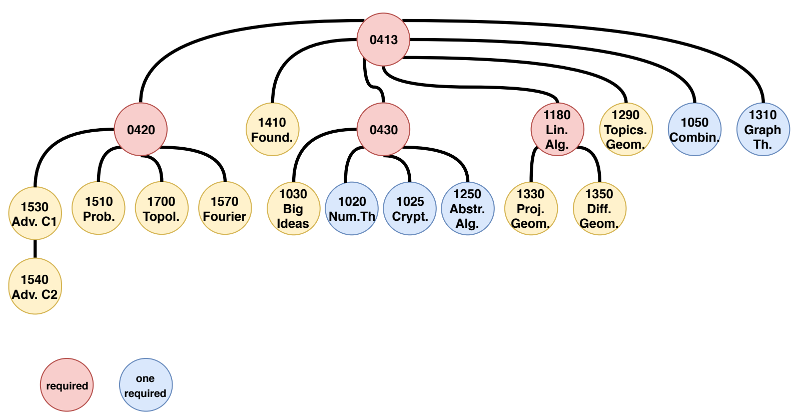
MATH 413 - INTRODUCTION TO THEORETICAL MATHEMATICS

1. Math 413 revision



The Pitt math department voted to revise the 413 syllabus as part of a package of undergraduate reforms presented in the strategic plan. Moving forward, there will be emphasis throughout the course on learning to write and analyze proofs and on developing abstract reasoning skills.

Currently Math 0413 serves as a prerequisite to Math 0420, Math 0430, and Math 1180 (which are all required for our major in Mathematics), as well as for a number of other courses in geometry and combinatorics, all of which are proof-based courses. See the dependency tree. The revisions are being made in order to allow all these courses to have a proper foundation and be served with equal efficiency, to strengthen the proof-writing ability of the students and increase the likelihood of their success in subsequent courses.

The course will no longer be as closely tied to 420 as it once was, and some material related to series and sequences will be moved from 413 to 420. Math 413 is not a course in analysis, but basic ideas in analysis may be used as a backdrop to illustrate proof techniques.

Catalog (red insertions, strikeout deletions):

- Description: This course is an introduction to the theoretical treatment of logic, sets, functions, relations, numbers, and proofs ~~sequences, and limits~~. Classwork and homework concentrate reading and writing of proofs of theorems centered on these topics.
- Course Requirements: PREQ: MATH 0230 or 0235 or 0231 or 0150; and ENG 0102 or ENGCMP (0002 or 0006 or 0020 or 0200 or 0203 or 0205 or 0207 or 0208 or 0210 or 0212) or ENGFLM 0210 or FP (0003 or 0006)
- Course Attributes: SCI Quantitative: Mathematics GE. Req., Writing Requirement Course

2. Course Organization

2.1. **Perspective.** This course is designed for a 13-week semester, covering logic and proof (4 weeks), sets and functions (3 weeks), and number systems (5 weeks). Students who enroll in the course are not required to have any previous experience in logic, reading and writing proofs, set theory, axiom systems, or careful reasoning about number systems. The hope is that students completing this course will have substantial practice and proficiency with all of these topics.

This course is a prerequisite for many other courses in the major. As such, the syllabus should be viewed as being as tightly controlled as it is for the courses in the calculus sequence.

Math 1410 (Foundations of Math) is intended as a more in-depth treatment of the foundations of set theory. In Math 413, students should become somewhat aware that set theory is ultimately based on a collection of axioms (the Zermelo Fraenkel axioms) as developed through logical argument, but it is not necessary for the students to learn the axioms. In short, Math 413 operates at the level of what is called *naive set theory*, while letting students know that there is a level of rigor beyond naive set theory. A few axioms (extensionality, specification, choice, and infinity) are specifically mentioned topics in the 413 syllabus. Again, these axioms can be treated at a naive level. For example, after defining what an infinite set is, the axiom of infinity can be explained by stating that none of the other axioms of set theory are powerful enough to prove that the existence of an infinite set, that a separate axiom is needed to do so, and that the axiom is invoked to prove the existence of the set of natural numbers and other infinite sets.

Students should learn that every real number is a set, but it is not necessary to teach how the real numbers are constructed. That is, constructions of the real numbers as Dedekind cuts or as equivalence classes of Cauchy sequences *are not included* in the syllabus. Similarly, when treating cardinality, there is no need to construct cardinals from the ground up. Cardinals can be taken as axiomatically given (as presented in Newstead's book, described in the next section).

2.2. **Textbook.** The recommended textbook is Clive Newstead's *Infinite Descent into Pure Mathematics* (ID) [New]. The book is regularly updated online. The instructor should use the most recent *unstable build* of the book and fix the version used in a given section by posting it on Canvas for the students to download. The book runs several hundred pages, and Math 413 only covers some of the sections. There is currently no printed version of the book, but because of Creative Commons licensing, Pitt has permission to print chapters or the entire book if desired.

An alternative textbook is *A Transition to Advanced Mathematics* (TAM), by Douglas Smith, Maurice Eggen, and Richard St. Andre [SESA]. The books TAM and ID are briefly reviewed in Sections 3.1 and 3.2. The alignment of the books with our syllabus is given in Section 4.

