A HIGHER ORDER ENSEMBLE SIMULATION ALGORITHM FOR FLUID FLOWS

NAN JIANG

ABSTRACT. This report presents an efficient, higher order method for fast calculation of an ensemble of solutions of the Navier-Stokes equations (NSE). We give a complete stability and convergence analysis of the method. For high Reynolds number flows, we propose and analyze an eddy viscosity model with a recent reparametrization of the mixing length. This turbulence model depends on a new definition of the ensemble mean, presented here for the first time, compatible with the higher order method. We show the turbulence model has superior stability, also demonstrated in numerical tests. We also give tests showing the potential of the new method for exploring flow problems to compute turbulence intensities, effective Lyapunov exponents, windows of predictability, and to verify the selective decay principle.

1. Introduction

Ensemble calculation is essential in uncertainty quantification, numerical weather prediction, sensitivity analysis, predicting probability distributions for quantities of interest and many other applications in Computational Fluid Dynamics (CFD), see for instance, [7], [14], [23], [25], [21], [26], [30]. Despite the fact that a lot of effort has been made to make reliable predictions with only a finite/small size ensemble, such as the bred vectors algorithm, [30], little progress has been made in developing efficient algorithms to compute the flow ensemble. A recent, first order accurate, ensemble algorithm was proposed in [19]. The algorithm results in J linear systems with the same coefficient matrix instead of J linear systems with J different coefficient matrices at each time step, which allows the use of block iterative methods, e.g., [29], [11], [12], [10], to reduce the computing time and required memory substantially. While efficient, the method of [19] is only first order accurate. In applications such as the climate and ocean forecasts, which involve both turbulent flows and long time integration, higher order methods incorporating turbulence models are indispensable.

In this paper, we extend the method in [19] to an efficient, *higher order*, ensemble time discretization and extend an ensemble eddy viscosity model of [20] to the higher order method. The new higher order method preserves the good algorithmic property of the method of [19], while being second order convergent. This advantage makes it a promising tool to increase ensemble size and improve data

---

Date: January 16, 2014.

2000 *Mathematics Subject Classification.* Primary 65M12; Secondary 65J08 .

Key words and phrases. Navier-Stokes Equations, Ensemble Calculation, Uncertainty Quantification, BDF2AB2, Eddy Viscosity Model.

The research of the author described herein was partially supported by NSF grant DMS 1216465 and Air Force grant FA 9550-12-1-0191.

quality, complementing the current ensemble techniques. The time stepping method we used here is a combination of a second order in time Backward-differentiation (BDF) and a special explicit Adams-Bashforth (AB) treatment of the advection term. The base ensemble algorithm naturally induces a new definition of the ensemble mean, compatible to the higher order time discretization, see Section 1.1 below.

We give comprehensive stability analysis and error analysis of the higher order method in Section 3, 4, respectively. This method, without any parametrizations of turbulence, requires a timestep restriction, (3.2) below, for stability. This condition degrades as the Reynolds number increases. For high Reynolds number flows, we analyze an ensemble eddy viscosity turbulence model based on the higher order ensemble method. Computing ensembles allows direct parameterization of the turbulence characteristic velocity $|u'|$ and the mixing length (recently redefined in [20]) $|u'|\Delta t$. The stability condition derived from the eddy viscosity model, (4.2) below, is far less restrictive than (3.2) and in our tests large/moderate timesteps are enough to give good stability results. In several important cases it implies unconditional stability.

Three numerical tests are presented in Section 5. Convergence of each ensemble member is verified and the convergence rate is calculated. The second numerical experiment tests the timestep conditions for stability and gives some insight into the usefulness of the proposed methods. Several important quantities in turbulence simulations, such as turbulence intensity (e.g., [33]), effective Lyapunov exponent (introduced in [3]) and Dirichlet quotient (e.g., [27]), are computed with our method. Lastly, for 3D Ethier-Steinman flow, the superiority of the turbulence model in stability is shown for high Reynolds number flows.

1.1. **Methods and Models.** In this paper, we consider a second order accurate method for computing an ensemble of J Navier-Stokes equations, $j = 1, ..., J$:

$$\begin{aligned} u_{j,t} + u_j \cdot \nabla u_j - \nu \triangle u_j + \nabla p_j &= f_j, \text{ in } \Omega, \\ \nabla \cdot u_j &= 0, \text{ in } \Omega, \\ u_j &= 0, \text{ on } \partial\Omega, \\ u_j(x,0) &= u_j^0(x), \text{ in } \Omega, \end{aligned} \tag{1.1}$$

The first important subtlety is that a new (but consistent) definition of the mean, (1.2) below, is needed to match the numerical method.

Definition 1. *Let $t^n = n\Delta t, n = 0, 1, 2, ..., N_T,$ and $T := N_T \Delta t$. Denote $u_j^n = u_j(t^n)$, $j = 1, ..., J$. We define the* **ensemble mean** *and* **fluctuation about the mean** *as follows.*

$$< u >^n := \frac{1}{J} \sum_{j=1}^{J} (2u_j^n - u_j^{n-1}), \tag{1.2}$$

$$u_j'^n := 2u_j^n - u_j^{n-1} - < u >^n . \tag{1.3}$$

Lemma 1. *The ensemble mean and fluctuation have the following properties.*

$$< u' >= 0, \qquad << u >>=< u >,$$
$$<< u > \cdot v >=< u > \cdot < v >, \qquad << u > \cdot v' >= 0.$$

In particular, if $u_j \equiv a$, then $< u >= a$.

Suppressing the spacial discretization until Section 3 for clarity, for laminar flows/small Re, the method is, for $j = 1, ..., J$, given u_j^0 and u_j^1,

$$\text{(EnB)} \qquad \begin{aligned} &\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} + < u >^n \cdot \nabla u_j^{n+1} \\ &\quad + u_j'^n \cdot \nabla \left(2u_j^n - u_j^{n-1}\right) + \nabla p_j^{n+1} - \nu \Delta u_j^{n+1} = f_j^{n+1}, \\ &\nabla \cdot u_j^{n+1} = 0. \end{aligned}$$

This method is second order accurate, Section 5.

For high Reynolds number flows, we incorporate an eddy viscosity model to the method. Following Prandtl's assumption that the eddy viscosity is proportional to the mixing length multiplied by a turbulence characteristic velocity, the eddy viscosity parameterization has the form

$$\nu_T = C_{\nu_T} \left(l \cdot |u'|\right).$$

Definition 2. *Let $|\cdot|$ denote the usual Euclidean length of a vector and the Frobenius norm of an array. Then the* **magnitude of fluctuation** *(the characteristic velocity) is defined to be*

$$|u'^n| := \left(\sum_{j=1}^{J} |u_j'^n|^2\right)^{1/2}.$$

The **mixing length** *(from* [20]*) is defined to be the distance that a fluctuating eddy travels in one timestep*

$$l^n = |u'^n| \Delta t.$$

Thus the eddy viscosity parameterization is

$$\nu_T = C_{\nu_T} |u'^n|^2 \Delta t,$$

The ensemble eddy viscosity model is, for $j = 1, ..., J$, given u_j^0 and u_j^1,

$$\text{(EVB)} \qquad \begin{aligned} &\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} + < u >^n \cdot \nabla u_j^{n+1} + u_j'^n \cdot \nabla \left(2u_j^n - u_j^{n-1}\right) \\ &\quad + \nabla p_j^{n+1} - \nu \Delta u_j^{n+1} - \nabla \cdot \left(2\nu_T \nabla^s u_j^{n+1}\right) = f_j^{n+1}, \\ &\nabla \cdot u_j^{n+1} = 0. \end{aligned}$$

Remark 1. *The time discretization was motivated by the scheme studied by Wang,* [32]*, for 2d Navier-Stokes problem in the vorticity-streamfunction formulation. The special explicit Adams-Bashforth treatment of the nonlinear term is different from the classical Adams-Bashforth treatment, i.e., $u_j'^n \cdot \nabla(2u_j^n - u_j^{n-1})$ herein vs. $u_j^{n+1} \cdot \nabla(2u_j^n - u_j^{n-1})$ and $2u_j^n \cdot \nabla u_j^n - u_j^{n-1} \cdot \nabla u_j^{n-1}$, see* [1], [22], [31].

2. Notation and preliminaries

Let Ω be an open, regular domain in $\mathbb{R}^d$ ($d = 2$ or 3). The $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$ respectively. For $p = 2$, the $L^2(\Omega)$ norm and the inner product are denoted by $\|\cdot\|$ and $(\cdot, \cdot)$. $H^k(\Omega)$ is used to denote the Sobolev space $W_2^k(\Omega)$, with norm $\|\cdot\|_k$. For functions $v(x, t)$ defined on $\Omega \times (0, T)$, define $(1 \le m < \infty)$

$$\|v\|_{\infty,k} := EssSup_{[0,T]}\|v(\cdot,t)\|_k \quad \text{and} \quad \|v\|_{m,k} := \left(\int_0^T \|v(\cdot,t)\|_k^m dt\right)^{1/m}.$$

We also introduce the following discrete norms:

$$|||v|||_{\infty,k} := \max_{0\le n\le N_T} \|v^n\|_k \quad \text{and} \quad |||v|||_{m,k} := \left(\sum_{n=0}^{N_T} \|v^n\|_k^m \Delta t\right)^{1/m}.$$

Let X, Q denote the velocity, pressure spaces:

$$X := H_0^1(\Omega)^d = \left\{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{d\times d} \text{ and } v = 0 \text{ on } \partial\Omega\right\},$$
$$Q := L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q\ dx = 0\right\}.$$

The dual space of X has the usual norm

$$\|f\|_{-1} = \sup_{0\neq v\in X} \frac{(f,v)}{\|\nabla v\|}.$$

A weak formulation of (1.1) is: find $u_j : [0,T] \to X$, $p_j : [0,T] \to Q$ satisfying, for $j = 1, ..., J$:

$$(u_{j,t}, v) + (u_j \cdot \nabla u_j, v) + \nu(\nabla u_j, \nabla v) - (p_j, \nabla\cdot v) = (f_j, v),\ \forall v \in X$$
$$u_j(x,0) = u_j^0(x) \text{ in } X \text{ and } (\nabla\cdot u_j, q) = 0,\ \forall q \in Q.$$

Define the usual skew symmetric trilinear form $b^*(u,v,w) := \frac{1}{2}(u\cdot\nabla v, w) - \frac{1}{2}(u\cdot\nabla w, v)$. By the divergence theorem

$$b^*(u,v,w) = \int_\Omega u\cdot\nabla v\cdot w\ dx + \frac{1}{2}\int_\Omega (\nabla\cdot u)(v\cdot w)\ dx. \tag{2.1}$$

In both 3d and 2d, $b^*(u,v,w)$ satisfies

$$|b^*(u,v,w)| \le C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\|, \tag{2.2}$$

and two sharper bounds (improvable in 2d)

$$|b^*(u,v,w)| \le C(\Omega)\|\nabla u\|^{1/2}\|u\|^{1/2}\|\nabla v\|\|\nabla w\|, \tag{2.3}$$

$$|b^*(u,v,w)| \le C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\|^{1/2}\|w\|^{1/2}. \tag{2.4}$$

Conforming velocity, pressure finite element spaces based on an edge to edge triangulation (if $d = 2$) or tetrahedralization (if $d = 3$) of Ω with maximum element diameter h are denoted by

$$X_h \subset X,\ Q_h \subset Q.$$

We assume the finite element spaces (X_h, Q_h) satisfy the usual discrete inf-sup $/LBB^h$ condition for stability of the discrete pressure, see [15]. Taylor-Hood elements, e.g., [5], [15], are one such choice used in the tests in Section 6. The discretely divergence free subspace of X_h is

$$V_h := \{v_h \in X_h : (\nabla\cdot v_h, q_h) = 0,\ \forall q_h \in Q_h\}.$$

We assume that the finite element spaces satisfy the inverse inequality (typical for quasi-uniform meshes, e.g., [5]), for all $v_h \in X_h$,

$$h\|\nabla v_h\| \le C\|v_h\|. \tag{2.5}$$

3. Stability of the method without eddy viscosity

For laminar flows, we prove (EnB) is stable under a timestep restriction, (3.2) below, relating the timestep to the size of the fluctuations about the mean. The fully discrete method is: given $u_{j,h}^{n-1}, u_{j,h}^{n}$, find $u_{j,h}^{n+1} \in X_h$, $p_{j,h}^{n+1} \in Q_h$ satisfying

$$
\begin{aligned}
(3.1)\qquad & \left(\frac{3u_{j,h}^{n+1} - 4u_{j,h}^{n} + u_{j,h}^{n-1}}{2\Delta t}, v_h\right) + b^*\left(< u_h >^n, u_{j,h}^{n+1}, v_h\right) \\
& + b^*\left(u_{j,h}^{\prime n}, 2u_{j,h}^{n} - u_{j,h}^{n-1}, v_h\right) - \left(p_{j,h}^{n+1}, \nabla \cdot v_h\right) \\
& + \nu\left(\nabla u_{j,h}^{n+1}, \nabla v_h\right) = \left(f_j^{n+1}, v_h\right), \qquad \forall v_h \in X_h, \\
& \left(\nabla \cdot u_{j,h}^{n+1}, q_h\right) = 0, \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Timestep condition of (EnB). With a standard spacial discretization with mesh size h, in both 2d and 3d (EnB) is stable under the CFL type condition:

$$
(3.2)\qquad C\frac{\Delta t}{\nu h}\|\nabla u_{j,h}^{\prime n}\|^2 \le 1, \qquad j = 1, ..., J.
$$

This is improvable in 2d following estimates in [19]. Note that the condition is explicit (i.e., the required information is available at t^n to determine Δt to compute $u_{j,h}^{n+1}$ stably) and depends on the size of the fluctuation $u_{j,h}^{\prime n}$. The constant C is independent of the timestep Δt but depends on the domain and minimum angle of the mesh. Pre-computations were used to determine C for different domains and meshes in our tests.

