1. (a) (3 pts.) Find an equation of the line that passes through the point (1, 2) and has slope m = -2.

Use point-slope form, $y - y_1 = m(x - x_1)$. Here, $m = -2$ and $(x_1, y_1) = (1, 2)$ are given.

$$\boxed{(y-2) = -2(x-1)}.$$

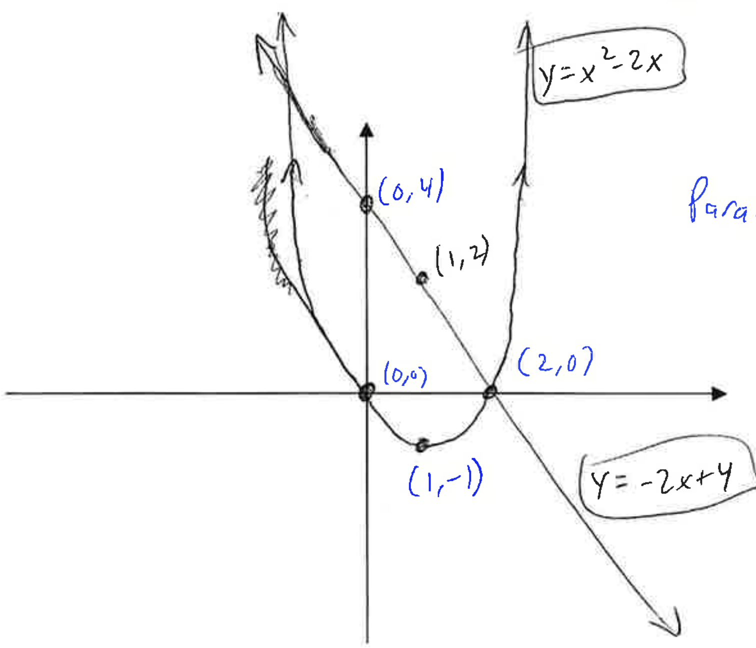
(b) (3 pts.) Find the vertex of the parabola given by y = f(x) = x^2 - 2x.

From algebra, the vertex is located at $x = \frac{-b}{2a}$ for $f(x) = ax^2 + bx + c$, so $x = \frac{-(-2)}{2(1)} = \boxed{1}$, $y = (1)^2 - 2(1) = \boxed{-1}$

From calculus, the vertex is located where $f'(x) = 0$, or $2x - 2 = 0 \Rightarrow \boxed{x = 1}$, $y = (1)^2 - 2(1) = \boxed{-1}$.

The vertex is located at $\boxed{(1, -1)}$.

(c) (4 pts.) Sketch the line and the parabola on the same set of axes, labeling the vertex of the parabola and the x- and y- intercepts of both.



Line: $y - 2 = -2(x-1)$

$y = -2x + 2 + 2$

$\boxed{y = -2x + 4}$

Parabola: $x^2 - 2x = 0$

$x(x-2) = 0$

$\boxed{x = 0}$, $\boxed{x = 2}$

2. (32 pts.) Find $f'(x)$. **You need not simplify.**

(a) $f(x) = e^{x^2} - \frac{1}{x} + e + \sqrt[4]{x^3}$ $= e^{(x^2)} - x^{-1} + e + x^{3/4}$

$f'(x) = e^{x^2} \cdot (2x) - (-1x^{-2}) + 0 + \frac{3}{4}x^{-1/4}$

(b) $f(x) = (\frac{1}{2x} + 3)^5$

$f'(x) = 5\left(\frac{1}{2x} + 3\right)^4 \cdot \left(\frac{1}{2}(-x^{-2})\right)$

(c) $f(x) = \frac{2x^3 - x}{1 - 2x}$

$f'(x) = \frac{[1-2x][6x^2-1] - [2x^3-x][-2]}{(1-2x)^2}$

(d) $f(x) = x\ln(1-x)$

$f'(x) = [x] \cdot \left[\frac{1}{1-x} \cdot (-1)\right] + [1] \cdot [\ln(1-x)]$

3. (a) (8 pts.) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$. Use this definition to find the derivative of $f(x) = \frac{1}{x^2}$.

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2 h} = \lim_{h\to 0} \frac{-2xh-h^2}{x^2(x+h)^2 h} = \lim_{h\to 0} \frac{h(-2x-h)}{h\left(x^2(x+h)^2\right)} = \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2}$$

$$= \frac{-2x}{4} = \boxed{\frac{-2}{x^3}}$$

(b) (8 pts.) A snowball is melting so that the radius is decreasing at a rate of 2 inches per hour. How fast is the volume decreasing at the moment when the radius is 10 inches? (The volume of a sphere in terms of the radius is $V = \frac{4}{3}\pi R^3$).

