MATH 0120 - BUSINESS CALCULUS

SAMPLE FINAL EXAM

1. Evaluate the following limits

(a) (5 points) $\lim_{x\to -1} \frac{3x^3-3x^2-6x}{x^2+x} \to \frac{0}{0}$

$$= \lim_{x\to -1} \frac{3x(x^2-x-2)}{x(x+1)} = \lim_{x\to -1} \frac{3x(x+1)(x-2)}{x(x+1)} = \lim_{x\to -1} \frac{3(x-2)}{1} = 3((-1)-2) = \boxed{-9}.$$

(b) (5 points) $\lim_{x\to 4} \frac{\sqrt{x}+2}{x+4} \to \frac{4}{8} = \boxed{\frac{1}{2}}$

In more detail: $\frac{\sqrt{4}+2}{4+4} = \frac{2+2}{4+4} = \frac{4}{8} = \frac{1}{2}.$

(c) (5 points) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \to \frac{0}{0}$

$$= \lim_{x\to 4} \frac{(\sqrt{x}-2)}{(x-4)} \cdot \frac{(\sqrt{x}+2)}{(\sqrt{x}+2)} = \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}.$$

(d) (5 points) $\lim_{x\to 3+} \frac{2}{3-x} \to \frac{2}{0}$

Note as $x \to 3^+$, $3-x \to 0^-$ (in other words, $3-x<0$).

Thus, our fraction is $\frac{\oplus}{\ominus}$, so $\lim_{x\to 3^+} \frac{2}{3-x}$ $\boxed{\text{DNE } (-\infty)}$

2. Find the derivatives of the following functions. **You do NOT need to simplify your answer.**

(a) (6 points) $f(x) = 3\sqrt{x}e^{7-x} + e^2$

$f'(x) = [3\sqrt{x}][e^{7-x} \cdot (-1)] + [3 \cdot \frac{1}{2} x^{-1/2}][e^{7-x}] + 0$

(b) (6 points) $g(x) = \frac{x}{\ln(x^2+1)}$

$$g'(x) = \frac{[\ln(x^2+1)][1] - [x][\frac{1}{x^2+1} \cdot 2x]}{[\ln(x^2+1)]^2}$$

(c) (6 points) $h(x) = \frac{x\sqrt[3]{x^4} - x + 2\sqrt{x} - 3}{x}$

$$h(x) = \frac{x^{7/3} - x + 2x^{1/2} - 3}{x} = x^{4/3} - 1 + 2x^{-1/2} - 3x^{-1}$$

$$\boxed{h'(x) = \frac{4}{3}x^{1/3} - 0 + 2(-\frac{1}{2}x^{-3/2}) - 3(-x^{-2})}$$

3. Evaluate the following integrals.

(a) (8 points) $\int \frac{x\sqrt[3]{x^4} - x + 2\sqrt{x} - 3}{x}\,dx$

$= \int x^{4/3} - 1 + 2x^{-1/2} - \frac{3}{x}\,dx$

$\boxed{= \frac{x^{7/3}}{7/3} - x + 2\frac{x^{1/2}}{1/2} - 3\ln|x| + C}$

(b) (8 points) $\int \frac{e^{2x}}{3+e^{2x}}\,dx$ U-substitution

$u = 3 + e^{2x}$

$du = 2e^{2x}dx$

$dx = \frac{du}{2e^{2x}}$

$= \int \frac{e^{2x}}{u} \cdot \frac{du}{2e^{2x}} = \int \frac{1}{2} \cdot \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|3+e^{2x}| + C}$

(c) (8 points) $\int_0^1 2xe^{2x^2}\,dx$ U-substitution; ignore bounds until the end.

$u = 2x^2$

$du = 4x\,dx$

$dx = \frac{du}{4x}$

$= \int 2xe^u \cdot \frac{du}{4x} = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + C = \boxed{\frac{1}{2}e^{2x^2} + C}$

(d) (8 points) $\int x^2 \ln x\,dx$ Integration by Parts: $\int f\cdot dg = f\cdot g - \int g\cdot df$

