1. (a) (3 pts.) Find an equation of the line that passes through the points (1, 2) and (3,-2).

Slope $= \frac{-2-2}{3-1} = \frac{-4}{2} = \boxed{-2}$. Use point-slope form $y - y_1 = m(x - x_1)$, so

$$\boxed{y - 2 = -2(x-1)}$$

(b) (3 pts.) Find the vertex of the parabola given by y = f(x) = 4x - x^2.

From algebra, the vertex is located at $x = \frac{-b}{2a}$ for $f(x) = ax^2 + bx + c$, so $x = \frac{-4}{2(-1)} = \boxed{2}$, $y = f(2) = 4(2) - (2)^2 = \boxed{4}$.

From calculus, the vertex is located where $f'(x) = 0$, so $4 - 2x = 0 \Rightarrow \boxed{x = 2}$, $y = f(2) = 4(2) - (2)^2 = \boxed{4}$.

The vertex is located at $\boxed{(2, 4)}$.

(c) (4 pts.) Sketch the line and the parabola on the same set of axes, labeling the vertex of the parabola and the x- and y- intercepts of both.

Line: $y - 2 = -2(x-1)$

$y = -2x + 2 + 2$

$\boxed{y = -2x + 4}$

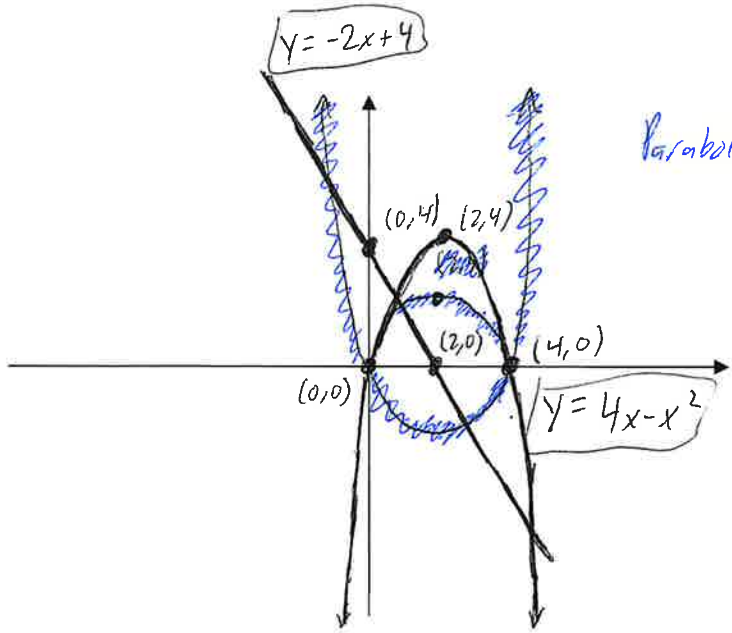
$0 = -2x + 4$

$\Rightarrow \boxed{x = 2}$

Parabola: $4x - x^2 = 0$

$x(4 - x) = 0$

$\boxed{x = 0}$, $\boxed{x = 4}$



2. (a) (5 pts.) Find $\lim_{x \to 2} \dfrac{x-2}{x^3-8}$. $\rightarrow \dfrac{0}{0}$

$$= \lim_{x \to 2} \frac{x-2}{(x-2)(x^2+2x+4)} = \lim_{x \to 2} \frac{1}{x^2+2x+4} = \frac{1}{(2)^2+2(2)+4} = \boxed{\frac{1}{12}}.$$

(b) (5 pts.) Let f(x) be a function. Write the definition of $f'(x)$, the derivative function.

