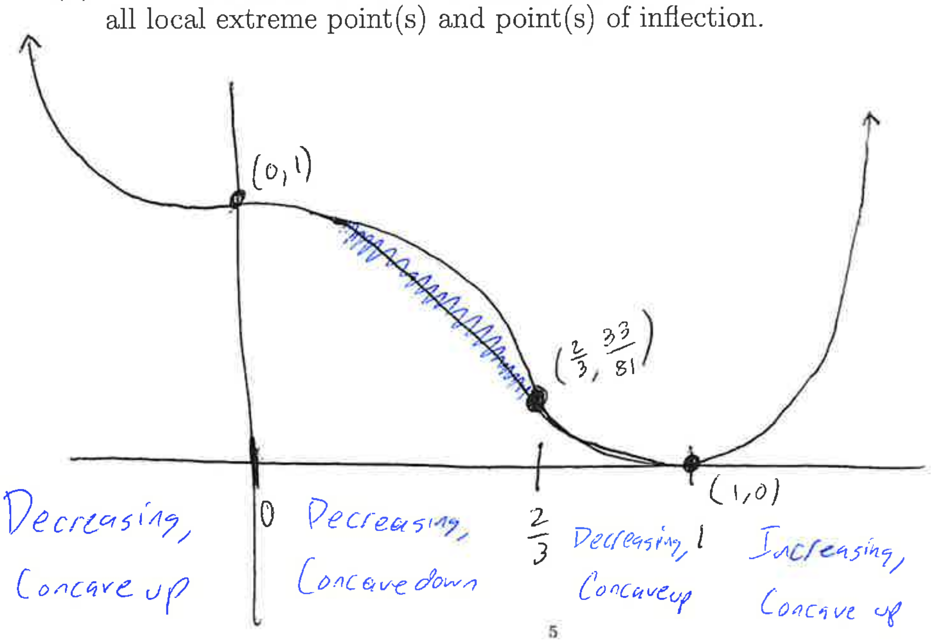
$f(x)$ is concave up on $(-\infty, 0) \cup (\frac{2}{3}, \infty)$ and concave down on $(0, \frac{2}{3})$.

(c) Determine whether $f(x)$ at each critical point has a local maximum value, a local minimum value or neither.

$x=0$ is neither; $x=1$ is a local min.

(d) Sketch the graph of $f(x)$ by hand, indicating a scale on both axes and labeling all local extreme point(s) and point(s) of inflection.

$f(1) = 3(1)^4 - 4(1)^3 + 1 = 0$.



(7) (a) A whitewater rafting company knows that at a price of \$80 for a half-day trip, they will attract 300 customers. For every \$5 decrease in price it is estimated that they will attract an additional 30 customers.
What price should the company charge and how many customers should they attract to maximize their revenue ? [10 points]

Revenue $R = \text{Price} \times \text{Quantity}$. Let

p = price decreases below \$80.

$\text{Price} = 80 - 5p$

$\text{Quantity} = 300 + 30p$

$R(p) = (80-5p)(300+30p)$

$= 24000 - 1500p + 2400p - 150p^2$

$= 24000 + 900p - 150p^2$.

To maximize, set $R'(p) = 0 \Rightarrow$

$900 - 300p = 0 \Rightarrow \boxed{p=3}$.

Since $R''(p) = -300 < 0$, $p=3$ corresponds to a local max.
Hence, the optimal price is $80 - 5(3) = \boxed{\$65}$ and the # of customers attracted is $300 + 30(3) = \boxed{390}$.

(b) Suppose the demand function for a certain product is given by

$$q = D(p) = 5000 - 3p^2$$

where p is the price in dollars.

(i) find the elasticity of demand, $E(p)$, at a price of \$30. Is the demand at this price elastic or inelastic ? [8 points]

$E(p) = \dfrac{-p\,D'(p)}{D(p)} = \dfrac{-p(-6p)}{5000-3p^2} = \dfrac{6p^2}{5000-3p^2}$. At $p=30$,

$E(30) = \dfrac{6(30)^2}{5000-3(30)^2} = \dfrac{5400}{2300} > 1$, so the demand is elastic.

(ii) Based on the elasticity found in (a), do you expect the price to maximize revenue to be above or below \$30 ? [2 points]

The max revenue should be attained at a price below \$30, because decrease in price results in a comparatively large increase in demand.

(8) Find the following integrals. Leave your answer in exact form [24 points]:

(a)

$$\int_{e}^{e^3} \left(\frac{t^2-1}{t}\right) dt$$

$$= \int_e^{e^3} \frac{t^2}{t} - \frac{1}{t} \, dt = \int_e^{e^3} t - \frac{1}{t} \, dt = \left[\frac{t^2}{2} - \ln|t|\right]_e^{e^3}$$

$$= \left(\frac{(e^3)^2}{2} - \ln|e^3|\right) - \left(\frac{(e)^2}{2} - \ln|e|\right) = \boxed{\frac{e^6}{2} - 3 - \frac{e^2}{2} + 1}$$