Theorem 1 (Stability of (EnB))**.** *Consider the method (3.1). Suppose the condition (3.2) holds. Then, for any $N > 1$*

$$
\begin{aligned}
(3.3)\qquad & \frac{1}{4}\|u_{j,h}^{N}\|^2 + \frac{1}{4}\|2u_{j,h}^{N} - u_{j,h}^{N-1}\|^2 + \frac{1}{8}\sum_{n=1}^{N-1}\|u_{j,h}^{n+1} - 2u_{j,h}^{n} + u_{j,h}^{n-1}\|^2 \\
& + \frac{\Delta t}{4}\sum_{n=1}^{N-1}\nu\|\nabla u_{j,h}^{n+1}\|^2 \le \sum_{n=1}^{N-1}\frac{\Delta t}{\nu}\|f_j^{n+1}\|_{-1}^2 + \frac{1}{4}\|u_{j,h}^{1}\|^2 + \frac{1}{4}\|2u_{j,h}^{1} - u_{j,h}^{0}\|^2 .
\end{aligned}
$$

Proof. Set $v_h = u_{j,h}^{n+1}$ in (3.1) and multiply through by Δt. This gives

$$
\begin{aligned}
(3.4)\qquad & \frac{1}{4}\left(\|u_{j,h}^{n+1}\|^2 + \|2u_{j,h}^{n+1} - u_{j,h}^{n}\|^2\right) - \frac{1}{4}\left(\|u_{j,h}^{n}\|^2 + \|2u_{j,h}^{n} - u_{j,h}^{n-1}\|^2\right) \\
& + \frac{1}{4}\|u_{j,h}^{n+1} - 2u_{j,h}^{n} + u_{j,h}^{n-1}\|^2 + \nu\Delta t\|\nabla u_{j,h}^{n+1}\|^2 \\
& + \Delta t b^*\left(u_{j,h}^{\prime n}, 2u_{j,h}^{n} - u_{j,h}^{n-1}, u_{j,h}^{n+1}\right) = \Delta t\left(f_j^{n+1}, u_{j,h}^{n+1}\right) .
\end{aligned}
$$

Applying Young's inequality to the right hand side gives

$$\frac{1}{4}\left(\|u_{j,h}^{n+1}\|^2+\|2u_{j,h}^{n+1}-u_{j,h}^{n}\|^2\right)-\frac{1}{4}\left(\|u_{j,h}^{n}\|^2+\|2u_{j,h}^{n}-u_{j,h}^{n-1}\|^2\right)
+\frac{1}{4}\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|^2+\nu\Delta t\|\nabla u_{j,h}^{n+1}\|^2
+\Delta t b^*\left(u_{j,h}'^{n},2u_{j,h}^{n}-u_{j,h}^{n-1},u_{j,h}^{n+1}\right)
\leq\frac{\nu\Delta t}{4}\|\nabla u_{j,h}^{n+1}\|^2+\frac{\Delta t}{\nu}\|f_j^{n+1}\|_{-1}^2\ . \tag{3.5}$$

Next, we bound the trilinear term using (2.4) and the inverse inequality (2.5).

$$\begin{aligned}
&\Delta t b^*\left(u_{j,h}'^{n},2u_{j,h}^{n}-u_{j,h}^{n-1},u_{j,h}^{n+1}\right)\\
&=\Delta t b^*\left(u_{j,h}'^{n},-u_{j,h}^{n+1}+2u_{j,h}^{n}-u_{j,h}^{n-1},u_{j,h}^{n+1}\right)\\
&=\Delta t b^*\left(u_{j,h}'^{n},u_{j,h}^{n+1},u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\right)\\
&\leq C\Delta t\|\nabla u_{j,h}'^{n}\|\|\nabla u_{j,h}^{n+1}\|\|\nabla(u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1})\|^{1/2}\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|^{1/2}\\
&\leq C\Delta t h^{-\frac{1}{2}}\|\nabla u_{j,h}'^{n}\|\|\nabla u_{j,h}^{n+1}\|\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|\ .
\end{aligned} \tag{3.6}$$

Using Young's inequality again yields

$$\begin{aligned}
&\Delta t b^*\left(u_{j,h}'^{n},2u_{j,h}^{n}-u_{j,h}^{n-1},u_{j,h}^{n+1}\right)\\
&\leq C\frac{\Delta t^2}{h}\|\nabla u_{j,h}'^{n}\|^2\|\nabla u_{j,h}^{n+1}\|^2+\frac{1}{8}\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|^2\ .
\end{aligned} \tag{3.7}$$

With this bound, combining like terms, (3.5) becomes

$$\frac{1}{4}\left(\|u_{j,h}^{n+1}\|^2+\|2u_{j,h}^{n+1}-u_{j,h}^{n}\|^2\right)-\frac{1}{4}\left(\|u_{j,h}^{n}\|^2+\|2u_{j,h}^{n}-u_{j,h}^{n-1}\|^2\right)
+\frac{\nu\Delta t}{4}\|\nabla u_{j,h}^{n+1}\|^2+\frac{\nu\Delta t}{2}\left(1-C\frac{\Delta t}{\nu h}\|\nabla u_{j,h}'^{n}\|^2\right)\|\nabla u_{j,h}^{n+1}\|^2
+\frac{1}{8}\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|^2\leq\frac{\Delta t}{\nu}\|f_j^{n+1}\|_{-1}^2\ . \tag{3.8}$$

With the restriction (3.2) assumed, we have

$$\frac{\nu\Delta t}{2}\left(1-C\frac{\Delta t}{\nu h}\|\nabla u_{j,h}'^{n}\|^2\right)\|\nabla u_{j,h}^{n+1}\|^2\geq 0\ .$$

Equation (3.8) reduces to

$$\frac{1}{4}\left(\|u_{j,h}^{n+1}\|^2+\|2u_{j,h}^{n+1}-u_{j,h}^{n}\|^2\right)-\frac{1}{4}\left(\|u_{j,h}^{n}\|^2+\|2u_{j,h}^{n}-u_{j,h}^{n-1}\|^2\right)
+\frac{\nu\Delta t}{4}\|\nabla u_{j,h}^{n+1}\|^2+\frac{1}{8}\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|^2\leq\frac{\Delta t}{\nu}\|f_j^{n+1}\|_{-1}^2\ . \tag{3.9}$$

Summing up (3.9) from $n = 1$ to $n = N - 1$ results in (3.3). This concludes the proof of Theorem 1. ∎

The necessity of a timestep condition of the form (3.2) is shown in Section 6.2. More extensive experiments for a similar lower order accurate method in [19] were consistent with the conclusion that some timestep condition is needed.

4. Stability of the method with eddy viscosity

In this section, we analyze the method (EVB) including the effect of the eddy viscosity term. The approximation we study of NSE is: given $u_{j,h}^{n-1}, u_{j,h}^{n}$, find $u_{j,h}^{n+1} \in X_h$, $p_{j,h}^{n+1} \in Q_h$ satisfying

$$\begin{aligned}(4.1)\qquad &\left(\frac{3u_{j,h}^{n+1} - 4u_{j,h}^{n} + u_{j,h}^{n-1}}{2\Delta t}, v_h\right) + b^*\left(< u_h >^n, u_{j,h}^{n+1}, v_h\right)\\ &+b^*\left(u_{j,h}'^{n}, 2u_{j,h}^{n} - u_{j,h}^{n-1}, v_h\right) - \left(p_{j,h}^{n+1}, \nabla \cdot v_h\right) + \nu\left(\nabla u_{j,h}^{n+1}, \nabla v_h\right)\\ &+\int_\Omega C_{\nu_T}\Delta t|u'^n|^2\left(\nabla u_{j,h}^{n+1}\cdot\nabla v_h\right)dx = \left(f_j^{n+1}, v_h\right), \qquad \forall v_h \in X_h,\\ &\left(\nabla\cdot u_{j,h}^{n+1}, q_h\right) = 0, \qquad \forall q_h \in Q_h.\end{aligned}$$

Timestep conditions of (EVB). With a standard spacial discretization with mesh size h, in both 2d and 3d (EVB) is stable, if the following condition holds

$$(4.2)\qquad C_{\nu_T} > 1 \qquad \text{and} \qquad \Delta t\|\nabla\cdot u_{j,h}'^{n}\|_{L^4}^2 \le \frac{(C_{\nu_T}-1)\,\nu}{2C_sC_{\nu_T}}, \qquad j = 1, ..., J.$$

If exactly divergence free elements (e.g., [6], [8], [13], [34]) are used, this implies unconditional stability if $C_{\nu_T} > 1$. The constant C_s in the second condition of (4.2) comes from the Sobolev embedding inequality and thus only depends on the domain.

Remark 2. (4.2) *is only one sufficient condition for stability. Other sufficient conditions can be derived, e.g., for some* θ *and* α, $0 \le \theta \le \frac{1}{2}$, $0 < \alpha < 1$,

$$\theta\nu + \Delta t\left(C_{\nu_T} - \frac{1}{\alpha}\right)|u_h'^n|^2 \ge 0 \qquad \textit{and}$$
$$\left(\frac{1}{2} - \theta\right)\nu\|\nabla u_{j,h}^{n+1}\|^2 - \frac{1}{1-\alpha}\Delta t\|\nabla\cdot u_{j,h}'^{n}\|_{L^4}^2\|u_{j,h}^{n+1}\|_{L^4}^2 \ge 0.$$

Theorem 2 (Stability of (EVB))**.** *Consider the method (4.1). Suppose the conditions in (4.2) hold. Then, for any* $N > 1$

$$\begin{aligned}(4.3)\qquad &\frac{1}{4}\|u_{j,h}^{N}\|^2 + \frac{1}{4}\|2u_{j,h}^{N} - u_{j,h}^{N-1}\|^2 + \frac{\Delta t}{4}\sum_{n=1}^{N-1}\nu\|\nabla u_{j,h}^{n+1}\|^2\\ &\le \sum_{n=1}^{N-1}\frac{\Delta t}{\nu}\|f_j^{n+1}\|_{-1}^2 + \frac{1}{4}\|u_{j,h}^{1}\|^2 + \frac{1}{4}\|2u_{j,h}^{1} - u_{j,h}^{0}\|^2 .\end{aligned}$$

Proof. Setting $v_h = u_{j,h}^{n+1}$ in (4.1) and multiplying through by Δt yields:

$$(4.4)\qquad \frac{1}{4}\left(\|u_{j,h}^{n+1}\|^2 + \|2u_{j,h}^{n+1} - u_{j,h}^{n}\|^2\right) - \frac{1}{4}\left(\|u_{j,h}^{n}\|^2 + \|2u_{j,h}^{n} - u_{j,h}^{n-1}\|^2\right)$$

$$+\frac{1}{4}\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|^2+\int_\Omega \Delta t\left(\nu+C_{\nu_T}\triangle t|u_h'^n|^2\right)|\nabla u_{j,h}^{n+1}|^2dx$$
$$+\Delta t b^*\left(u_{j,h}'^n,2u_{j,h}^n-u_{j,h}^{n-1},u_{j,h}^{n+1}\right)=\Delta t\left(f_j^{n+1},u_{j,h}^{n+1}\right)\ .$$

