1. (32 pts.) Find $f'(x)$. **You need not simplify**.

(a) $f(x) = e^{x^2} - \frac{1}{x} + e + \sqrt[4]{x^3}$ $= e^{(x^2)} - x^{-1} + e + x^{3/4}$

$f'(x) = e^{x^2} \cdot (2x) - (-1x^{-2}) + 0 + \frac{3}{4} x^{-1/4}$

(b) $f(x) = (\frac{1}{x} + 3)^5$

$f'(x) = 5\left(\frac{1}{x} + 3\right)^4 \cdot (-x^{-2})$

(c) $f(x) = \frac{2x^3 - x}{1-2x}$

$f'(x) = \frac{[1-2x][6x^2-1] - [2x^3-x][-2]}{(1-2x)^2}$

(d) $f(x) = x\ln(1-x)$

$f'(x) = \{x\} \cdot \left[\frac{1}{1-x} \cdot (-1)\right] + \{1\} \cdot [\ln(1-x)]$

2. (12 pts.) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Use this definition to find the derivative of $f(x) = \frac{1}{x^2}$.

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2 h} = \lim_{h\to 0} \frac{-2xh-h^2}{x^2(x+h)^2 h} = \lim_{h\to 0} \frac{h(-2x-h)}{h(x^2(x+h)^2)} = \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2}$$

$$= \frac{-2x}{x^4} = \boxed{\frac{-2}{x^3}}$$

3. (10 pts.) $x^2y + xy^2 = 0$. Find $\frac{dy}{dx}$ at x=1, y = −1.

Differentiate both sides w.r.t. x:

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0.$$

Plug in (1, −1) for (x, y):

$$2(1)(-1) + (1)^2\frac{dy}{dx} + (-1)^2 + 2(1)(-1)\frac{dy}{dx} = 0$$

$$-2 + \frac{dy}{dx} + 1 - 2\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$-1 - \frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -1}$$

4. a. (8 pts.) If the national debt of a certain country (in billions of dollars) t years from now is given by $N(t) = 4 + e^{0.01t}$, find the instantaneous rate of change and the relative rate of change at t = 0. Include units in your answers.

Instantaneous rate of change is $N'(t) = 0.01e^{0.01t}$ billions of dollars / year

Relative rate of change is $\frac{N'(t)}{N(t)} = \frac{0.01e^{0.01t}}{4+e^{0.01t}}$ $\frac{\text{billions of dollars / year}}{\text{billions of dollars}}$, so

the unit for R.R.O.C. is $\boxed{\text{years}^{-1}}$.

b. (10 pts.) Find an equation of the tangent line to $f(x) = x^2$ at x = 2.

We will use point-slope form: Note $f'(x) = 2x$.

Point: $x = 2$, $y = (2)^2 = 4$.

Slope: $f'(2) = 2(2) = 4$.

So the equation is $\boxed{y - 4 = 4(x-2)}$.

5. (15 pts.) A cherry tree will yield 100 pounds of cherries now, which will sell for 80 cents a pound. Each week that the farmer waits will increase the yield by 10 pounds, but the selling price will decrease by 4 cents per pound. How long should the farmer wait to pick the fruit in order to maximize the revenue?

Revenue R = Price x Quantity. Let w be # of weeks waited.

Then Price = $80 - 4w$ and Quantity = $100 + 10w$.

So our objective is to maximize $R(w) = (80-4w)(100+10w) = 8000 + 400w - 40w^2$.

$R'(w) = 400 - 80w$, so $R'(w) = 0$ when $400 = 80w \Rightarrow \boxed{w = 5}$.

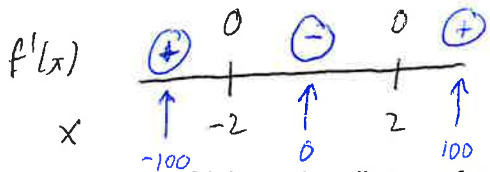
Also, since $R''(w) = -80$, $R''(5) = -80 < 0$ indicates a local max.

Hence, the farmer should wait 5 weeks.

6. (20 pts.) Given $f(x) = x^3 - 12x$, do the following:

(a) Make a sign diagram for the first derivative of f(x). $f'(x) = 3x^2 - 12 = 3(x^2-4) = 3(x-2)(x+2)$.

Conclude $f'(x) = 0$ when $\boxed{x=2}$ or $\boxed{x=-2}$:



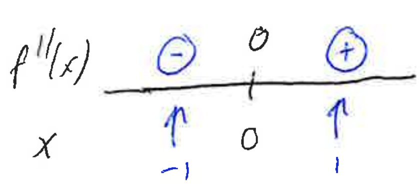
$f'(0) = 3(0)^2 - 12 = -12$

$f'(100) = 3(100)^2 - 12 = 29988$

$f'(-100) = 3(-100)^2 - 12 = 29988$

(b) Make a sign diagram for the second derivative of f(x). $f''(x) = 6x$.

Conclude $f''(x) = 0$ when $\boxed{x=0}$.