$$f'(x) = \boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

(c) (7 pts.) Use this definition to find the derivative of $f(x) = \dfrac{1}{x-3}$.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{(x+h)-3} - \frac{1}{x-3}}{h} = \lim_{h \to 0} \frac{\frac{x-3}{(x+h-3)(x-3)} - \frac{x+h-3}{(x+h-3)(x-3)}}{h} = \lim_{h \to 0} \frac{\left(\frac{x-3-(x+h-3)}{(x+h-3)(x-3)}\right)}{h}$$

$$= \lim_{h \to 0} \frac{x-3-x-h+3}{(x+h-3)(x-3)h} = \lim_{h \to 0} \frac{-h}{h(x+h-3)(x-3)} = \lim_{h \to 0} \frac{-1}{(x+h-3)(x-3)} = \boxed{\frac{-1}{(x-3)^2}}.$$

3. (24 pts.) Find the derivatives. (**You need not simplify**):

(a) $g(x) = \dfrac{x}{\ln(1-x)}$.

$$g'(x) = \frac{[\ln(1-x)][1] - [x]\cdot\left[\frac{1}{1-x}\cdot(-1)\right]}{(\ln(1-x))^2}$$

(b) $f(x) = (x^2 + x)(3x^2 - 2)^{-2}$.

$$f'(x) = [x^2+x]\left[-2(3x^2-2)^{-3}\cdot(6x)\right] + [2x+1]\left[(3x^2-2)^{-2}\right].$$

(c) $x^3y - xy^3 = 0$. Find $\dfrac{dy}{dx}$.

Differentiate both sides with respect to x:

$$3x^2y + x^3\frac{dy}{dx} - \left(y^3 + 3xy^2\frac{dy}{dx}\right) = 0.$$

Now, solve for $\frac{dy}{dx}$:

$$3x^2y - y^3 = -x^3\frac{dy}{dx} + 3xy^2\frac{dy}{dx}$$

$$3x^2y - y^3 = \frac{dy}{dx}\left(3xy^2 - x^3\right)$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2y - y^3}{3xy^2 - x^3}}$$

4. (a) (12 pts.) An orchard contains 300 peach trees with each tree yielding 800 peaches. For every additional tree planted, the yield per tree decreases by 2 peaches. How many trees should be planted to maximize the total yield of the orchard?

Yield = (# trees) x (# peaches/tree). Let t = # additional trees:

trees = $(300+t)$

peaches/tree = $(800-2t)$.

So, maximize $Y(t)$

$= (300+t)(800-2t)$

$= 240000 + 800t - 600t - 2t^2$

$Y(t) = 240000 + 200t - 2t^2$

$Y'(t) = 200 - 4t = 0$

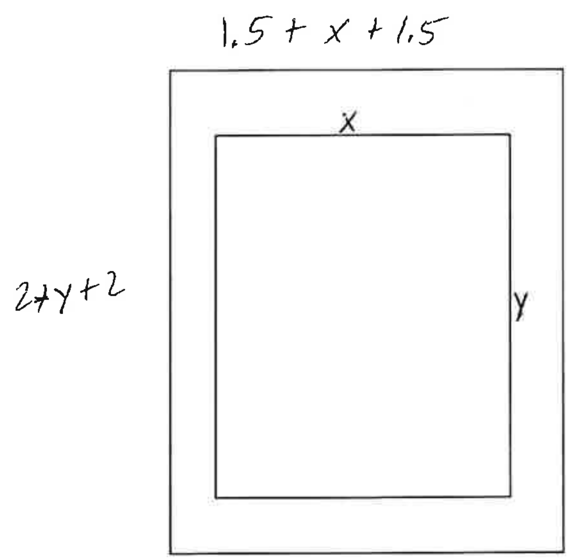
$\Rightarrow \boxed{t = 50}$

Also, since $Y''(t) = -4 < 0$,

$t=50$ gives a local max.

Conclude $t=50$ gives the max yield, so 50 additional trees should be planted for a total of 350 trees.

(b) (13 pts.) A poster is to have 2-inch margins at the top and bottom and 1½-inch margins on the sides. The total area is to be 300 square inches. Find the dimensions that will maximize the print area of the poster.