Applying Young's inequality to the right hand side gives

$$\frac{1}{4}\left(\|u_{j,h}^{n+1}\|^2+\|2u_{j,h}^{n+1}-u_{j,h}^{n}\|^2\right)-\frac{1}{4}\left(\|u_{j,h}^{n}\|^2+\|2u_{j,h}^{n}-u_{j,h}^{n-1}\|^2\right)$$
$$+\frac{1}{4}\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|^2+\int_\Omega \Delta t\left(\nu+C_{\nu_T}|u_h'^n|^2\Delta t\right)|\nabla u_{j,h}^{n+1}|^2dx$$
$$+\Delta t b^*\left(u_{j,h}'^n,2u_{j,h}^n-u_{j,h}^{n-1},u_{j,h}^{n+1}\right)\le\frac{\nu\Delta t}{4}\|\nabla u_{j,h}^{n+1}\|^2+\frac{\Delta t}{\nu}\|f_j^{n+1}\|_{-1}^2\ . \tag{4.5}$$

Next, for any $0<\alpha<1$, we bound the trilinear term as follows

$$\Delta t b^*\left(u_{j,h}'^n,2u_{j,h}^n-u_{j,h}^{n-1},u_{j,h}^{n+1}\right) \tag{4.6}$$
$$=\Delta t b^*\left(u_{j,h}'^n,-u_{j,h}^{n+1}+2u_{j,h}^n-u_{j,h}^{n-1},u_{j,h}^{n+1}\right)$$
$$=\Delta t b^*\left(u_{j,h}'^n,u_{j,h}^{n+1},u_{j,h}^{n+1}-2u_{j,h}^n+u_{j,h}^{n-1}\right)$$
$$=\Delta t\left(u_{j,h}'^n\cdot\nabla u_{j,h}^{n+1},u_{j,h}^{n+1}-2u_{j,h}^n+u_{j,h}^{n-1}\right)$$
$$+\frac{1}{2}\triangle t\left(\nabla\cdot u_{j,h}'^n,u_{j,h}^{n+1}\cdot\left(u_{j,h}^{n+1}-2u_{j,h}^n+u_{j,h}^{n-1}\right)\right)$$
$$\le\frac{\Delta t^2}{\alpha}\int_\Omega|u_{j,h}'^n|^2|\nabla u_{j,h}^{n+1}|^2dx+\frac{1}{4}\alpha\|u_{j,h}^{n+1}-2u_{j,h}^n+u_{j,h}^{n-1}\|^2$$
$$+\frac{\Delta t^2}{1-\alpha}\int_\Omega|\nabla\cdot u_{j,h}'^n|^2|u_{j,h}^{n+1}|^2dx+\frac{1}{4}\left(1-\alpha\right)\|u_{j,h}^{n+1}-2u_{j,h}^n+u_{j,h}^{n-1}\|^2\ .$$

The stability follows provided for some α, $0<\alpha<1$,

$$\int_\Omega\Big\{\left(\frac{\nu}{2}+C_{\nu_T}|u_h'^n|^2\Delta t\right)|\nabla u_{j,h}^{n+1}|^2$$
$$-\triangle t\left(\frac{1}{\alpha}|u_{j,h}'^n|^2|\nabla u_{j,h}^{n+1}|^2+\frac{1}{1-\alpha}|\nabla\cdot u_{j,h}'^n|^2|u_{j,h}^{n+1}|^2\right)\Big\}dx\ge 0. \tag{4.7}$$

We rewrite above inequality with $0\le\theta\le\frac{1}{2}$ as

$$\int_\Omega\Big\{\left(\theta\nu+\Delta t\left(C_{\nu_T}-\frac{1}{\alpha}\right)|u_h'^n|^2\right)|\nabla u_{j,h}^{n+1}|^2$$
$$+\left(\left(\frac{1}{2}-\theta\right)\nu|\nabla u_{j,h}^{n+1}|^2-\frac{1}{1-\alpha}\Delta t|\nabla\cdot u_{j,h}'^n|^2|u_{j,h}^{n+1}|^2\right)\Big\}dx\ge 0. \tag{4.8}$$

A sufficient condition for (4.8) is

$$\theta\nu+\Delta t\left(C_{\nu_T}-\frac{1}{\alpha}\right)|u_h'^n|^2\ge 0\qquad\text{and} \tag{4.9}$$
$$\left(\frac{1}{2}-\theta\right)\nu\|\nabla u_{j,h}^{n+1}\|^2-\frac{1}{1-\alpha}\Delta t\|\nabla\cdot u_{j,h}'^n\|_{L^4}^2\|u_{j,h}^{n+1}\|_{L^4}^2\ge 0.$$

By Sobolev embedding theorem, (4.9) holds if

$$\theta\nu + \Delta t\left(C_{\nu_T} - \frac{1}{\alpha}\right)|u_h'^n|^2 \geq 0 \qquad \text{and} \tag{4.10}$$
$$\left(\frac{1}{2}-\theta\right)\nu\|\nabla u_{j,h}^{n+1}\|^2 - \frac{C_s}{1-\alpha}\Delta t\|\nabla\cdot u_{j,h}'^n\|_{L^4}^2\|\nabla u_{j,h}^{n+1}\|^2 \geq 0.$$

In particular, let $C_{\nu_T} > 1$, $\alpha = 1/C_{\nu_T}$ and $\theta = 0$, then (4.10) reduces to

$$C_{\nu_T} > 1 \qquad \text{and} \qquad \frac{1}{2}\nu - \frac{C_s C_{\nu_T}}{C_{\nu_T}-1}\Delta t\|\nabla\cdot u_{j,h}'^n\|_{L^4}^2 \geq 0,$$

which is equivalent to (4.2). Assume (4.2) holds, then (4.5) becomes

$$\frac{1}{4}\left(\|u_{j,h}^{n+1}\|^2 + \|2u_{j,h}^{n+1} - u_{j,h}^n\|^2\right) - \frac{1}{4}\left(\|u_{j,h}^n\|^2 + \|2u_{j,h}^n - u_{j,h}^{n-1}\|^2\right) \tag{4.11}$$
$$+\frac{\nu\Delta t}{4}\|\nabla u_{j,h}^{n+1}\|^2 \leq \frac{\Delta t}{\nu}\|f_j^{n+1}\|_{-1}^2 .$$

Summing up (4.11) from $n = 1$ to $N-1$ completes the proof. ∎

The question arises: how restrictive is the ν in the RHS of the second conditon in (4.2) vs the L^4 accuracy of the weakly imposed divergence free condition (the LHS of the second condition in (4.2)). This is explored in numerical tests in Section 6.2, 6.3. In these tests, (4.2) did not appear to be restrictive.

5. Error Analysis

In this section we give a detailed error analysis of (EnB). Assume X_h and Q_h satisfy the usual (LBB^h) condition, then the method is equivalent to: *for* $n = 1, ..., N_T - 1$, *find* $u_{j,h}^{n+1} \in V_h$ *such that*

$$\left(\frac{3u_{j,h}^{n+1} - 4u_{j,h}^n + u_{j,h}^{n-1}}{2\Delta t}, v_h\right) + b^*\left(< u_h >^n, u_{j,h}^{n+1}, v_h\right) \tag{5.1}$$
$$+b^*\left(u_{j,h}'^n, 2u_{j,h}^n - u_{j,h}^{n-1}, v_h\right) + \nu\left(\nabla u_{j,h}^{n+1}, \nabla v_h\right) = \left(f_j^{n+1}, v_h\right) \ , \forall v_h \in V_h.$$

To analyze the rate of convergence of the approximation we assume that the following regularity assumptions on the NSE

$$u_j \in L^\infty\left(0,T;H^1(\Omega)\right) \cap H^1\left(0,T;H^{k+1}(\Omega)\right) \cap H^2\left(0,T;H^1(\Omega)\right),$$
$$p_j \in L^2\left(0,T;H^{s+1}(\Omega)\right), \text{and } f_j \in L^2\left(0,T;L^2(\Omega)\right).$$

Let $e_j^n = u_j^n - u_{j,h}^n$ be the error between the true solution and the approximate solution, then we have the following error estimates.

Theorem 3 (Convergence of (EnB))**.** *Consider the method (*EnB*). If the following condition holds*

$$C_e\frac{\Delta t}{\nu h}\|\nabla u_{j,h}'^n\|^2 \leq 1, \qquad j = 1, ..., J, \tag{5.2}$$

where C_e *is a constant that depends on the domain and the minimum angle of the mesh, but independent of the timestep, then for any* $1 < N \leq N_T$*, there is a positive* *constant C independent of the mesh width and timestep such that*

$$
\begin{aligned}
(5.3)\qquad & \frac{1}{2}\|e_j^N\|^2 + \frac{1}{2}\|2e_j^N - e_j^{N-1}\|^2 + \frac{1}{4}\sum_{n=1}^{N-1}\|e_j^{n+1} - 2e_j^n + e_j^{n-1}\|^2 \\
& + \frac{\nu\Delta t}{4}\|\nabla e_j^N\|^2 + \frac{\nu\Delta t}{8}\|\nabla e_j^{N-1}\|^2 \\
& \leq exp\left(\frac{CT}{\nu^2}\right)\Big\{\frac{1}{2}\|e_j^1\|^2 + \frac{1}{2}\|2e_j^1 - e_j^0\|^2 + \frac{\nu\Delta t}{4}\|\nabla e_j^1\|^2 + \frac{\nu\Delta t}{8}\|\nabla e_j^0\|^2 \\
& + C\frac{h^{2k}}{\nu}|||\nabla u_j|||_{\infty,0}^2|||u_j|||_{2,k+1}^2 + C\frac{\Delta t^4}{\nu}|||\nabla u_{j,tt}|||_{2,0}^2 \\
& + C\frac{h^{2k}}{\nu}|||\nabla u_j|||_{2,k+1}^2 + C\Delta t^2 h^{2k+1}|||\nabla u_{j,tt}|||_{2,k}^2 + C\Delta t^3 h|||\nabla u_{j,tt}|||_{2,0}^2 \\
& + C\frac{h^{2s+2}}{\nu}|||p_j|||_{2,s+1}^2 + Ch^{2k+2}\nu^{-1}|||u_{t,j}|||_{2,k+1}^2 \\
& + C\nu h^{2k}|||\nabla u_j|||_{2,k}^2 + \frac{C\Delta t^4}{\nu}|||u_{j,ttt}|||_{2,0}^2\Big\}\ .
\end{aligned}
$$

Remark 3. *The condition* (5.2) *is the same type of condition as the stability condition* (3.2) *for (*ENB*). Only the constant C_e in* (5.2) *is possibly different from the one in* (3.2)*.*

Corollary 1. *Under the assumptions of Theorem 3, with (X_h, Q_h) given by the Taylor-Hood approximation elements ($k = 2, s = 1$), i.e., C^0 piecewise quadratic velocity space X_h and C^0 piecewise linear pressure space Q_h, we have the following error estimate*

$$
\begin{aligned}
(5.4)\qquad & \frac{1}{2}\|e_j^N\|^2 + \frac{1}{2}\|2e_j^N - e_j^{N-1}\|^2 + \frac{1}{4}\sum_{n=1}^{N-1}\|e_j^{n+1} - 2e_j^n + e_j^{n-1}\|^2 \\
& + \frac{\nu\Delta t}{4}\|\nabla e_j^N\|^2 + \frac{\nu\Delta t}{8}\|\nabla e_j^{N-1}\|^2 \leq C\left(h^4 + \Delta t^4 + h\Delta t^3 + \|\nabla e_j^0\|^2 + \|\nabla e_j^1\|^2\right)\ .
\end{aligned}
$$

Proof. The true solution(u_j, p_j) of the NSE satisfies

$$
\begin{aligned}
(5.5)\qquad & \left(\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t}, v_h\right) + b^*\left(u_j^{n+1}, u_j^{n+1}, v_h\right) + \nu\left(\nabla u_j^{n+1}, \nabla v_h\right) \\
& - \left(p_j^{n+1}, \nabla\cdot v_h\right) = \left(f_j^{n+1}, v_h\right) + Intp\left(u_j^{n+1}; v_h\right), \qquad \text{for all } v_h \in V_h,
\end{aligned}
$$

where $Intp\left(u_j^{n+1}; v_h\right)$ is defined as

$$
Intp\left(u_j^{n+1}; v_h\right) = \left(\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} - u_{j,t}(t^{n+1}), v_h\right)\ .
$$