$f''(1) = 6(1) = 6$

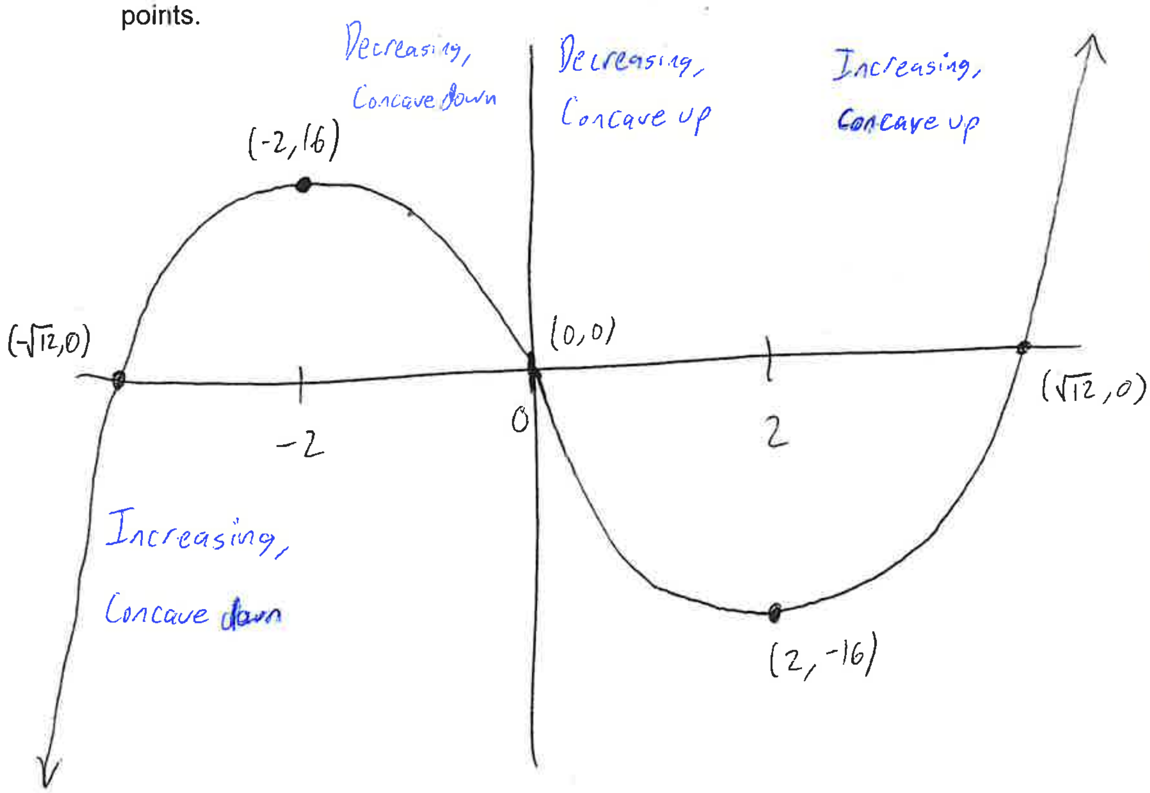
$f''(-1) = 6(-1) = -6$

(c) State the open intervals on which f(x) is increasing, decreasing, concave up and concave down.

f(x) is increasing on $(-\infty, -2) \cup (2, \infty)$ and decreasing on $(-2, 2)$.

f(x) is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

(d) Sketch the graph of y =f(x) by hand, labeling all relative extreme points and inflection points.



$f(0) = (0)^3 - 12(0) = 0$

$f(2) = (2)^3 - 12(2) = -16$

$f(-2) = (-2)^3 - 12(-2) = 16$

$f(x) = 0$ when

$x^3 - 12x = 0$

$x(x^2 - 12) = 0$

$\boxed{x=0}$, $\boxed{x=\sqrt{12}}$, $\boxed{x=-\sqrt{12}}$

7. (40 pts.) Find the following integrals:

(a) $\int (5x^3 + \pi - \sqrt[3]{x^4}\,dx = \int 5x^3 + \pi + x^{4/3}\,dx$

$$= \boxed{\frac{5x^4}{4} + \pi x + \frac{x^{7/3}}{7/3} + C}$$

(b) $\int \frac{\sqrt{x}+1}{x}\,dx = \int \frac{x^{1/2}}{x} + \frac{1}{x}\,dx$

$$= \int x^{-1/2} + \frac{1}{x}\,dx$$

$$= \boxed{\frac{x^{1/2}}{1/2} + \ln|x| + C}$$

(c) $\int x\, e^{2x}\,dx$ Integration by Parts: $\int f\,dg = f \cdot g - \int g \cdot df$

$f = x \qquad dg = e^{2x}dx$

$df = dx \qquad g = \frac{1}{2}e^{2x} \quad (u\text{-sub}: u = 2x, du = 2dx)$

$$= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx = \boxed{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

(d) $\int (x^2 + 1)\sqrt{x^3 + 3x}\,dx$ U-Sub

$u = x^3 + 3x$

$du = (3x^2 + 3)\,dx$

$\frac{du}{3} = (x^2 + 1)\,dx$

$$= \int \frac{1}{3}\sqrt{u}\,du = \int \frac{1}{3}u^{1/2}\,du = \frac{1}{3}\frac{u^{3/2}}{3/2} = \boxed{\frac{2(x^3+3x)^{3/2}}{9} + C}$$

8. (a) (10 pts.) Find the **area bounded by** the curves **y = x³** and **y = 4x**.