We are given $\frac{dR}{dt} = -2$. Since $V = \frac{4}{3}\pi R^3$, differentiate both sides with respect to time: $\frac{dV}{dt} = 4\pi R^2 \cdot \frac{dR}{dt}$. Then, since $R = 10$ is given,

conclude $\frac{dV}{dt} = 4\pi (10)^2 \cdot (-2) = \boxed{-800\pi \frac{in^3}{hour}}$

(c) (8 pts.) Find an equation of the tangent line to $f(x) = \ln(x^3)$ at $x = 1$.

Use point-slope form: $y - y_1 = m(x - x_1)$.

$x_1 = 1$ (given)

$y_1 = f(x_1) = \ln((1)^3) = \ln(1) = 0$

$m = f'(x_1)$, and $f'(x) = \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}$, so $f'(x_1) = \frac{3}{1} = 3$.

Conclude the tangent line is $\boxed{y - 0 = 3(x-1)}$

4. (12 pts.) City Computers Incorporated finds that it costs $800 to manufacture each PC, and fixed costs are $500 per day. The price function is p(x) = 1600 – 10x, where p(x) is the price (in dollars) at which exactly x PCs will be sold. Find the number of PCs that City Computers should produce per day and the price it should charge to maximize profit.

Profit = Revenue − Costs = Price × Quantity − Costs.

Price: $p(x) = 1600 - 10x$

Quantity: x

Costs: $800x + 500$

Objective: Maximize Profit

$P(x) = (1600-10x)x - (800x+500)$

$= 1600x - 10x^2 - 800x - 500$

$= -10x^2 + 800x - 500.$

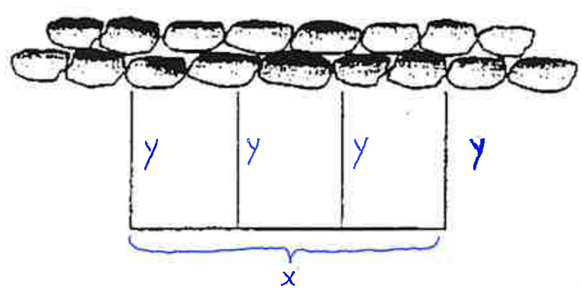
$P'(x) = -20x + 800 = 0$

$\Rightarrow \boxed{x = 40}$

Also, $P''(x) = -20 < 0$, so $x = 40$ indicates a local max.

Conclude the number of PCs to produce is 40, at a price of $p(40) = 1600 - 10(40) = \boxed{\$1200}$

5. (13 pts.) A homeowner wishes to enclose three adjacent rectangular pens of equal size along a straight wall, as in the diagram. If the side along the wall needs no fence, what is the largest total area that can be enclosed using 320 feet of fencing?



Max Area = xy subject to $P = 320 = x + 4y$.

Conclude $x = 320 - 4y$, so Area $A = (320-4y)y = 320y - 4y^2$.

$A'(y) = 320 - 8y = 0 \Rightarrow \boxed{y = 40}$. Thus, $x = 320 - 4(40) = \boxed{160}$, so

$A = 160(40) = \boxed{6400 \text{ ft}^2}$

6. (20 pts.) Given $f(x) = x^3 - 12x$, do the following:

(a) Make a sign diagram for the first derivative of f(x). $f'(x) = 3x^2 - 12 = 3(x^2-4) = 3(x-2)(x+2)$.

Conclude $f'(x) = 0$ when $\boxed{x=2}$ or $\boxed{x=-2}$:

$f'(x)$	$+$	0	$-$	0	$+$
x	-100	-2	0	2	100

$f'(0) = 3(0)^2 - 12 = -12$

$f'(100) = 3(100)^2 - 12 = 29988$

$f'(-100) = 3(-100)^2 - 12 = 29988$

(b) Make a sign diagram for the second derivative of f(x). $f''(x) = 6x$.