Let $e_j^n = u_j^n - u_{j,h}^n = \left(u_j^n - I_h u_j^n\right) + \left(I_h u_j^n - u_{j,h}^n\right) = \eta_j^n + \xi_{j,h}^n$, where $I_h u_j^n \in V_h$ is an interpolant of u_j^n in V_h. Subtracting (5.1) from (5.5) gives

$$
(5.6)\qquad \left(\frac{3\xi_{j,h}^{n+1} - 4\xi_{j,h}^n + \xi_{j,h}^{n-1}}{2\Delta t}, v_h\right) + b^*\left(u_j^{n+1}, u_j^{n+1}, v_h\right) + \nu\left(\nabla\xi_{j,h}^{n+1}, \nabla v_h\right)
$$

$$-b^*\left(2u_{j,h}^n - u_{j,h}^{n-1} - u_{j,h}'^n, u_{j,h}^{n+1}, v_h\right) - b^*\left(u_{j,h}'^n, 2u_{j,h}^n - u_{j,h}^{n-1}, v_h\right) - \left(p_j^{n+1}, \nabla\cdot v_h\right)$$
$$= -\left(\frac{3\eta_j^{n+1} - 4\eta_j^n + \eta_j^{n-1}}{2\Delta t}, v_h\right) - \nu\left(\nabla\eta_j^{n+1}, \nabla v_h\right) + Intp\left(u_j^{n+1}; v_h\right)\ .$$

Set $v_h = \xi_{j,h}^{n+1} \in V_h$, and rearrange the nonlinear terms, then we have

$$\frac{1}{4\Delta t}\left(\|\xi_{j,h}^{n+1}\|^2 + \|2\xi_{j,h}^{n+1} - \xi_{j,h}^n\|^2\right) - \frac{1}{4\Delta t}\left(\|\xi_{j,h}^n\|^2 + \|2\xi_{j,h}^n - \xi_{j,h}^{n-1}\|^2\right)$$
$$+\frac{1}{4\Delta t}\|\xi_{j,h}^{n+1} - 2\xi_{j,h}^n + \xi_{j,h}^{n-1}\|^2 + \nu\|\nabla\xi_{j,h}^{n+1}\|^2$$
$$= -b^*\left(u_j^{n+1}, u_j^{n+1}, \xi_{j,h}^{n+1}\right) + b^*\left(2u_{j,h}^n - u_{j,h}^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$+b^*\left(u_{j,h}'^n, 2u_{j,h}^n - u_{j,h}^{n-1} - u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) + \left(p_j^{n+1}, \nabla\cdot\xi_{j,h}^{n+1}\right)$$
$$-\left(\frac{3\eta_j^{n+1} - 4\eta_j^n + \eta_j^{n-1}}{2\Delta t}, \xi_{j,h}^{n+1}\right) - \nu\left(\nabla\eta_j^{n+1}, \nabla\xi_{j,h}^{n+1}\right) + Intp\left(u_j^{n+1}; \xi_{j,h}^{n+1}\right)\ . \tag{5.7}$$

Now we bound the right hand side of equation (5.7). First, for the nonlinear term, adding and subtracting $b^*(u_j^{n+1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1})$ (Step 1), $b^*(2u_j^n - u_j^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1})$ (Step 2) and $b^*(u_{j,h}'^n, 2u_j^n - u_j^{n-1} - u_j^{n+1}, \xi_{j,h}^{n+1})$ (Step3) respectively, we have

$$-b^*\left(u_j^{n+1}, u_j^{n+1}, \xi_{j,h}^{n+1}\right) + b^*\left(2u_{j,h}^n - u_{j,h}^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$+b^*\left(u_{j,h}'^n, 2u_{j,h}^n - u_{j,h}^{n-1} - u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) \tag{5.8}$$

(Step 1)
$$= -b^*\left(u_j^{n+1}, e_j^{n+1}, \xi_{j,h}^{n+1}\right) - b^*\left(u_j^{n+1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$+b^*\left(2u_{j,h}^n - u_{j,h}^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) + b^*\left(u_{j,h}'^n, 2u_{j,h}^n - u_{j,h}^{n-1} - u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right)$$

(Step 2)
$$= -b^*\left(u_j^{n+1}, e_j^{n+1}, \xi_{j,h}^{n+1}\right) - b^*\left(u_j^{n+1} - (2u_j^n - u_j^{n-1}), u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$-b^*\left(2e_j^n - e_j^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) + b^*\left(u_{j,h}'^n, 2u_{j,h}^n - u_{j,h}^{n-1} - u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right)$$

(Step 3)
$$= -b^*\left(u_j^{n+1}, e_j^{n+1}, \xi_{j,h}^{n+1}\right) - b^*\left(u_j^{n+1} - \left(2u_j^n - u_j^{n-1}\right), u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$-b^*\left(2e_j^n - e_j^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) - b^*\left(u_{j,h}'^n, 2e_j^n - e_j^{n-1} - e_j^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$+b^*\left(u_{j,h}'^n, 2u_j^n - u_j^{n-1} - u_j^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$= -b^*\left(u_j^{n+1}, \eta_j^{n+1}, \xi_{j,h}^{n+1}\right) - b^*\left(u_j^{n+1} - (2u_j^n - u_j^{n-1}), u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$-b^*\left(2\eta_j^n - \eta_j^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) - b^*\left(2\xi_{j,h}^n - \xi_{j,h}^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$-b^*\left(u_{j,h}'^n, 2\xi_{j,h}^n - \xi_{j,h}^{n-1} - \xi_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) - b^*\left(u_{j,h}'^n, 2\eta_j^n - \eta_j^{n-1} - \eta_j^{n+1}, \xi_{j,h}^{n+1}\right)$$
$$+b^*\left(u_{j,h}'^n, 2u_j^n - u_j^{n-1} - u_j^{n+1}, \xi_{j,h}^{n+1}\right)\ .$$

We estimate the nonlinear terms using (2.2), (2.3) and Young's inequality as follows.

$$b^*\left(u_j^{n+1}, \eta_j^{n+1}, \xi_{j,h}^{n+1}\right) \le C\|\nabla u_j^{n+1}\|\|\nabla\eta_j^{n+1}\|\|\nabla\xi_{j,h}^{n+1}\| \tag{5.9}$$

$$\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\|\nabla u_j^{n+1}\|^2\|\nabla\eta_j^{n+1}\|^2 \ .$$

$$\begin{aligned}
&b^*\left(u_j^{n+1} - \left(2u_j^n - u_j^{n-1}\right), u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) \\
&\leq C\|\nabla\left(u_j^{n+1} - 2u_j^n + u_j^{n-1}\right)\|\|\nabla u_{j,h}^{n+1}\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\|\nabla\left(u_j^{n+1} - 2u_j^n + u_j^{n-1}\right)\|^2\|\nabla u_{j,h}^{n+1}\|^2 \\
&\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\Delta t^3\left(\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|^2 dt\right)\|\nabla u_{j,h}^{n+1}\|^2 \ .
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
b^*\left(2\eta_j^n - \eta_j^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) &\leq C\|\nabla\left(2\eta_j^n - \eta_j^{n-1}\right)\|\|\nabla u_{j,h}^{n+1}\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\left(\|\nabla\eta_j^n\|^2 + \|\nabla\eta_j^{n-1}\|^2\right)\|\nabla u_{j,h}^{n+1}\|^2.
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
2b^*\left(\xi_{j,h}^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) &\leq C\|\nabla\xi_{j,h}^n\|^{\frac{1}{2}}\|\xi_{j,h}^n\|^{\frac{1}{2}}\|\nabla u_{j,h}^{n+1}\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\leq C\|\nabla\xi_{j,h}^n\|^{\frac{1}{2}}\|\xi_{j,h}^n\|^{\frac{1}{2}}\|\nabla\xi_{j,h}^{n+1}\| \\
&\leq C\left(\epsilon\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{1}{\epsilon}\|\nabla\xi_{j,h}^n\|\|\xi_{j,h}^n\|\right) \\
&\leq C\left(\epsilon\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{1}{\epsilon}\left(\delta\|\nabla\xi_{j,h}^n\|^2 + \frac{1}{\delta}\|\xi_{j,h}^n\|^2\right)\right) \\
&\leq \left(\frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{\nu}{16}\|\nabla\xi_{j,h}^n\|^2\right) + C\nu^{-3}\|\xi_{j,h}^n\|^2.
\end{aligned} \tag{5.12}$$

Similarly,

$$\begin{aligned}
b^*\left(\xi_{j,h}^{n-1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) &\leq C\|\nabla\xi_{j,h}^{n-1}\|^{\frac{1}{2}}\|\xi_{j,h}^{n-1}\|^{\frac{1}{2}}\|\nabla u_{j,h}^{n+1}\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\leq C\|\nabla\xi_{j,h}^{n-1}\|^{\frac{1}{2}}\|\xi_{j,h}^{n-1}\|^{\frac{1}{2}}\|\nabla\xi_{j,h}^{n+1}\| \\
&\leq C\left(\epsilon\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{1}{\epsilon}\|\nabla\xi_{j,h}^{n-1}\|\|\xi_{j,h}^{n-1}\|\right) \\
&\leq C\left(\epsilon\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{1}{\epsilon}\left(\delta\|\nabla\xi_{j,h}^{n-1}\|^2 + \frac{1}{\delta}\|\xi_{j,h}^{n-1}\|^2\right)\right) \\
&\leq \left(\frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{\nu}{16}\|\nabla\xi_{j,h}^{n-1}\|^2\right) + C\nu^{-3}\|\xi_{j,h}^{n-1}\|^2.
\end{aligned} \tag{5.13}$$

By skew symmetry

$$\begin{aligned}
b^*\left(u'^n_{j,h}, 2\xi_{j,h}^n - \xi_{j,h}^{n-1} - \xi_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) &= -b^*\left(u'^n_{j,h}, \xi_{j,h}^{n+1} - 2\xi_{j,h}^n + \xi_{j,h}^{n-1}, \xi_{j,h}^{n+1}\right) \\
&= b^*\left(u'^n_{j,h}, \xi_{j,h}^{n+1}, \xi_{j,h}^{n+1} - 2\xi_{j,h}^n + \xi_{j,h}^{n-1}\right).
\end{aligned}$$

Using (2.4) and inverse inequality (2.5) gives

$$\begin{aligned}
&b^*\left(u'^n_{j,h}, 2\xi_{j,h}^n - \xi_{j,h}^{n-1} - \xi_{j,h}^{n+1}, \xi_{j,h}^{n+1}\right) \\
&\leq C\|\nabla u'^n_{j,h}\|\|\nabla\xi_{j,h}^{n+1}\|\|\nabla(\xi_{j,h}^{n+1} - 2\xi_{j,h}^n + \xi_{j,h}^{n-1})\|^{1/2}\|\xi_{j,h}^{n+1} - 2\xi_{j,h}^n + \xi_{j,h}^{n-1}\|^{1/2} \\
&\leq C\|\nabla u'^n_{j,h}\|\|\nabla\xi_{j,h}^{n+1}\|\left(h^{-1/2}\right)\|\xi_{j,h}^{n+1} - 2\xi_{j,h}^n + \xi_{j,h}^{n-1}\|
\end{aligned} \tag{5.14}$$

$$\leq \frac{1}{8\Delta t}\|\xi_{j,h}^{n+1}-2\xi_{j,h}^{n}+\xi_{j,h}^{n-1}\|^2+\frac{C_e}{16}\frac{\Delta t}{h}\|\nabla u_{j,h}'^{n}\|^2\|\nabla\xi_{j,h}^{n+1}\|^2.$$

$$\begin{aligned}(5.15)\qquad & b^*\left(u_{j,h}'^{n},\eta_j^{n+1}-2\eta_j^{n}+\eta_j^{n-1},\xi_{j,h}^{n+1}\right)\\ &\leq C\|\nabla u_{j,h}'^{n}\|\|\nabla\left(\eta_j^{n+1}-2\eta_j^{n}+\eta_j^{n-1}\right)\|\|\nabla\xi_{j,h}^{n+1}\|\\ &\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2+C\nu^{-1}\|\nabla u_{j,h}'^{n}\|^2\|\nabla\left(\eta_j^{n+1}-2\eta_j^{n}+\eta_j^{n-1}\right)\|^2\\ &\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2+\frac{C\Delta t^3}{\nu}\|\nabla u_{j,h}'^{n}\|^2\left(\int_{t^{n-1}}^{t^{n+1}}\|\nabla\eta_{j,tt}\|^2\,dt\right).\end{aligned}$$

$$\begin{aligned}(5.16)\qquad & b^*\left(u_{j,h}'^{n},u_j^{n+1}-2u_j^{n}+u_j^{n-1},\xi_{j,h}^{n+1}\right)\\ &\leq C\|\nabla u_{j,h}'^{n}\|\|\nabla\left(u_j^{n+1}-2u_j^{n}+u_j^{n-1}\right)\|\|\nabla\xi_{j,h}^{n+1}\|\\ &\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2+C\nu^{-1}\|\nabla u_{j,h}'^{n}\|^2\|\nabla\left(u_j^{n+1}-2u_j^{n}+u_j^{n-1}\right)\|^2\\ &\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2+C\nu^{-1}\Delta t^3\|\nabla u_{j,h}'^{n}\|^2\left(\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|^2\,dt\right).\end{aligned}$$