$x^3 = 4x$ when $x^3 - 4x = 0 \Rightarrow x(x-2)(x+2) = 0 \Rightarrow \boxed{x=0}, \boxed{x=2}, \boxed{x=-2}$.

** Therefore, these curves actually create two regions, and the question is poorly-phrased. **

One way I would solve this is to add the areas of both regions, where the top/bottom functions could change between regions: on $[0,2]$, $4x > x^3$ (test $x=1$) and on $[-2,0]$, $4x < x^3$ (test $x=-1$). Therefore, get $A = \int_{-2}^{0} x^3 - 4x \, dx + \int_{0}^{2} 4x - x^3 \, dx$.

Conclude $A = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} + \left[2x^2 - \frac{x^4}{4}\right]_{0}^{2} = \left(\frac{(0)^4}{4} - 2(0)^2\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right) + \left(2(2)^2 - \frac{(2)^4}{4}\right) - \left(2(0)^2 - \frac{(0)^4}{4}\right) = \boxed{8}$.

(b) (10 pts.) Find the **producers' surplus** for the supply function s(x) = 0.04 x at the demand level x = 100.

$PS = \int_0^{x^*} p^* - s(x) \, dx$. We have $x^* = 100$ given, so $p^* = s(x^*) = 0.04(100) = 4$.

Conclude $PS = \int_0^{100} 4 - 0.04x \, dx = \left[4x - 0.02x^2\right]_0^{100} = \left(4(100) - 0.02(100)^2\right) - \left(4(0) - 0.02(0)^2\right)$

$= 400 - 200 - 0 = \boxed{200}$.

9. (15 pts.) Find all critical points of $f(x.y) = 6xy - x^3 - 3y^2$ and classify each as a relative maximum, relative minimum, or saddle point.

Set $f_x = 0$ and $f_y = 0$: Note $f_x = 6y - 3x^2$ and $f_y = 6x - 6y$.

From f_y, conclude we must have $x = y$. Thus, $6x - 3x^2 = 0 \Rightarrow 3x(2-x) = 0$

$\Rightarrow \boxed{x=0}$ or $\boxed{x=2}$. The critical points are therefore $\boxed{(0,0)}$ and $\boxed{(2,2)}$.

To classify each, compute $D = f_{xx} f_{yy} - (f_{xy})^2$ for each critical point.

$f_{xx} = -6x$, $f_{yy} = -6$, $f_{xy} = 6$. Get $D = 36x - 36$. At $(0,0)$, $D = -36$ (saddle point)

and at $(2,2)$, $D = 36(2) - 36 = 36$ with $f_{xx} = -6(2) = -12$. Hence, $(2,2)$ is a relative maximum.

10. (18 pts.) Use the method of Lagrange multipliers to maximize **and** minimize f(x,y) = 2x + y subject to the constraint $x^2 + 2y^2 = 72$. (Both extreme values exist.)

Compute $\nabla f(x,y) = \lambda \cdot \nabla g(x,y)$. Write $g(x,y) = x^2 + 2y^2 - 72 = 0$.

$f_x = 2$ and $f_y = 1$, and $g_x = 2x$ and $g_y = 4y$.

Conclude $2 = \lambda \cdot 2x$ and $1 = \lambda \cdot 4y$.

Hence, $x = \frac{2}{2\lambda} = \frac{1}{\lambda}$ and $y = \frac{1}{4\lambda}$, so

$$\left(\frac{1}{\lambda}\right)^2 + 2\left(\frac{1}{4\lambda}\right)^2 = 72 \Rightarrow \frac{1}{\lambda^2} + \frac{1}{8\lambda^2} = 72 \Rightarrow \frac{8}{8\lambda^2} + \frac{1}{8\lambda^2} = 72$$

$$\Rightarrow \frac{9}{8\lambda^2} = 72 \Rightarrow \frac{9}{72} = 8\lambda^2 \Rightarrow \frac{1}{64} = \lambda^2 \Rightarrow \boxed{\lambda = \pm \frac{1}{8}}.$$

When $\lambda = \frac{1}{8}$, $x = 8$ and $y = 2$, giving $f(x,y) = 2(8) + (2) = \boxed{18}$

When $\lambda = \frac{-1}{8}$, $x = -8$ and $y = -2$, giving $f(x,y) = 2(-8) + (-2) = \boxed{-18}$.

Conclude <u>(8,2) is the absolute max</u> and <u>(-8,-2) is the absolute min.</u>