Objective: Maximize $A = xy$

subject to $(x+3)(y+4) = 300$

$\Rightarrow \boxed{x = \dfrac{300}{y+4} - 3}$

So, $A(y) = \left(\dfrac{300}{y+4} - 3\right) y = \dfrac{300y}{(y+4)} - 3y$, and

$A'(y) = \dfrac{(y+4)(300) - (300y)}{(y+4)^2} - 3 = 0$

$\Rightarrow \dfrac{1200}{(y+4)^2} = 3 \Rightarrow (y+4)^2 = 400$

$\Rightarrow y+4 = 20 \Rightarrow \boxed{y=16}$. (Note $y+4=-20$ gives a negative dimension, not reasonable)

Since $A''(y) = \dfrac{-2400}{(y+4)^3} < 0$ for $y=16$, we obtain a local max at $y=16$. Conclude $\boxed{y=16}$, $\boxed{x = \dfrac{300}{(16)+4} - 3}$ gives the optimal dimensions.

5. (20 pts.) $f(x) = x^4 + 4x^3 = x^3(x+4)$, $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$, and $f''(x) = 12x^2 + 24x = 12x(x+2)$.
Give a specific answer to each part:

(a) Construct sign charts for the first and second derivatives. (b) Find the critical numbers and the inflection points of f. (c) Find all open intervals of increase and decrease and open intervals on which the graph is concave up and concave down. (d) Classify each critical point as a relative maximum, relative minimum or neither. (e) Sketch the graph of y = f(x) by hand, plotting and labeling **only** the relative extreme points, inflection points and intercepts. **Use the factored form of f(x) to evaluate the functional values**.

(a) $f'(x)=0 \Rightarrow 4x^2(x+3)=0 \Rightarrow \boxed{x=0,\ x=-3}$. $f''(x)=0 \Rightarrow 12x(x+2)=0 \Rightarrow \boxed{x=0,\ x=-2}$