Next, consider the pressure term. Since $\xi_{j,h}^{n+1}\in V_h$ we have

$$\begin{aligned}(5.17)\qquad & \left(p_j^{n+1},\nabla\cdot\xi_{j,h}^{n+1}\right)=\left(p_j^{n+1}-q_{j,h}^{n+1},\nabla\cdot\xi_{j,h}^{n+1}\right)\\ &\leq \|p_j^{n+1}-q_{j,h}^{n+1}\|\|\nabla\cdot\xi_{j,h}^{n+1}\|\\ &\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2+C\nu^{-1}\|p_j^{n+1}-q_{j,h}^{n+1}\|^2,\qquad \forall q_{j,h}^{n+1}\in Q_h.\end{aligned}$$

The other terms, are bounded as

$$\begin{aligned}(5.18)\qquad & \left(\frac{3\eta_j^{n+1}-4\eta_j^{n}+\eta_j^{n-1}}{2\Delta t},\xi_{j,h}^{n+1}\right)\leq C\|\frac{3\eta_j^{n+1}-4\eta_j^{n}+\eta_j^{n-1}}{2\Delta t}\|\|\nabla\xi_{j,h}^{n+1}\|\\ &\leq C\nu^{-1}\|\frac{3\eta_j^{n+1}-4\eta_j^{n}+\eta_j^{n-1}}{2\Delta t}\|^2+\frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2\\ &\leq C\nu^{-1}\|\frac{1}{\Delta t}\int_{t^{n-1}}^{t^{n+1}}\eta_{j,t}\,dt\|+\frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2\\ &\leq \frac{C}{\nu\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\eta_{j,t}\|^2\,dt+\frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2.\end{aligned}$$

$$\begin{aligned}(5.19)\qquad & \nu\left(\nabla\eta_j^{n+1},\nabla\xi_{j,h}^{n+1}\right)\leq\nu\|\nabla\eta_j^{n+1}\|\|\nabla\xi_{j,h}^{n+1}\|\\ &\leq C\nu\|\nabla\eta_j^{n+1}\|^2+\frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2.\end{aligned}$$

Finally,

$$(5.20)\qquad Intp\left(u_j^{n+1};\xi_{j,h}^{n+1}\right)=\left(\frac{3u_j^{n+1}-4u_j^{n}+u_j^{n-1}}{2\Delta t}-u_{j,t}(t^{n+1}),\xi_{j,h}^{n+1}\right)$$

$$\leq C\|\frac{3u_j^{n+1}-4u_j^n+u_j^{n-1}}{2\Delta t}-u_{j,t}(t^{n+1})\|\|\nabla\xi_{j,h}^{n+1}\|$$
$$\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2+\frac{C}{\nu}\|\frac{3u_j^{n+1}-4u_j^n+u_j^{n-1}}{2\Delta t}-u_{j,t}(t^{n+1})\|^2$$
$$\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2+\frac{C\Delta t^3}{\nu}\int_{t^{n-1}}^{t^{n+1}}\|u_{j,ttt}\|^2dt\ .$$

Combining, we now have the following inequality

$$\begin{aligned}(5.21)\quad &\frac{1}{4\Delta t}\left(\|\xi_{j,h}^{n+1}\|^2+\|2\xi_{j,h}^{n+1}-\xi_{j,h}^n\|^2\right)-\frac{1}{4\Delta t}\left(\|\xi_{j,h}^n\|^2+\|2\xi_{j,h}^n-\xi_{j,h}^{n-1}\|^2\right)\\
&+\frac{1}{8\Delta t}\|\xi_{j,h}^{n+1}-2\xi_{j,h}^n+\xi_{j,h}^{n-1}\|^2+\frac{\nu}{16}\left(\|\nabla\xi_{j,h}^{n+1}\|^2-\|\nabla\xi_{j,h}^n\|^2\right)\\
&+\frac{\nu}{16}\left(\left(\|\nabla\xi_{j,h}^{n+1}\|^2+\|\nabla\xi_{j,h}^n\|^2\right)-\left(\|\nabla\xi_{j,h}^n\|^2+\|\nabla\xi_{j,h}^{n-1}\|^2\right)\right)\\
&+\left(\frac{\nu}{16}-\frac{C_e}{16}\frac{\Delta t}{h}\|\nabla u_{j,h}^{\prime n}\|^2\right)\|\nabla\xi_{j,h}^{n+1}\|^2\leq C\nu^{-3}\left(\|\xi_{j,h}^n\|^2+\|\xi_{j,h}^{n-1}\|^2\right)\\
&+C\nu^{-1}\|\nabla u_j^{n+1}\|^2\|\nabla\eta_j^{n+1}\|^2+\frac{C\Delta t^3}{\nu}\left(\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|^2dt\right)\|\nabla u_{j,h}^{n+1}\|^2\\
&+C\nu^{-1}\left(\|\nabla\eta_j^n\|^2+\|\nabla\eta_j^{n-1}\|^2\right)\|\nabla u_{j,h}^{n+1}\|^2+\frac{C\Delta t^3}{\nu}\|\nabla u_{j,h}^{\prime n}\|^2\left(\int_{t^{n-1}}^{t^{n+1}}\|\nabla\eta_{j,tt}\|^2\ dt\right)\\
&+\frac{C\Delta t^3}{\nu}\|\nabla u_{j,h}^{\prime n}\|^2\left(\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|^2\ dt\right)+C\nu^{-1}\|p_j^{n+1}-q_{j,h}^{n+1}\|^2\\
&+\frac{C}{\nu\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\eta_{j,t}\|^2\ dt+C\nu\|\nabla\eta_j^{n+1}\|^2+\frac{C\Delta t^3}{\nu}\int_{t^{n-1}}^{t^{n+1}}\|u_{j,ttt}\|^2dt.\end{aligned}$$

Under the assumption of (5.2), $(\frac{\nu}{16}-\frac{C_e}{16}\frac{\Delta t}{h}\|\nabla u_{j,h}^{\prime n}\|^2)$ is nonnegative and thus can be eliminated from the LHS of (5.21). We then take the sum of (5.21) from $n=1$ to $n=N-1$ and multiply through by $2\Delta t$. This yields

$$\begin{aligned}(5.22)\quad &\frac{1}{2}\|\xi_{j,h}^N\|^2+\frac{1}{2}\|2\xi_{j,h}^N-\xi_{j,h}^{N-1}\|^2+\frac{1}{4}\sum_{n=1}^{N-1}\|\xi_{j,h}^{n+1}-2\xi_{j,h}^n+\xi_{j,h}^{n-1}\|^2\\
&+\frac{\nu\Delta t}{8}\|\nabla\xi_{j,h}^N\|^2+\frac{\nu\Delta t}{8}\left(\|\nabla\xi_{j,h}^N\|^2+\|\nabla\xi_{j,h}^{N-1}\|^2\right)\\
&\leq\frac{1}{2}\|\xi_{j,h}^1\|^2+\frac{1}{2}\|2\xi_{j,h}^1-\xi_{j,h}^0\|^2+\frac{\nu\Delta t}{8}\|\nabla\xi_{j,h}^1\|^2+\frac{\nu\Delta t}{8}\left(\|\nabla\xi_{j,h}^1\|^2+\|\nabla\xi_{j,h}^0\|^2\right)\\
&+\Delta t\sum_{n=0}^{N-1}C\nu^{-3}\|\xi_{j,h}^n\|^2+\Delta t\sum_{n=0}^{N-1}\Big\{C\nu^{-1}\|\nabla u_j^{n+1}\|^2\|\nabla\eta_j^{n+1}\|^2\\
&+\frac{C\Delta t^3}{\nu}\left(\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|^2\ dt\right)+C\nu^{-1}\|\nabla\eta_j^n\|^2\\
&+C\Delta t^2h\left(\int_{t^{n-1}}^{t^{n+1}}\|\nabla\eta_{j,tt}\|^2dt\right)+C\Delta t^2h\left(\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|^2dt\right)\end{aligned}$$

$$+C\nu^{-1}\|p_j^{n+1}-q_{j,h}^{n+1}\|^2+\frac{C}{\nu\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\eta_{j,t}\|^2dt$$
$$+C\nu\|\nabla\eta_j^{n+1}\|^2+\frac{C\Delta t^3}{\nu}\int_{t^{n-1}}^{t^{n+1}}\|u_{j,ttt}\|^2\ dt\Big\}.$$

Applying interpolation inequalities gives

$$\begin{aligned}
&\frac{1}{2}\|\xi_{j,h}^N\|^2+\frac{1}{2}\|2\xi_{j,h}^N-\xi_{j,h}^{N-1}\|^2+\frac{1}{4}\sum_{n=1}^{N-1}\|\xi_{j,h}^{n+1}-2\xi_{j,h}^n+\xi_{j,h}^{n-1}\|^2\\
&\quad+\frac{\nu\Delta t}{4}\|\nabla\xi_{j,h}^N\|^2+\frac{\nu\Delta t}{8}\|\nabla\xi_{j,h}^{N-1}\|^2\\
&\leq\frac{1}{2}\|\xi_{j,h}^1\|^2+\frac{1}{2}\|2\xi_{j,h}^1-\xi_{j,h}^0\|^2+\frac{\nu\Delta t}{4}\|\nabla\xi_{j,h}^1\|^2+\frac{\nu\Delta t}{8}\|\nabla\xi_{j,h}^0\|^2+\Delta t\sum_{n=0}^{N-1}C\nu^{-3}\|\xi_{j,h}^n\|^2\\
&\quad+C\frac{h^{2k}}{\nu}|||\nabla u_j|||_{\infty,0}^2|||u_j|||_{2,k+1}^2+C\frac{\Delta t^4}{\nu}|||\nabla u_{j,tt}|||_{2,0}^2\\
&\quad+C\frac{h^{2k}}{\nu}|||\nabla u_j|||_{2,k+1}^2+C\Delta t^2h^{2k+1}|||\nabla u_{j,tt}|||_{2,k}^2+C\Delta t^3h|||\nabla u_{j,tt}|||_{2,0}^2\\
&\quad+C\frac{h^{2s+2}}{\nu}|||p_j|||_{2,s+1}^2+Ch^{2k+2}\nu^{-1}|||u_{t,j}|||_{2,k+1}^2\\
&\quad+C\nu h^{2k}|||\nabla u_j|||_{2,k}^2+\frac{C\Delta t^4}{\nu}|||u_{j,ttt}|||_{2,0}^2.
\end{aligned}\tag{5.23}$$

The next step will be the application of the discrete Gronwall inequality (Girault and Raviart [16], p. 176).