$f = \ln(x) \quad dg = x^2dx$

$df = \frac{1}{x}dx \quad g = \frac{1}{3}x^3$

$= \frac{1}{3}x^3\ln(x) - \int \frac{1}{x}\cdot\frac{1}{3}x^3\,dx$

$= \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^2\,dx$

$\boxed{= \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C}$

4. (10 points) Use **the limit definition** of the derivative to find the derivative of $f(x) = x^2 - 7x + 3$.
NO CREDIT will be given if the limit definition is not used.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 7(x+h) + 3 - (x^2 - 7x + 3)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 7x - 7h + 3 - x^2 + 7x - 3}{h} = \lim_{h\to 0} \frac{2xh + h^2 - 7h}{h} = \lim_{h\to 0} \frac{h(2x+h-7)}{h}$$

$$= \lim_{h\to 0} (2x + h - 7) = \boxed{2x - 7}.$$

5. If total costs are given by $C(x) = 10x + 12$ and total revenues are given by $R(x) = 18x - x^2$, both in thousands of dollars, where x is the number of units.

(a) (3 points) Find the break-even points (break-even points are the numbers of units of production where a companys costs are equal to its revenue).

$C(x) = R(x)$ when $10x+12 = 18x - x^2 \Rightarrow x^2 - 8x + 12 = 0$

$\Rightarrow (x-6)(x-2) = 0 \Rightarrow \boxed{x=6}, \boxed{x=2}$.

(b) (5 points) What is the maximum profit?

Profit $= R(x) - C(x) = 18x - x^2 - (10x+12) = -x^2 + 8x - 12$.

Maximum is achieved when $P'(x) = 0$, or $-2x+8 = 0 \Rightarrow \boxed{x=4}$.

Note $P''(x) = -2 < 0$, indicating a local max.

6. Follow the steps to graph the stated function.

$$f(x) = x^4 - 4x^3, \quad f'(x) = 4x^3 - 12x^2, \quad \text{and} \quad f''(x) = 12x^2 - 24x.$$

(a) (8 points) Make a sign diagram (or sign chart) for the first derivative of $f(x)$, and find all open intervals of increase and all open intervals of decrease.

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$, so $f'(x) = 0$ when $\boxed{x=0}$ or $\boxed{x=3}$.

$f'(x)$	$-$	0	$-$	0	$+$
x	-1	0	1	3	4

$f(-1) = 4(-1)^2((-1)-3) = -16$

$f(1) = 4(1)^2((1)-3) = -8$

$f(4) = 4(4)^2((4)-3) = 64$

$f(x)$ is increasing on $(3, \infty)$ and decreasing on $(-\infty, 0) \cup (0, 3)$.

(b) (8 points) Make a sign diagram (or sign chart) for the second derivative of $f(x)$ and find all open intervals on which the graph is concave up and all open intervals on which the graph is concave down.

$f''(x) = 12x^2 - 24x = 12x(x-2)$, so $f''(x) = 0$ when $\boxed{x=0}$ or $\boxed{x=2}$.

$f''(x)$	$+$	0	$-$	0	$+$
x	$\uparrow$ -1	0	$\uparrow$ 1	2	$\uparrow$ 3

$f(-1) = 12(-1)((-1)-2) = 36$

$f(1) = 12(1)((1)-2) = -12$

$f(3) = 12(3)((3)-2) = 36$

$f(x)$ is concave up on $(-\infty, 0) \cup (2, \infty)$ and concave down on $(0, 2)$.

(c) (6 points) Find the critical numbers and the inflection points of $f(x)$ and classify each critical point as a relative maximum, relative minimum, or inflection point.

Critical #s: $x=0$ and $x=3$. $x=3$ is a local min and $x=0$ is neither a max or min.