$f'(x)$	⊖	0	⊕	0	⊕
x	↑	-3	↑	0	↑
	-100		-1		1

(b) Critical #s

$f''(x)$	⊕	0	⊖	0	⊕
x	↑	-2	↑	0	↑
	-3		-1		1

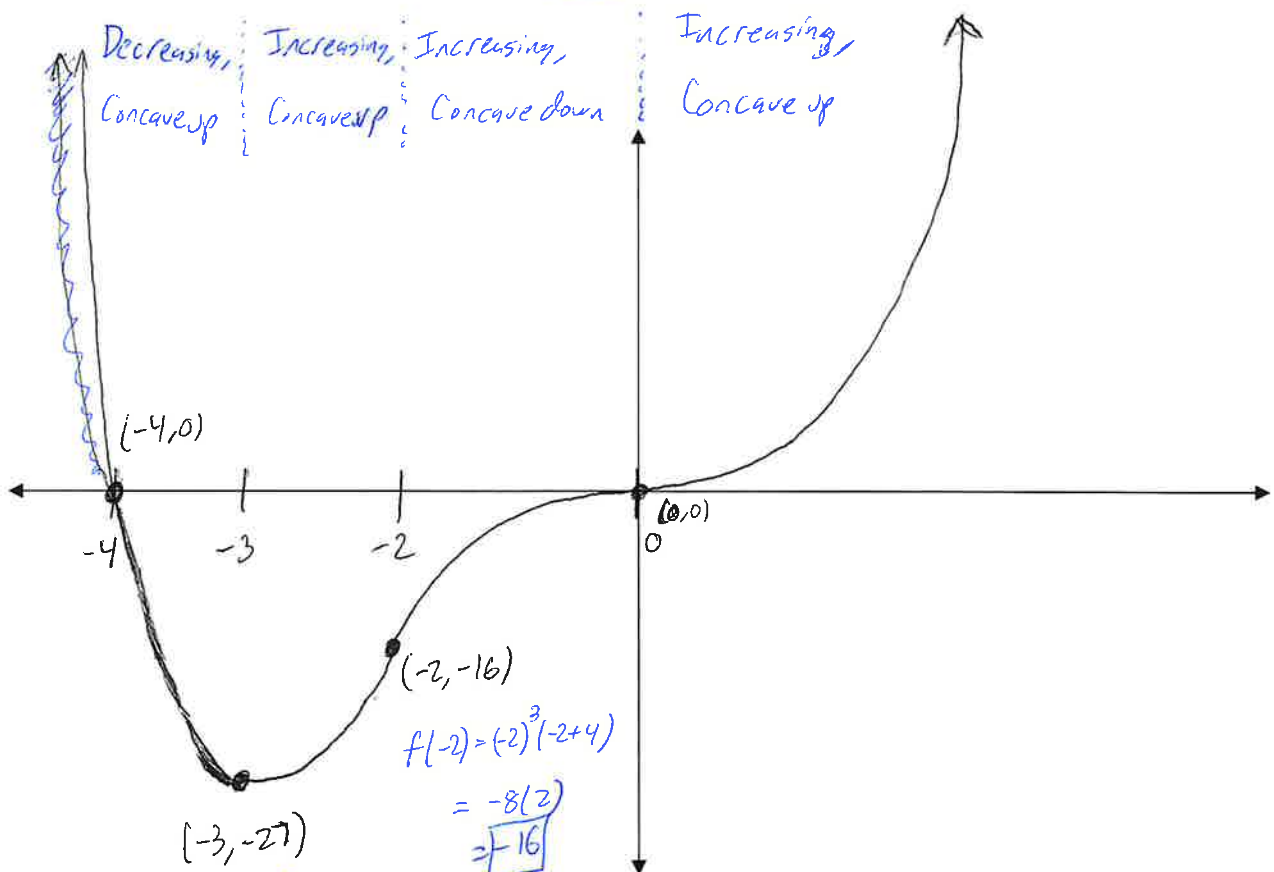
(b) ~~Critical #s~~ Inflection points

(c) f(x) is increasing on $(-3,0) \cup (0,\infty)$ and decreasing on $(-\infty,-3)$

f(x) is concave up on $(-\infty,-2) \cup (0,\infty)$ and concave down on $(-2,0)$

(d) $x=-3$ is a local min, $x=0$ is neither

(e) $f(x)=0$ when $x^3(x+4)=0 \Rightarrow \boxed{x=0,\ x=-4}$



$f(-3)=(-3)^3(-3+4)$
$= -27(1) = \boxed{-27}$

6 (a) (7 pts.) $f(x) = 3 + e^{3-x}$. Find an equation of the tangent line at x = 3.

Use point-slope form $y - y_1 = m(x - x_1)$. $x_1 = 3$ given, $y_1 = f(x_1) = 3 + e^{3-3} = 3 + e^0 = 4$, $m = f'(x_1)$. Note $f'(x) = e^{3-x} \cdot (-1)$, so $m = e^0 \cdot (-1) = -1$.

Conclude the line is $\boxed{y - 4 = -1(x-3)}$

(b) (6 pts.) When a cold roast is placed into a 375° F oven, its temperatuture T, in degrees Fahrenheit at time t hours is $T(t) = 375 - 325e^{-.2t}$. Find the instantaneous and relative rates of change of T at the time the roast is placed in the oven (t = 0). **Include proper units in your answers**.

Instantaneous rate of change: $T'(t) = -325e^{-0.2t}(-0.2)$ $\frac{\text{degrees F}}{\text{hour}}$.

Relative rate of change: $\frac{T'(t)}{T(t)} = \frac{-325e^{-0.2t}(-0.2)}{375 - 325e^{-0.2t}}$ $\frac{\left(\frac{\text{degrees F}}{\text{hour}}\right)}{\text{degrees F}}$

The units for R.R.O.C. are $\boxed{(\text{hour})^{-1}}$

7. (a) (6 pts.) State the Fundamental Theorem of Calculus.