$$\begin{aligned}
&\frac{1}{2}\|\xi_{j,h}^N\|^2+\frac{1}{2}\|2\xi_{j,h}^N-\xi_{j,h}^{N-1}\|^2+\frac{1}{4}\sum_{n=1}^{N-1}\|\xi_{j,h}^{n+1}-2\xi_{j,h}^n+\xi_{j,h}^{n-1}\|^2\\
&\quad+\frac{\nu\Delta t}{4}\|\nabla\xi_{j,h}^N\|^2+\frac{\nu\Delta t}{8}\|\nabla\xi_{j,h}^{N-1}\|^2\\
&\leq exp\left(\frac{CN\Delta t}{\nu^2}\right)\Big\{\frac{1}{2}\|\xi_{j,h}^1\|^2+\frac{1}{2}\|2\xi_{j,h}^1-\xi_{j,h}^0\|^2+\frac{\nu\Delta t}{4}\|\nabla\xi_{j,h}^1\|^2+\frac{\nu\Delta t}{8}\|\nabla\xi_{j,h}^0\|^2\\
&\quad+C\frac{h^{2k}}{\nu}|||\nabla u_j|||_{\infty,0}^2|||u_j|||_{2,k+1}^2+C\frac{\Delta t^4}{\nu}|||\nabla u_{j,tt}|||_{2,0}^2\\
&\quad+C\frac{h^{2k}}{\nu}|||\nabla u_j|||_{2,k+1}^2+C\Delta t^2h^{2k+1}|||\nabla u_{j,tt}|||_{2,k}^2+C\Delta t^3h|||\nabla u_{j,tt}|||_{2,0}^2\\
&\quad+C\frac{h^{2s+2}}{\nu}|||p_j|||_{2,s+1}^2+Ch^{2k+2}\nu^{-1}|||u_{t,j}|||_{2,k+1}^2\\
&\quad+C\nu h^{2k}|||\nabla u_j|||_{2,k}^2+\frac{C\Delta t^4}{\nu}|||u_{j,ttt}|||_{2,0}^2\Big\}\ .
\end{aligned}\tag{5.24}$$

Using triangle inequality on the error and absorbing constants into a new constant C, we obtain (5.3). ∎

Δt	$\|u_1 - u_{1,h}\|_{\infty,0}$	rate	$\|\nabla u_1 - \nabla u_{1,h}\|_{2,0}$	rate
0.05	$4.85642 \cdot 10^{-4}$	–	$5.11092 \cdot 10^{-3}$	–
0.025	$1.26128 \cdot 10^{-4}$	1.9450	$1.18810 \cdot 10^{-3}$	2.1049
0.0125	$3.21716 \cdot 10^{-5}$	1.9710	$2.92502 \cdot 10^{-4}$	2.0221
0.00625	$8.12342 \cdot 10^{-6}$	1.9856	$7.31031 \cdot 10^{-5}$	2.0004
0.003125	$2.04078 \cdot 10^{-6}$	1.9930	$1.83094 \cdot 10^{-5}$	1.9974

TABLE 1. (EnB): Errors and convergence rates for the first ensemble member

6. Numerical Tests

In this section, we give three tests to verify the convergence rate, test the effect of the EV term, test stability of the methods, test the severity of the timestep conditions, and explore the use of ensemble methods to interrogate flows.

6.1. **Convergence.**

We check the convergence rate on a simple test problem from [18], with known exact solution. This problem preserves spacial patterns of the Green-Taylor solution, [4], [17], but the vortices do not decay as $t \to \infty$. The analytical solution of the Navier-Stokes equations in the unit square $\Omega = [0,1]^2$ is given by

$$u_{true} = (-g(t)\cos x \sin y, +g(t)\sin x \cos y)^T,$$
$$p_{true} = -\frac{1}{4}[\cos(2x) + \cos(2y)]g^2(t), \quad \text{where } g(t) = \sin(2t),$$

with source term $f(x,y,t) = [g'(t) + 2\nu g(t)](-\cos x \sin y, \sin x \cos y)^T$. The boundary condition on the problem is taken to be inhomogeneous Dirichlet: $u = u_{true}$ on $\partial\Omega$.

The generation of perturbations to initial conditions and source terms is application dependable. In this simple test, we generate perturbations to initial conditions in the same way as in [19]. Consider an ensemble of two members $u_{1,2} = (1 \pm \epsilon)u_{true}$, $\epsilon = 10^{-3}$, which are the solutions to NSE corresponding to two different initial conditions $u^0_{1,2} = (1 \pm \epsilon)u^0_{true}$, respectively. Note the source term and boundary condition are adjusted accordingly.

Taking $\nu = 0.01$, $T = 1$, $h = 2\Delta t$, we compute approximations to the test problem with both (EnB) and (EVB) on 5 successive mesh refinements and corresponding timestep reductions. From Table 1 and Table 2 we can see u_1 and u_2 computed with (EnB) are second order convergent as predicted. The eddy viscosity term in (EVB) results in extra errors that depend on the magnitude of the fluctuations. A comparison of data from Table 1, 2, 3 and 4 shows that the errors from (EVB) are comparable to although slightly bigger than errors from (EnB).

6.2. **Flow between two offset cylinders.**

We test the stability of our algorithm on a problem of flow between two offset circles, given in [19]. The domain is a disk with a smaller off center obstacle inside. Let $r_1 = 1$, $r_2 = 0.1$, $c = (c_1, c_2) = (\frac{1}{2}, 0)$, then the domain is given by

$$\Omega = \{(x,y) : x^2 + y^2 \le r_1^2 \text{ and } (x - c_1)^2 + (y - c_2)^2 \ge r_2^2\}.$$

Δt	$\|u_2 - u_{2,h}\|_{\infty,0}$	rate	$\|\nabla u_2 - \nabla u_{2,h}\|_{2,0}$	rate
0.05	$4.84794 \cdot 10^{-4}$	–	$5.09708 \cdot 10^{-3}$	–
0.025	$1.25913 \cdot 10^{-4}$	1.9449	$1.18528 \cdot 10^{-3}$	2.1044
0.0125	$3.21161 \cdot 10^{-5}$	1.9711	$2.91837 \cdot 10^{-4}$	2.0220
0.00625	$8.10943 \cdot 10^{-6}$	1.9856	$7.29391 \cdot 10^{-5}$	2.0004
0.003125	$2.03726 \cdot 10^{-6}$	1.9930	$1.82684 \cdot 10^{-5}$	1.9973

TABLE 2. (EnB): Errors and convergence rates for the second ensemble member

Δt	$\|u_1 - u_{1,h}\|_{\infty,0}$	rate	$\|\nabla u_1 - \nabla u_{1,h}\|_{2,0}$	rate
0.05	$4.85644 \cdot 10^{-4}$	–	$5.11094 \cdot 10^{-3}$	–
0.025	$1.26129 \cdot 10^{-4}$	1.9450	$1.18812 \cdot 10^{-3}$	2.1049
0.0125	$3.21721 \cdot 10^{-5}$	1.9710	$2.92508 \cdot 10^{-4}$	2.0221
0.00625	$8.12367 \cdot 10^{-6}$	1.9856	$7.31064 \cdot 10^{-5}$	2.0004
0.003125	$2.0409 \cdot 10^{-6}$	1.9929	$1.83110 \cdot 10^{-5}$	1.9973

TABLE 3. (EVB): Errors and convergence rates for the first ensemble member

Δt	$\|u_2 - u_{2,h}\|_{\infty,0}$	rate	$\|\nabla u_2 - \nabla u_{2,h}\|_{2,0}$	rate
0.05	$4.84796 \cdot 10^{-4}$	–	$5.09709 \cdot 10^{-3}$	–
0.025	$1.25915 \cdot 10^{-4}$	1.9449	$1.185291 \cdot 10^{-3}$	2.1044
0.0125	$3.21166 \cdot 10^{-5}$	1.9711	$2.91844 \cdot 10^{-4}$	2.0220
0.00625	$8.10968 \cdot 10^{-6}$	1.9856	$7.29423 \cdot 10^{-5}$	2.0004
0.003125	$2.03738 \cdot 10^{-6}$	1.9929	$1.82701 \cdot 10^{-5}$	1.9973

TABLE 4. (EVB): Errors and convergence rates for the second ensemble member

The flow is driven by a counterclockwise rotational body force

$$f(x, y, t) = (-4y * (1 - x^2 - y^2), 4x * (1 - x^2 - y^2))^T,$$

with no-slip boundary conditions imposed on both circles. The flow between the two circles shows interesting structures interacting with the inner circle. A Von Kármán vortex street is formed and then reinteracts with the inner circle and itself generating complex flow patterns.

We generate perturbations of the initial conditions in the same way given in [19] for the same test problem. Two perturbed initial conditions are obtained by by solving steady Stokes problem with perturbed body forces given by

$$f_1(x, y, t) = f(x, y, t) + \epsilon_1 * (sin(3\pi x)sin(3\pi y), cos(3\pi x)cos(3\pi y))^T,$$

$$f_2(x, y, t) = f(x, y, t) + \epsilon_2 * (sin(3\pi x)sin(3\pi y), cos(3\pi x)cos(3\pi y))^T.$$

We compute approximations to the test problem with both (EnB) and (EVB), with perturbation parameters $\epsilon_1 = 10^{-3}$, $\epsilon_2 = -10^{-3}$. The mesh is generated by Delaunay triangulation with 40 mesh points on the outer circle and 10 mesh points on the inner circle.

For (EnB), we cut Δt according to the following specific timestep condition

$$\Delta t \|\nabla u'^{n}_{j,h}\|^2 \leq 80000\nu h, \qquad (j = 1, 2). \tag{6.1}$$

For (EVB), we cut Δt according to the following specific timestep condition

$$\Delta t \|\nabla \cdot u'^{n}_{j,h}\|^2_{L^4} \leq 40000\nu, \qquad (j = 1, 2). \tag{6.2}$$

Pre-computations were used to determine the coefficients $(80,000$ and $40,000)$ in the conditions. The timestep is cut in half if the condition is violated and doubled if the magnitude of the fluctuations gets small enough (specifically, $\Delta t \|\nabla u'^{n}_{j,h}\|^2 \leq 40000\nu h$ for (EnB); $\Delta t \|\nabla \cdot u'^{n}_{j,h}\|^2_{L^4} \leq 20000\nu$ for (EVB)). In all cases, Δt is enforced to not exceed 0.05. Figure 1 shows the kinetic energy of the average velocity from different methods. The curve marked with 'no perturbation' is computed using the linearly implicit Backward Euler method with no perturbation on the initial condition. The other two curves are computed by (EnB) (marked with 'noEV') and (EVB) (marked with 'EV', with eddy viscosity coefficient $C_{\nu_T} = 1$) respectively. *Figure 1 shows that the choice* $C_{\nu_T} = 1.0$ *results in too much damping. EV models have this common sensitivity to the precise values of the EV coefficients.*

There is a significant difference between the simulations with the averaged initial conditions and the averaged simulations with 10^{-3} perturbation of the initial conditions. We can see from Figure 1 that the (EnB) gives a better approximation for $\nu = 0.02$ than (EVB) which is somewhat over-diffused. For $\nu = 0.02$, both (EnB) and (EVB) are stable under the timestep conditions given above with timestep reduced to 0.00625 for (EnB) while timestep kept to be 0.05 all the time for (EVB).

Figure 2 shows a comparison of kinetic energy of the average velocity approximated by (EVB) with different eddy viscosity parameters. For $\nu = 0.001$, (EnB) fails with timestep cut to $< 10^{-7}$, while (EVB) stays stable with large/moderate timesteps. The upper picture in Figure 2 is obtained by adapting timestep according to the timestep restriction (6.2) and the statistics in the lower picture are obtained by computing with constant timestep $\Delta t = 0.0125$. The approximations are sensitive to the choice of the eddy viscosity parameter C_{ν_T}, an observation consistent with results in [2].

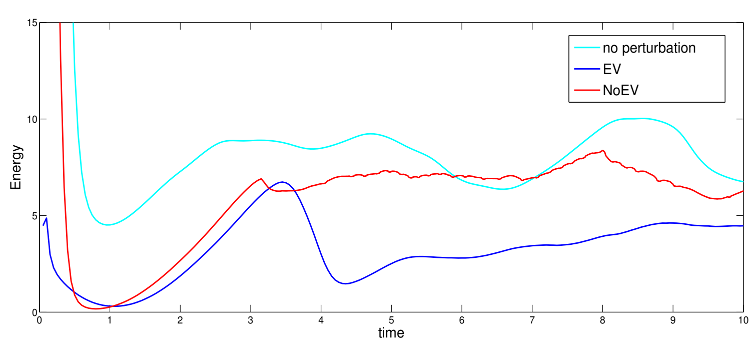

Figure 1. Kinematic Energy for $\nu = 0.02$.