Conclude $f''(x) = 0$ when $\boxed{x=0}$:

$f''(x)$	$-$	0	$+$
x	-1	0	1

$f''(1) = 6$

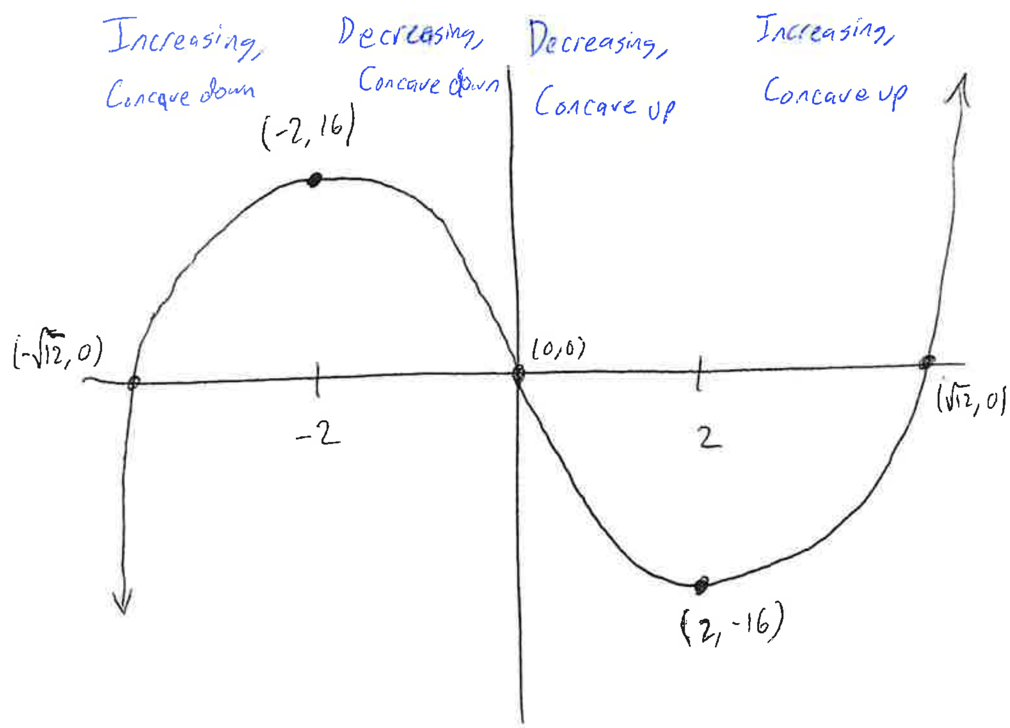
$f''(-1) = -6$

(c) State the open intervals on which f(x) is increasing, decreasing, concave up and concave down.

$f(x)$ is increasing on $(-\infty, 2) \cup (2, \infty)$ and decreasing on $(-2, 2)$.

$f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

(d) Sketch the graph of y =f(x) by hand, labeling all relative extreme points, inflection points and intercepts.



$f(0) = (0)^3 - 12(0) = 0$

$f(2) = (2)^3 - 12(2) = -16$

$f(-2) = (-2)^3 - 12(-2) = 16$

$f(x) = 0$ when

$x^3 - 12 = 0$

$\Rightarrow x(x^2 - 12) = 0$

$\boxed{x=0}$, $\boxed{x=\sqrt{12}}$, $\boxed{x=-\sqrt{12}}$.

7. (40 pts.) Find the following integrals:

(a) $\int (5x^3 + \pi^2 - \sqrt[3]{x^4})dx = \int 5x^3 + \pi^2 - x^{4/3}\, dx$

$= \boxed{\dfrac{5x^4}{4} + \pi^2 x - \dfrac{x^{7/3}}{7/3} + C}$

(b) $\int \frac{\sqrt{x}+1}{x}dx = \int \frac{\sqrt{x}}{x} + \frac{1}{x}\, dx = \int x^{-1/2} + \frac{1}{x}\, dx$

$= \boxed{\dfrac{x^{1/2}}{1/2} + \ln|x| + C}$

(c) $\int x\, e^{0.2x} dx$ Integration by Parts: $\int f\, dg = f \cdot g - \int g \cdot df$

$f = x \qquad dg = e^{0.2x} dx$

$df = dx \qquad g = 5e^{0.2x}$ (u-sub: $u = 0.2x$, $du = 0.2dx$)

$= 5xe^{0.2x} - \int 5e^{0.2x} dx = \boxed{5xe^{0.2x} - 25e^{0.2x} + C}$

(d) $\int (x^2+1)\sqrt{x^3+3x}\, dx$ u-sub

$u = x^3 + 3x$

$du = (3x^2+3)\, dx$

$\frac{du}{3} = (x^2+1)\, dx$

$= \int \frac{1}{3}\sqrt{u}\, du = \dfrac{\frac{1}{3} u^{3/2}}{3/2} + C = \boxed{\frac{2}{9}(x^3+3x)^{3/2} + C}$

8. (a) (12 pts.) Find the **area bounded by** the curves **$y = x^3$** and **$y = 4x$**.