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F(x) \text{ is an anti-derivative of } f(x).$$

(b) (7 pts.) Find f(x) given that $f'(x) = e + e^x + \frac{1}{x}$ and f(1) = 1 + 2e.

$f(x) = \int e + e^x + \frac{1}{x}\,dx = ex + e^x + \ln|x| + C$. Also, $1 + 2e = e(1) + e^{(1)} + \ln|1| + C$

$\Rightarrow 1 + 2e = 2e + 0 + C \Rightarrow \boxed{C = 1}$. Conclude $\boxed{f(x) = ex + e^x + \ln|x| + 1}$

(c) (14 pts.) Find the **area between** the curves $\mathbf{f(x) = x^2 - x}$ and $\mathbf{g(x) = x - x^2}$ on [-1,3].

** The interval [-1,3] captures ~~three~~ distinct regions, since $f(x) = g(x)$ when $x^2 - x = x - x^2 \Rightarrow 2x^2 - 2x = 0 \Rightarrow 2x(x-1) = 0 \Rightarrow \boxed{x=0}, \boxed{x=1}$. The top/bottom functions could change at each intersection. Here, $x^2 - x \geq x - x^2$ on $[-1,0]$ and $[1,3]$ (test $x = \frac{-1}{2}$ and $x = 2$) and $x^2 - x \leq x - x^2$ on $[0,1]$ (test $x = \frac{1}{2}$). So, the total area is $A = \int_{-1}^{0} (x^2 - x) - (x - x^2)\,dx + \int_0^1 (x - x^2) - (x^2 - x)\,dx + \int_1^3 (x^2 - x) - (x - x^2)\,dx$. Conclude since

$\int (x^2 - x) - (x - x^2)\,dx = \int 2x^2 - 2x\,dx = \frac{2}{3}x^3 - x^2 + C$ that $A = \left[\frac{2}{3}x^3 - x^2\right]_{-1}^{0} + \left[x^2 - \frac{2}{3}x^3\right]_0^1 + \left[\frac{2}{3}x^3 - x^2\right]_1^3$

$= \left(\frac{2}{3}(0)^3 - (0)^2\right) - \left(\frac{2}{3}(-1)^3 - (-1)^2\right) + \left((1)^2 - \frac{2}{3}(1)^3\right) - \left((0)^2 - \frac{2}{3}(0)^3\right) + \left(\frac{2}{3}(3)^3 - (3)^2\right) - \left(\frac{2}{3}(1)^3 - (1)^2\right) = \boxed{\frac{34}{3}}$

8. (36 pts.) Find the following integrals:

(a) $\int (\sqrt[3]{x^5} - e^{-2x} - \frac{3}{x^2} + 3)\, dx$ u-sub: $u = -2x$, $du = -2dx$

$= \int x^{5/3} dx - \boxed{\int e^{-2x} dx} - \int 3x^{-2} dx + \int 3\, dx$

$$\boxed{= \frac{x^{8/3}}{8/3} - \frac{e^{-2x}}{-2} - \frac{3x^{-1}}{-1} + 3x + C}$$

(b) $\int x \ln x\, dx$ Integration by Parts: $\int f \cdot dg = f \cdot g - \int g \cdot df$

$f = \ln(x)$ $dg = x\,dx$

$df = \frac{1}{x} dx$ $g = \frac{x^2}{2}$

$= \frac{x^2}{2} \ln(x) - \int \frac{1}{x} \cdot \frac{x^2}{2} dx = \frac{x^2}{2} \ln(x) - \int \frac{1}{2} x\, dx$

$$\boxed{= \frac{x^2}{2} \ln(x) - \frac{1}{4} x^2 + C}$$

(c) $\int \frac{e^{\frac{1}{x}}}{x^2} dx$ u-sub: $u = \frac{1}{x}$, $du = \frac{-1}{x^2} dx$