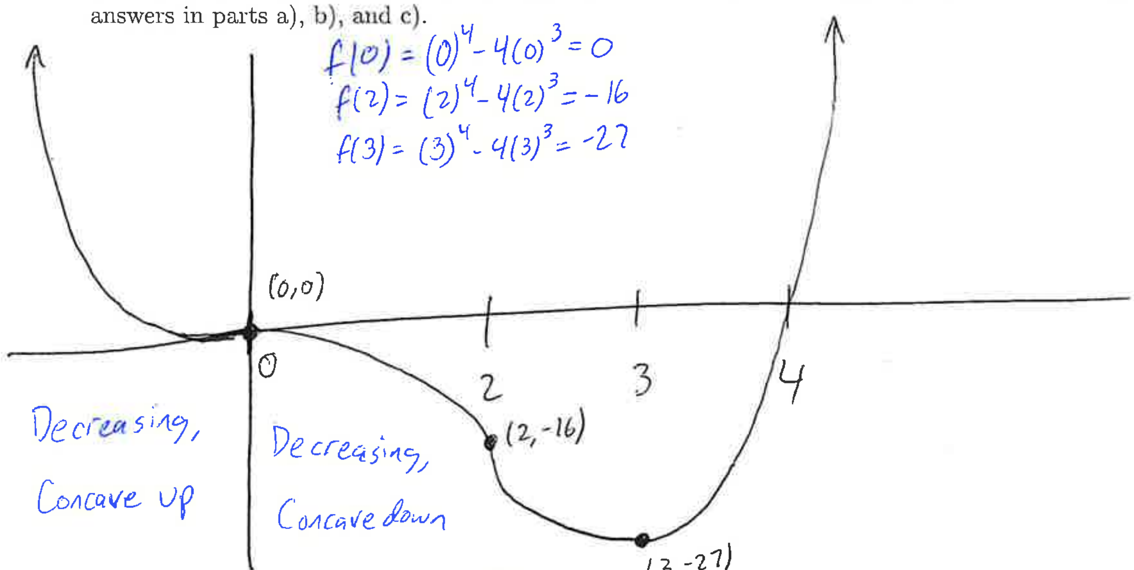
Inflection Points: $x=0$ and $x=2$.

(d) (4 points) Sketch the graph of $y = f(x)$ by hand, plotting and labeling **only** the relative extreme points, inflection points and the y-intercept. To be considered correct, your graph must match your answers in parts a), b), and c).

$f(0) = (0)^4 - 4(0)^3 = 0$

$f(2) = (2)^4 - 4(2)^3 = -16$

$f(3) = (3)^4 - 4(3)^3 = -27$



Decreasing,
Concave up

Increasing,
Concave up

7. (8 points) Find the equation of the line tangent to $x^4 + y^4 - 2x^2y^2 = 0$ at the point $(2,1)$.

Differentiate both sides with respect to x:

$$4x^3 + 4y^3\frac{dy}{dx} - 4xy^2 - 4x^2y\frac{dy}{dx} = 0$$

Plug in point:

$$4(2)^3 + 4(1)^3\frac{dy}{dx} - 4(2)(1)^2 - 4(2)^2(1)\frac{dy}{dx} = 0$$

$$32 + 4\frac{dy}{dx} - 8 - 16\frac{dy}{dx} = 0.$$

$$-12\frac{dy}{dx} = -24 \Rightarrow \boxed{\frac{dy}{dx} = 2}$$

8. (8 points) Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 40 cm? **Indicate units in your answer**. Show all work. (NOTE: For a sphere, $V = \frac{4}{3}\pi r^3$).

We are given $\frac{dV}{dt} = 100$ and $d = 40$, so $r = 20$.

Differentiate both sides with respect to time:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

$$100 = 4\pi(20)^2\frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{100}{1600\pi}}$$

9. Through some marketing research, Megadodo Publications knows that its best-selling book "The Hitchhikers Guide to the Galaxy" has the demand function $D(p) = 300 - p^2$ where p is the price of the book and D is the daily U.S. demand.

(a) (5 points) Find the Elasticity of Demand, $E(p)$, for this book.

$$E(p) = \frac{-p\,D'(p)}{D(p)} = \frac{-p(-2p)}{300-p^2} = \boxed{\frac{2p^2}{300-p^2}}$$

(b) (3 points) If the current price of the book is \$5 determine if the demand is elastic, inelastic or unit-elastic.

$$E(5) = \frac{2(5)^2}{300-(5)^2} = \frac{50}{250} = \boxed{\frac{1}{5}}$$, so demand is inelastic.

(c) (2 points) In order to raise revenue, should Megadodo Publishing raise or lower the price?

They should raise the price.

(d) (4 points) How much should Megadodo Publications charge for each book if it wants to maximize revenue?

Solve $E(p) = 1 \Rightarrow \frac{2p^2}{300-p^2} = 1 \Rightarrow 2p^2 = 300 - p^2$

$\Rightarrow 3p^2 = 300 \Rightarrow p^2 = 100 \Rightarrow \boxed{p = 10}$ ($p = -10$ doesn't make sense).

Conclude they should charge \$10/book.

10. (6 points) State **but do not evaluate** the expression which gives the area bounded by the two curves.

$$f(x) = x^2 - 4 \qquad\qquad g(x) = 8 - 2x^2$$

$f(x) = g(x)$ when $x^2 - 4 = 8 - 2x^2 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow \boxed{x=-2}, \boxed{x=2}$.