$x^3=4x$ when $x^3-4x=0 \Rightarrow x(x-2)(x+2)=0 \Rightarrow \boxed{x=0}, \boxed{x=2}, \boxed{x=-2}$.

Therefore, these curves actually create two regions, and the question is poorly-phrased.

One way I would solve this is to add the areas of both regions, where the top/bottom functions could change between regions: On $[0,2]$, $4x > x^3$ (test $x=1$) and on $[-2,0]$, $4x < x^3$ (test $x=-1$). Therefore, get $A=\int_{-2}^{0} x^3-4x\,dx+\int_{0}^{2} 4x-x^3\,dx$.

Conclude $A=\left[\frac{x^4}{4}-2x^2\right]_{-2}^{0}+\left[2x^2-\frac{x^4}{4}\right]_{0}^{2} = \left(\frac{(0)^4}{4}-2(0)^2\right)-\left(\frac{(-2)^4}{4}-2(-2)^2\right)+\left(2(2)^2-\frac{(2)^4}{4}\right)-\left(2(0)^2-\frac{(0)^4}{4}\right)=\boxed{8}$.

(b) (10 pts.) Find the average value of $f(x) = e^x$ on $[0, \ln 8]$

The average value is $\int_{0}^{\ln(8)} e^x\,dx \cdot \frac{1}{\ln(8)-0} = \left[e^x\right]_{0}^{\ln(8)} \cdot \frac{1}{\ln(8)}$

$= \left[e^{\ln(8)}-(e^0)\right] \cdot \frac{1}{\ln(8)} = \frac{8-1}{\ln(8)} = \boxed{\frac{7}{\ln(8)}}$

9. (12 pts.) Find all critical points of $f(x.y) = 6xy - x^3 - 3y^2$ and classify each as a relative maximum, relative minimum, or saddle point.

Set $f_x=0$ and $f_y=0$: Note $f_x=6y-3x^2$ and $f_y=6x-6y$.

For f_y, conclude we must have $x=y$. Thus, $6x-3x^2=0 \Rightarrow 3x(2-x)=0$

$\Rightarrow \boxed{x=0}$ or $\boxed{x=2}$. The critical points are therefore $\boxed{(0,0)}$ and $\boxed{(2,2)}$.

To classify each, compute $D=f_{xx}f_{yy}-(f_{xy})^2$ for each critical point.

$f_{xx}=-6x, f_{yy}=-6, f_{xy}=6$. Get $D=36x-36$. At $(0,0)$, $D=-36$ (saddle point)

and at $(2,2)$, $D=36(2)-36=36$ with $f_{xx}=-6(2)=-12$. Hence,

$(2,2)$ is a relative maximum.

10. (15 pts.) Use the method of Lagrange multipliers to maximize **and** minimize f(x,y) = 2x + y subject to the constraint $x^2 + 2y^2 = 72$. (Both extreme values exist.)

Compute $\nabla f(x,y) = \lambda \cdot \nabla g(x,y)$. Write $g(x,y) = x^2 + 2y^2 - 72 = 0$.

$f_x = 2$, $f_y = 1$ and $g_x = 2x$, $g_y = 4y$.

Conclude $2 = \lambda \cdot 2x$ and $1 = \lambda \cdot 4y$.

Hence, $x = \frac{2}{2\lambda} = \frac{1}{\lambda}$ and $y = \frac{1}{4\lambda}$, so

$\left(\frac{1}{\lambda}\right)^2 + 2\left(\frac{1}{4\lambda}\right)^2 = 72 \Rightarrow \frac{1}{\lambda^2} + \frac{1}{8\lambda^2} = 72 \Rightarrow \frac{8}{8\lambda^2} + \frac{1}{8\lambda^2} = 72$

$\Rightarrow \frac{9}{8\lambda^2} = 72 \Rightarrow \frac{9}{72} = 8\lambda^2 \Rightarrow \frac{1}{64} = \lambda^2 \Rightarrow \lambda = \pm\frac{1}{8}$.

When $\lambda = \frac{1}{8}$, $x = 8$ and $y = 2$, giving $f(x,y) = 2(8) + (2) = \boxed{18}$.

When $\lambda = -\frac{1}{8}$, $x = -8$ and $y = -2$, giving $f(x,y) = 2(-8) + (-2) = \boxed{-18}$.

Conclude (8,2) gives the absolute max and (-8,-2) gives the absolute min.