(b)

$$\int \frac{x^3}{(1+x^4)^{1/3}} \, dx$$ u-substitution

$u = 1+x^4$

$du = 4x^3 dx$

$dx = \frac{du}{4x^3}$

$$= \int \frac{x^3}{u^{1/3}} \cdot \frac{du}{4x^3} = \int \frac{1}{4} u^{-1/3} du = \frac{1}{4} \cdot \frac{u^{2/3}}{2/3} + C$$

$$\boxed{= \frac{1}{4} \frac{(1+x^4)^{2/3}}{2/3} + C}$$

(c)

$$\int \left(\frac{\ln x}{x^2}\right) dx$$ Integration by Parts: $\int f \cdot dg = f \cdot g - \int g \cdot df$

$f = \ln(x) \quad dg = x^{-2} dx$

$df = \frac{1}{x} dx \quad g = -x^{-1}$

$$= -\frac{\ln(x)}{x} - \int \frac{1}{x} \cdot -x^{-1} dx = -\frac{\ln(x)}{x} + \int x^{-2} dx = \boxed{\frac{-\ln(x)}{x} + \frac{x^{-1}}{-1} + C}$$

(9) (a) A company's marginal cost function is $C'(x) = 0.015x^2 - 2x + 80$ dollars where x is the number of units produced in one day. The company has a fixed cost of \$2000 per day.

(i) Find the total cost of producing x units per day. [7 points]

$C(x) = \int C'(x)\,dx = \int 0.015x^2 - 2x + 80 \; dx = \frac{0.015}{3}x^3 - x^2 + 80x + D$.

Also, $C(0) = 2000$, so $2000 = \frac{0.015}{3}(0)^3 - (0)^2 + 80(0) + D \Rightarrow \boxed{D = 2000}$.

Conclude $\boxed{C(x) = \frac{0.015}{3}x^3 - x^2 + 80x + 2000}$

(ii) If the current production level is $x = 10$ units per day, determine the total change in cost if the production level is raised to $x = 20$ units per day.[5 points]

Change in cost $= C(20) - C(10)$

$= \left[\frac{0.015}{3}(20)^3 - (20)^2 + 80(20) + 2000\right] - \left[\frac{0.015}{3}(10)^3 - (10)^2 + 80(10) + 2000\right]$

$= 3600 - 2750 = \boxed{\$850}$.

(b) Set up but **do not evaluate** the integral that gives the area of the region bounded by the two curves[10 points] :

$$y_1 = 2x^2 \qquad \text{and} \qquad y_2 = x^3 - 3x$$

Set $y_1 = y_2$: $2x^2 = x^3 - 3x \Rightarrow x^3 - 2x^2 - 3x = 0$

$\Rightarrow x(x^2 - 2x - 3) = 0 \Rightarrow x(x-3)(x+1) = 0 \Rightarrow \boxed{x = 0,\ x = 3,\ x = -1}$

With three intersection points, the curves create two regions, so the question is poorly phrased.

One way to solve this question is to add the areas of both regions, where the top function could change between each set of bounds. So, on $[-1, 0]$, $2x^2 < x^3 - 3x$ (test $x = \frac{-1}{2}$) and on $[0, 3]$, $2x^2 > x^3 - 3x$ (test $x = 1$).

So the area is

$$\boxed{A = \int_{-1}^{0} \left((x^3 - 3x) - (2x^2)\right) dx + \int_{0}^{3} \left((2x^2) - (x^3 - 3x)\right) dx}$$

(10) (a) A company sells two types of electric blenders, Model A and Model B, where x and y represent the number of units sold of Model A and Model B respectively. The daily revenue function is given by:

$$R(x, y) = -0.02x^2 + 80x - 0.05y^2 + 60y - 0.02xy$$

(i) Find $R_x(x, y)$ and $R_y(x, y)$. [8 points]

$R_x(x,y) = -0.04x + 80 - 0.02y$

$R_y(x,y) = -0.1y + 60 - 0.02x$

(ii) Calculate $R_x(100, 300)$ [3 points]

$R_x(100, 300) = -0.04(100) + 80 - 0.02(300)$

$= -4 + 80 - 6 = \boxed{70}$

(iii) Calculate $R_y(100, 300)$ [3 points]

$R_y(100, 300) = -0.1(100) + 60 - 0.02(300)$

$= -10 + 60 - 6 = \boxed{44}$

(b) For the function:

$$f(x,y) = 4y^3 + x^2 - 12y^2 - 36y + 2$$

Find all the critical point(s) of $f(x,y)$ and classify each as either a relative maximum, relative minimum or saddle point.[14 points]

Set $f_x = 0$ and $f_y = 0$: $f_x = 2x$ and $f_y = 12y^2 - 24y - 36$.

From f_x, get $x = 0$, and from f_y, get $12(y^2 - 2y - 3) = 0 \Rightarrow 12(y-3)(y+1) = 0$

$\Rightarrow \boxed{y = 3,\ y = -1}$. The critical points are thus $\boxed{(0,3) \text{ and } (0,-1).}$

To classify each, ~~t~~est $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$.

Note $f_{xx} = 2$, $f_{yy} = 24y - 24$, $f_{xy} = 0$, so $D = 48y - 48$.

At $(0,3)$, $D = 48(3) - 48 = 96 > 0$ and $f_{xx} = 2 > 0$, so $\underline{(0,3) \text{ is a local min}}$.

At $(0,-1)$, $D = 48(-1) - 48 = -96 < 0$, so $\underline{(0,-1) \text{ is a saddle point}}$.