$= \int -e^u du = -e^u + C = \boxed{-e^{\frac{1}{x}} + C}$

(d) $\int \frac{\sqrt{x} + 3}{x} dx$

$= \int \frac{\sqrt{x}}{x} + \frac{3}{x}\, dx$

$= \int x^{-1/2} + 3 \cdot \frac{1}{x}\, dx$

$$= \boxed{\frac{x^{1/2}}{1/2} + 3 \ln|x| + C}$$

9. (13 pts.) Find all critical point(s) of **$f(x, y) = 2x^3 + 2y^3 - 12xy + 5$** and classify each as a relative maximum, relative minimum, or saddle point.

Set $f_x = 0$ and $f_y = 0$: Note $f_x = 6x^2 - 12y$ and $f_y = 6y^2 - 12x$.

From f_x, $6x^2 = 12y \Rightarrow y = \frac{1}{2}x^2$, and from f_y, $x = \frac{1}{2}y^2$.

So, $y = \frac{1}{2}\left(\frac{1}{2}y^2\right)^2 \Rightarrow y = \frac{1}{8}y^4 \Rightarrow \frac{1}{8}y^4 - y = 0 \Rightarrow y\left(\frac{1}{8}y^3 - 1\right) = 0$

$\Rightarrow \boxed{y = 0}$ or $\frac{1}{8}y^3 = 1 \Rightarrow y^3 = 8 \Rightarrow \boxed{y = 2}$.

When $y = 0$, $x = \frac{1}{2}(0)^2 = \boxed{0}$.

When $y = 2$, $x = \frac{1}{2}(2)^2 = \boxed{2}$.

So the critical points are $\boxed{(0,0) \text{ and } (2,2).}$

To classify each, note that $D = f_{xx}f_{yy} - (f_{xy})^2$.

Here, $f_{xx} = 12x$, $f_{yy} = 12y$, $f_{xy} = -12$. So, $D = 144xy - 144$.

At (0,0), $D = 144(0)(0) - 144 = -144 < 0$, so we get a saddle point.

At (2,2), $D = 144(2)(2) - 144 > 0$, and $12(2) > 0$, so we get a local min.

10. (15 pts.) Use the method of Lagrange multipliers to maximize and minimize **f(x, y) = 2x + y** subject to the constraint $\mathbf{x^2 + 2y^2 = 72}$. (Both extreme values exist).

Compute $\nabla f(x,y) = \lambda \cdot g(x,y)$. Write $g(x,y) = x^2 + 2y^2 - 72 = 0$,

$f_x = 2, f_y = 1$ and $g_x = 2x, g_y = 4y$.

Conclude $2 = \lambda \cdot 2x$ and $1 = \lambda \cdot 4y$.

Hence, $x = \frac{2}{2\lambda} = \frac{1}{\lambda}$ and $y = \frac{1}{4\lambda}$, so

$$\left(\frac{1}{\lambda}\right)^2 + 2\left(\frac{1}{4\lambda}\right)^2 = 72 \Rightarrow \frac{1}{\lambda^2} + \frac{1}{8\lambda^2} = 72 \Rightarrow \frac{8}{8\lambda^2} + \frac{1}{8\lambda^2} = 72$$

$$\Rightarrow \frac{9}{8\lambda^2} = 72 \Rightarrow \frac{9}{72} = 8\lambda^2 \Rightarrow \frac{1}{64} = \lambda^2 \Rightarrow \boxed{\lambda = \pm \frac{1}{8}}.$$

When $\lambda = \frac{1}{8}$, $x = 8$ and $y = 2$, giving $f(x,y) = 2(8) + (2) = \boxed{18}$.

When $\lambda = \frac{-1}{8}$, $x = -8$ and $y = -2$, giving $f(x,y) = 2(-8) + (-2) = \boxed{-18}$.

Conclude <u>(8,2) gives the absolute max</u> and <u>(-8,-2) gives the absolute min</u>.