It is believed that 2d (unforced) turbulent flow has a larger window of predictability than similar 3d flows due to the trend for energy to cascade to large,

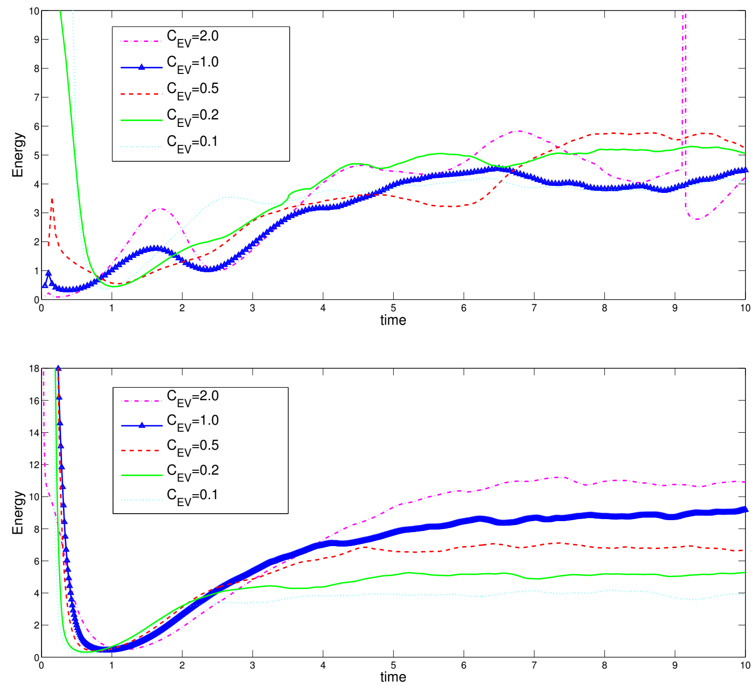

FIGURE 2. Kinematic Energy for $\nu = 0.001$. UPPER: adapted timestep; LOWER: constant timestep $\Delta t = 0.0125$.

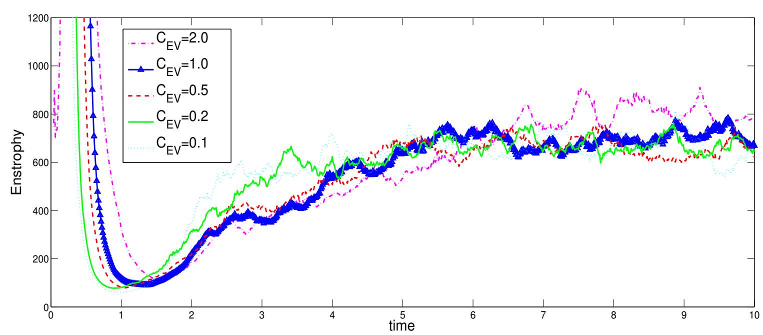

FIGURE 3. Enstrophy for $\nu = 0.001$ with constant timestep $\Delta t = 0.0125$.

coherent structures. This trend may be connected to the selective decay principle and convergence of Dirichlet quotients; see Majda and Wang [27] for elaborations. We test this for the above 2d flow between offset circles as follows. We solve the problem with the same initial conditions generated by the above perturbed body forces until complex, small scale structures appear at time T^* using (EVB) with eddy viscosity parameter $C_{EV} = 1.0$. The time T^* is selected to be $T^* = 9.6$, a time of (near) maximal enstrophy from Figure 3. Thereafter (for $t > T^*$) we set $f(x, t) \equiv 0$ and study the decay of the flow thereafter.

Definition 3. *The* **turbulence intensity** $I(t)$ *is*

$$I(t) := \frac{\left\langle \|u_j'\|^2 \right\rangle^{1/2}}{\| \langle u \rangle \|}(t),$$

and the **Dirichlet quotients** *are*

$$D(t) := \frac{\|\nabla \times \langle u \rangle \|^2}{\| \langle u \rangle \|^2}(t), \text{ and } \langle D \rangle (t) := \left\langle \frac{\|\nabla \times u_j\|^2}{\|u_j\|^2} \right\rangle (t).$$

Definition 4. *The* **relative energy fluctuation** *is*

$$r(t) := \frac{\|u_1 - u_2\|^2}{\|u_1\|\|u_2\|}(t),$$

and the **average, effective Lyapunov exponent** *over* $0 \le t \le T$ *is*

$$\gamma_T(t) := \frac{1}{2T} log\left(\frac{r(t+T)}{r(t)}\right).$$

Remark 4. *The definition of* $\gamma_T(t)$*, where* T *is chosen to be the simulation time, is from* [3].

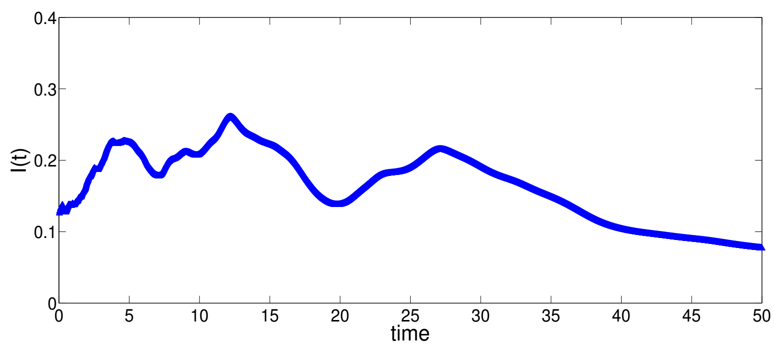

FIGURE 4. Turbulence Intensity, $\nu = 0.001$, $\Delta t = 0.0125$.

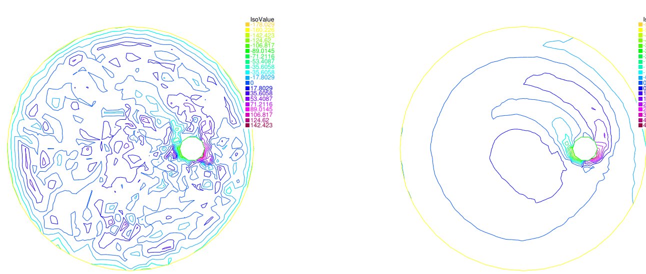

FIGURE 5. Vorticity, $\nu = 0.001$, $\Delta t = 0.0125$. LEFT: t=0; RIGHT: t=50.

We solve the problem as described and give plots of $I(t), D(t), \langle D \rangle (t)$, $r(t)$ and $\gamma_T(t)$ versus time in Figure 4, 6, 7, 8. The turbulence intensity is above 5% during the entire simulation time, see Figure 4, which is normally associated with significant turbulent fluctuations. Figure 5 shows vorticity contours of the freely evolving turbulent flows at $t = 0$ (reinitiation) and $t = 50$ respectively. At $t = 0$, the domain is filled with vortices of a wide range of scales. As time evolves, the size of eddies continually grows, associated with the cascade of energy from small scales to large scales, and then a large scale, coherent structure starts to show up. We can see clearly from Figure 5 a large scale, coherent structure formed at $t = 50$. Figure 6 shows that the Dirichlet quotients approach a constant around 32, consistent with the selective decay theorem for freely decaying 2d Navier-Stokes flows in [27]. The relative energy fluctuation is plotted in Figure 7. For only a short time $r(t)$ is above 0.25, which is the threshold used in [3] to define the predictability time. The difference here is that $r(t)$ grows exponentially for only a short time period ($t = 0 \sim 5$, approximately) and after which $r(t)$ fluctuates with time and is actually decreasing to small values after time $t = 30$. Our explanation is that the dissipation is already active after $t = 5$ and the separation of the two trajectories is slowed down due to lost of energy. The approximated Lyapunov exponent corresponding to different simulation times is plotted in Fig. 8. It is positive until around $t = 44$. After this point (until $t = 50$), the two trajectories are actually closer than they initially were.

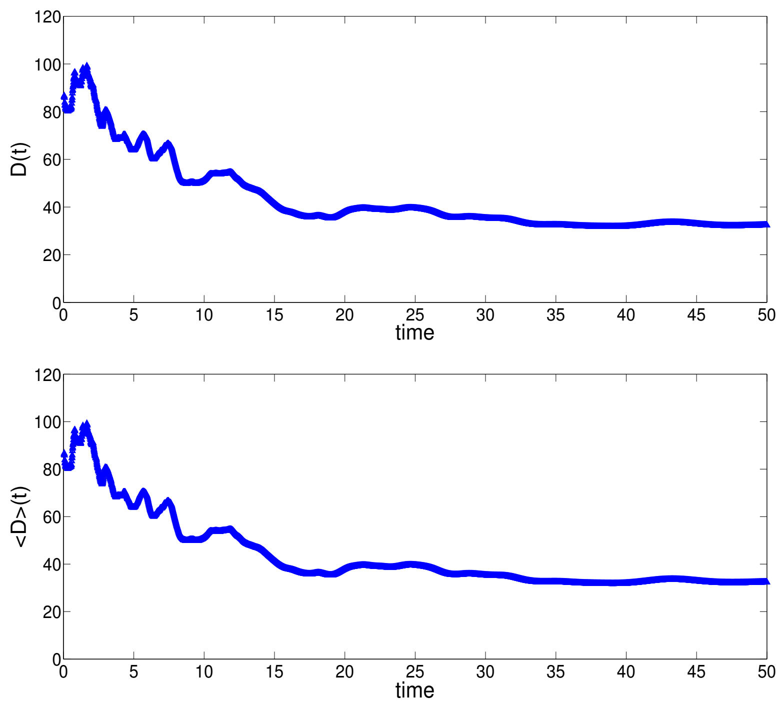

FIGURE 6. Dirichlet Quotients, $\nu = 0.001$, $\Delta t = 0.0125$.

6.3. **3D Ethier-Steinman Flow.** We give a 3D test to confirm the positive effect of the eddy viscosity model for high Reynolds number flows compared to the laminar flow model.

The well-known 3D Ethier-Steinman analytical solutions to the incompressible Navier-Stokes euqations are used in [9] to provide benchmarks for testing Navier-Stokes solvers. The solutions are valid for all Reynolds numbers and complex

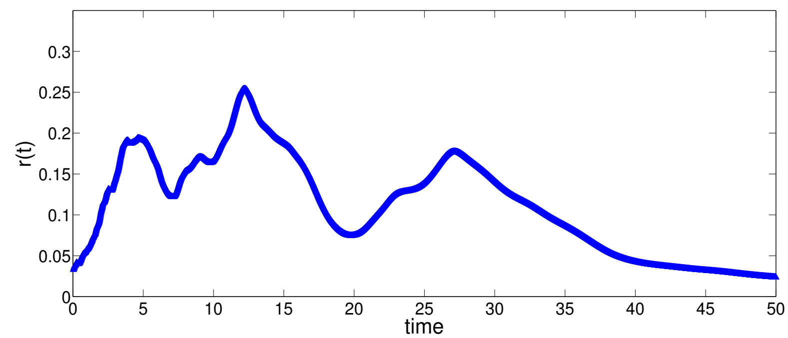

FIGURE 7. Relative Energy Fluctuation, $\nu = 0.001$, $\Delta t = 0.0125$.