Now, the top function can be found by testing $x = 0$:

$f(0) = (0)^2 - 4 = -4$, $g(0) = 8 - 2(0)^2 = 8$, so $g(x)$ is the top function on $[-2, 2]$.

Conclude

$$\boxed{A = \int_{-2}^{2} (8-2x^2) - (x^2-4)\, dx}$$

11. (8 points) A company's marginal cost function is $MC(x) = 10e^{-0.01x}$ where x is the number of units. Supposing that fixed costs are \$500, find the cost function.

$TC(x) = 10e^{-0.01x} \cdot \frac{1}{-0.01} + C$, where $TC(0) = 500$.

So, $500 = -1000e^{-0.01(0)} + C$

$\Rightarrow 500 = -1000(1) + C$

$\Rightarrow \boxed{1500 = C}$

Conclude $\boxed{TC(x) = -1000e^{-0.01x} + 1500}$.

12. Suppose for a certain product the demand function is $d(x) = 600 - 10x^2$ and the supply function is $s(x) = 40x$.

(a) (4 points) What is the market demand for x?

We find this by solving $d(x) = s(x) \Rightarrow 600 - 10x^2 = 40x$

$\Rightarrow 0 = 10x^2 + 40x - 600 \Rightarrow 10(x^2 + 4x - 60) = 0 \Rightarrow 10(x-6)(x+10) = 0$

$\Rightarrow \boxed{x^* = 6}$, $\cancel{x = -10}$ (note negative demand doesn't make sense).

(b) (2 points) What is the market price for x?

At $x = 6$, $s(x) = 40(6) = \boxed{240}$ (we also could have used $d(x)$).

So, $p^* = 240$.

(c) (6 points) State **but do not evaluate** the expression that gives the consumer's surplus at the market demand.

$$CS = \int_0^{x^*} (d(x) - p^*)\,dx = \boxed{\int_0^6 ((600 - 10x^2) - 240)\,dx}$$

(d) (6 points) State **but do not evaluate** the expression that gives the producer's surplus at the market demand.

$$PS = \int_0^{x^*} (p^* - s(x))\,dx = \boxed{\int_0^6 (240 - 40x)\,dx}$$

13. (12 points) Find all critical points extreme of the function below and classify each as a relative maximum, relative minimum, or saddle point.

$$f(x, y) = x^2 + y^3 - 6x - 12y.$$

Set $f_x = 0$ and $f_y = 0$:

$f_x = 2x - 6$, $f_y = 3y^2 - 12$.

Conclude $x = 3$ from f_x and $y = \pm 2$ from f_y.

So, the critical points are $(3,-2)$ and $(3,2)$.

To classify the points, test $D = f_{xx} f_{yy} - (f_{xy})^2$.

Note $f_{xx} = 2$, $f_{yy} = 6y$, $f_{xy} = 0$, so $D = 12y - (0)^2 = 12y$.

At $(3,-2)$, $D = 12(-2) = -24 < 0$, so $(3,-2)$ is a saddle point.

At $(3,2)$, $D = 12(2) = 24 > 0$ and $f_{xx} > 0$, so $(3,2)$ is a local min.

14. (12 points) A company manufactures two products with $x =$ the number of units of product A produced and $y =$ the number of units of product B produced. Because of limited materials and capital, the quantities produced must satisfy the equation $4x + 2y = 80$ (this is called a *production possibilities curve*). Given the company's profit function is $P = 4x^2 + y^2$, use Lagrange Multipliers to find the production levels of products A and B that maximize the company's profit.

Solve $\nabla f(x,y) = \lambda \cdot \nabla g(x,y)$, where $f(x,y) = 4x^2 + y^2$ and $g(x,y) = 4x + 2y - 80 = 0$:

$f_x = 8x$, $f_y = 2y$, $g_x = 4$, $g_y = 2$, so get

$8x = \lambda \cdot 4$ and $2y = \lambda \cdot 2$.

Conclude $x = \frac{1}{2}\lambda$ and $y = \lambda$, so

$4(\frac{1}{2}\lambda) + 2(\lambda) = 80 \Rightarrow 4\lambda = 80 \Rightarrow \boxed{\lambda = 20}$.

Hence, $\boxed{x = \frac{1}{2}(20) = 10}$ and $y = 20$ maximizes profit.