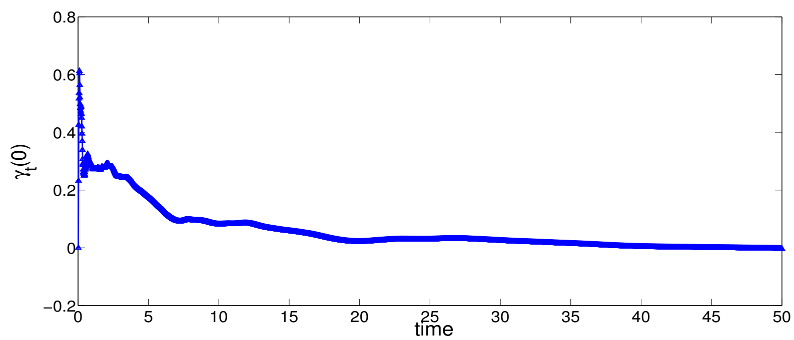

FIGURE 8. Effective Lyapunov Exponent, $\nu = 0.001$, $\Delta t = 0.0125$.

structures may also be expected due to the nontrivial helicty, see [24], [28]. The exact 3D NSE solutions on a $[0,1]^3$ box is given by

$$
\begin{aligned}
u_1 &= -a(e^{ax} sin(ay+dz) + e^{az} cos(ax+dy))e^{-\nu d^2 t} \\
u_2 &= -a(e^{ay} sin(az+dx) + e^{ax} cos(ay+dz))e^{-\nu d^2 t} \\
u_3 &= -a(e^{az} sin(ax+dy) + e^{ay} cos(az+dx))e^{-\nu d^2 t} \\
p &= -\frac{a^2}{2}(e^{2ax} + e^{2ay} + e^{2az} + 2sin(ax+dy)cos(az+dx)e^{a(y+z)} \\
&\quad +2sin(ay+dz)cos(ax+dy)e^{a(z+x)} + 2sin(az+dx)cos(ay+dz)e^{a(x+y)})e^{-2\nu d^2 t}.
\end{aligned} \tag{6.3}
$$

We compute approximations to (6.3) with parameters $a = 1.25, d = 2.25$, kinematic viscosity $\nu = 0.001$, mesh size $h = 0.1$ and end time $T = 1$. Perturbations are generated in the same way as in Section 6.1 with the same parameters $\epsilon_1 = 10^{-3}$, $\epsilon_2 = -10^{-3}$. (EnB) fails for all time steps $\Delta t = 0.05, \Delta t = 0.02$ and $\Delta t = 0.01$, while (EVB) gives acceptable approximations. To visualize the flow structure of the test problem, we plot streamribbons in the box, velocity streamlines and speed contours on the sides. The exact velocity field and the average velocity calculated from (EVB) are given in Figure 9, with $\Delta t = 0.02$.

We plot energy versus time in Figure 10. From the figure, we can easily see, (EnB) becomes unstable at time around $t = 0.5$ for all cases $\Delta t = 0.05, \Delta t = 0.02$ and $\Delta t = 0.01$, while (EVB) stays stable for all cases, giving better approximations as timestep reduces.

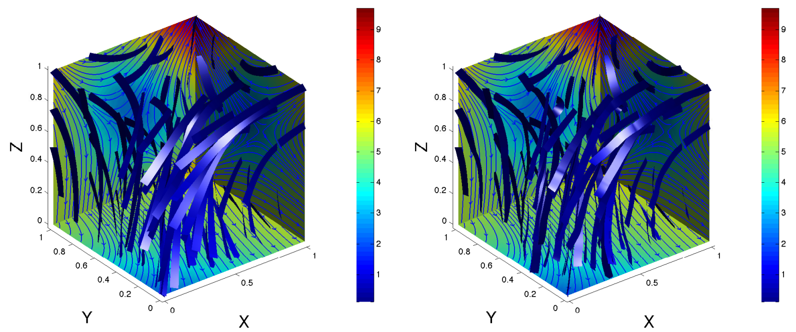

FIGURE 9. Flow structure for $\nu = 0.001$, $\Delta t = 0.02$. LEFT: Exact solution. Right: EV model average velocity.

7. Conclusion

We propose an efficient, second order accurate ensemble method to calculate Navier-Stokes equations and a turbulence model to simulate high Reynolds number flows. We believe the method has great potential for many important applications, in which the size of the ensemble used has a great impact on the reliability of prediction but is limited by computer resources. We emphasize the importance of the new definition of ensemble mean, adapted to be compatible with the time discretization. Numerical tests show the superiority of the ensemble turbulence model and give some indication as to its potential use in exploring various aspects of turbulent flows such as turbulence intensity, Dirichlet quotients and effective Lyapunov exponent. The study of the efficiency of the method proposed is the next step we will take.

References

[1] U.M. Ascher, S.J. Ruuth and W.T.R. Wetton, *Implicit-explicit methods for time-dependent partial differential equations*, SIAM J. Numer. Anal., 32 (1995), 797-823.

[2] M. Anitescu and W.J. Layton, *Sensitivities in large eddy simulation and improved estimates of turbulent flow functionals*, SIAM Journal of Scientific Computing, 29 (2007), 1650-1667.

[3] G. Boffetta, A. Celani, A. Crisanti and A. Vulpiani, *Predictability in two-dimensional decaying turbulence*, Phys. Fluids, 9 (3) (1997), 724-734.

[4] L.C. Berselli, *On the large eddy simulation of the Taylor-Green vortex*, J. Math. Fluid Mech., 7 (2005), S164-S191.

[5] S. Brenner and R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 3rd edition, 2008.

[6] E. Burman and A. Linke, *Stabilized finite element schemes for incompressible flow using Scott–Vogelius elements*, Applied Numerical Mathematics, 58 (2008), 1704-1719.

[7] M. Carney, P. Cunningham, J. Dowling and C. Lee, *Predicting Probability Distributions for Surf Height Using an Ensemble of Mixture Density Networks*, International Conference on Machine Learning, (2005).

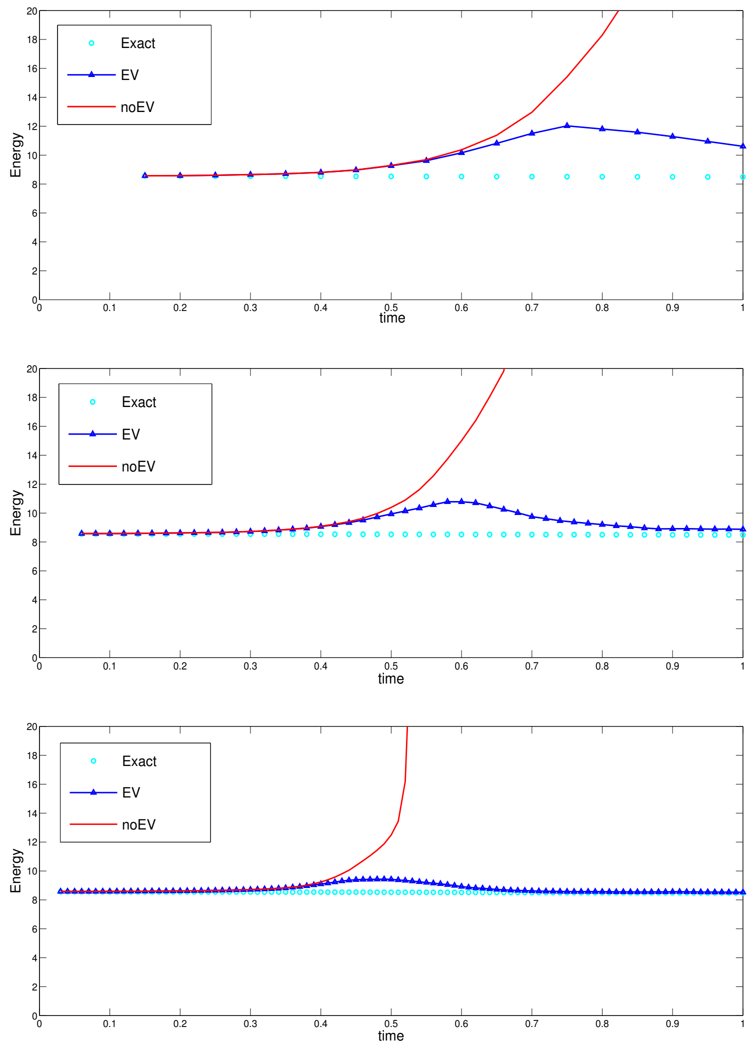

FIGURE 10. Kinematic Energy for $\nu = 0.001$. FIRST: $\Delta t = 0.05$; SECOND: $\Delta t = 0.02$; THIRD: $\Delta t = 0.01$

[8] B. Cockburn, G. Kanschat and D. Schötzau, *A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations*, J. Sci. Comput., 31 (2007), 61-73.

[9] C. Ethier and D. Steinman, *Exact fully 3D Navier-Stokes solutions for benchmarking*, Int. J. Numer. Methods Fluids, 19 (5) (1994), 369-375.

[10] Y. T. Feng, D. R. J. Owen and D. Peric, *A block conjugate gradient method applied to linear systems with multiple right hand sides*, Comp. Meth. Appl. Mech. & Engng. 127 (1995), 203-215.

[11] R. W. Freund and M. Malhotra, *A block QMR algorithm for non-Hermitian linear systems with multiple right-hand sides*, Linear Algebra and its Applications, 254 (1997), 119-157.

[12] E. Gallopulos and V. Simoncini, *Convergence of BLOCK GMRES and matrix polynomials*, Lin. Alg. Appl., 247 (1996), 97-119.

[13] C. Greif, D. Li, D. Schötzau and X. Wei, *A mixed finite element method with exactly divergence-free velocities for incompressible magnetohydrodynamics*, Comput. Methods Appl. Mech. Engrg., 199 (2010), 2840-2855.
[14] J.D. Giraldo and S.G. García Galiano, *Building hazard maps of extreme daily rainy events from PDF ensemble, via REA method, on Senegal River Basin*, Hydrology and Earth System Sciences, 15 (2011), 3605-3615.
[15] M.D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows - A Guide to Theory, Practices, and Algorithms*, Academic Press, (1989).
[16] V. Girault and P. Raviart, *Finite element approximation of the Navier-Stokes equations*, Lecture Notes in Mathematics, Vol. 749, Springer, Berlin, 1979.
[17] A.E. Green and G.I. Taylor, *Mechanism of the production of small eddies from larger ones*, Proc. Royal Soc. A., 158 (1937), 499-521.
[18] J.L. Guermond and L. Quartapelle, *On stability and convergence of projection methods based on pressure Poisson equation*, IJNMF, 26 (1998), 1039-1053.
[19] N. Jiang and W. Layton, *An algorithm for fast calculation of flow ensembles*, International Journal for Uncertainty Quantification, accepted, Dec 2013, available at: TR-MATH 13-06, Technical Report, Department of Mathematics, University of Pittsburgh.
[20] N. Jiang and W. Layton, *Numerical Analysis of two Ensemble Eddy Viscosity Models of Fluid Motion*, submitted, July 2013, available at: available at: TR-MATH 13-09, Technical Report, Department of Mathematics, University of Pittsburgh.
[21] J.M. Lewis, *Roots of ensemble forecasting*, Monthly Weather Rev., 133 (2005), 1865-1885.
[22] G. Karniadakis, M. Israeli and S. Orszag, *High-order splitting methods for the incompressible Navier-Stokes equations*, J. Comput. Phys., 97 (1991), 414-443.
[23] O.P Le Maitre and O.M. Kino, *Spectral methods for uncertainty quantification*, Springer, Berlin, 2010.
[24] H.K. Moffatt and A. Tsinober, *Helicity in laminar and turbulent flow*, Ann. Rev. Fluid Mech., 24(1992), 281-312.
[25] M. Leutbecher and T.N. Palmer, *Ensemble forecasting*, J. Comp. Phys., 227 (2008), 3515-3539.
[26] W.J. Martin and M. Xue, *Initial condition sensitivity analysis of a mesoscale forecast using very-large ensembles*, Mon. Wea. Rev., 134 (2006), 192-207.
[27] A. Majda and X. Wang, *Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows*, Cambridge, UK, 2006.
[28] M.A. Olshanskii and L.G. Rebholz, *Velocity-vorticity-helicity for formulation and a solver for the Navier-Stokes equations*, J. Comp, Phys., 229 (2010), 4291-4303.
[29] D. P. O'Leary, *The block conjugate gradient algorithm and related methods*, Linear Algebra and its Applications, 29 (1980), 293–322.
[30] Z. Toth and E. Kalney, *Ensemble forecasting at NMC: The generation of perturbations*, Bull. Amer. Meteor. Soc., 74 (1993), 2317-2330.
[31] J. Varah, *Stability restrictions on second-order, three level finite difference schemes for parabolic equations*, SIAM J. Numer. Anal., 17 (1980), 300-309.
[32] X. Wang, *An efficient second order in time scheme for approximating long time statistical properties of the two dimensional Navier-Stokes equations*, Numer. Math., 121 (2012), 753-779.
[33] F.J. Walburn, H.N. Sabbah and P.D. Stein, *An experimental evaluation of the use of an ensemble average for the calculation of turbulence in pulsatile flow*, Ann Biomed Eng., 11 (1983), 385-399.
[34] S. Zhang, *A family of Qk+1, k×Qk, k+1 divergence-free finite elements on rectangular grids*, SIAM J. Numer. Anal., 47 (2009), 2090-2107.

(All authors) Dept. of Mathematics, Univ. of Pittsburgh, Pittsburgh, PA 15260, USA
E-mail address: naj24@pitt.edu
URL: www.math.pitt.edu/~naj24