Influential factors in choosing Newstead's ID are that it is free online and that Clive Newstead (a CMU Teaching Assistant Professor) has been very willing to help in the planning stages of this course. About a thousand undergraduates per year take some version of this course at CMU, where the course is called *Concepts of Mathematics*.

Other books that have been considered in earlier reform 413 proposals include [ALvD], [Lip], [Leb]. There are several textbooks that are entirely focused on writing proofs [Fuc], [TR], [Nic], [Ham], [Sol]. Sundstrom has two books on proofs that have been released under a Creative Commons license that are appropriate resources for the first part of the semester [Sun20], [Sun14]. Other textbooks on proof that haven't been reviewed in relation to 413 include [CPZ13], [Kra], [Bie] [Vel19] [Cum] [Ger12]. *Proofs from the Book* is a recommended collection of model proofs [AZ99].

2.3. **Assessment.** The course will assess students frequently in the form of quizzes, homework and exams. The recommended assessment is as follows.

- 5% quizzes after every lecture. (Note: eventually, this will be automated in WebWork.) The quizzes will focus on drills, mechanical skills, and lecture comprehension. They are intended as a study aid for students, and are only a small part of the grade.
- 20% Weekly homework. The homework will focus on writing skills.
- 50%=10%+20%+20% Three midterm exams
- 25% Cumulative final exam

Note: The next stage of preparation for this course will be to develop a large collection of problems suitable for quizzes, homework, and exams. Newstead's *Infinite Descent* is a poor source of exercises.

2.4. **Writing Requirement.** This course fulfills a writing requirement. Lectures and assessment should emphasize principles of *writing well*, and not merely the correctness of proofs.

Fall 2024 project: we will ask for Trofimov's teaching seminar to make it an agenda item to draft a several page document that describes the standards for writing proofs that apply to undergraduates at Pitt. This should be a document standardized and approved by the faculty.

The references contain some recommended guides on writing mathematics well [Knu], [Gil], [Su], [Lee], [Ser], [Tao], [Poo], [Con]. Many of these references are online and can be worked into the syllabus.

3. Book Reviews

3.1. **Review: A Transition to Advanced Mathematics (TAM).** The book is based on a course at Central Michigan University.

From the preface: "This text is intended to bridge the gap between calculus and advanced courses in at least three ways. First, it provides a firm foundation in the major ideas needed for continued work. Second, it guides students to think and to express themselves mathematically – to analyze a situation, extract pertinent facts, and draw appropriate conclusions. Finally, we present introductions to modern algebra and analysis in sufficient depth to capture some of their spirit and characteristics."

- Authors: Douglas Smith, Maurice Eggen, Richard St. Andre
- This is a popular book, now in the 8th edition.
- The book has an attractive layout, and concepts are explained in simple terms.
- The book is expensive, list price $290, Amazon prize $92.
- `https://archive.org/details/transitiontoadva0000smit` (one-hour checkout on Internet Archive)
- If we use this book, we might need to write small supplements to this text on writing proofs and set theory.

3.2. **Review: Infinite Descent (ID) by Newstead.** From the preface: "The goal of this book is to help the reader make the transition from being a consumer of mathematics to a producer of it. This is what is meant by 'pure? mathematics. While a consumer of mathematics might learn the chain rule and use it to compute a derivative, a producer of mathematics might derive the chain rule from the rigorous definition of a derivative, and then prove more abstract versions of the chain rule in more general contexts (such as multivariate analysis).. . . "

"It is this transition from consumption to production of mathematics that guided the principles I used to design and write this book. In particular

- **Communication.** Above all, this book aims to help the reader to obtain mathematical literacy and express themselves mathematically. . .
- **Inquiry.** The research is clear that people learn more when they find things out for themselves. . .
- **Strategy.** Mathematical proof is much like a puzzle. At any given stage in a proof, you will have some definitions, assumptions and results that are available to be used, and you must piece them together using the logical rules at your disposal. Throughout the book, and particularly in the early chapters, I have made an effort to highlight useful proof strategies whenever they arise.
- **Content.** There isn't much point learning about mathematics if you don't have any concepts to prove results about. With this in mind, Part II includes several chapters dedicated to introducing some topic areas in pure mathematics, such as number theory, combinatorics, analysis and probability theory.
- **LaTeX.** The de facto standard for typesetting mathematics is LaTeX. I think it is important for mathematicians to learn this early in a guided way, so I wrote a brief tutorial in Appendix D and have included LaTeX code for all new notation as it is defined throughout the book."

- Author: Clive Newstead (CMU). The author of this book Clive Newstead is a PhD from Carnegie Mellon (under Steve Awodey) and is now an instructor there.
- Latest edition: January 10th 2024. (This *unstable* version has more exercises the the official *stable* version.)

4. Coverage of Topics

The page numbers in the first two columns show the correspondence between 413 topics and pages in the books TAM and ID (unstable build, Jan 10, 2024).

Logic and Proof (4 weeks)

TAM	ID	Topic
✓	p26	**Propositional logic** (1 week)
✓		syntax,
p1		truth value,
p4	p66	truth tables,
p4	p77	tautology, fallacy,
p9		conditional statement,
p19	p63	logical equivalence,
	p72	De Morgan's laws,
p29		rules of inference,
✓		**First order logic** (1 week)
p18	p49	variables,
p18		predicate,
p19-20	p49,54	quantifiers,
p20		syntax,
	p49	free versus bound variables,
p18		truth domain,
✓	p513	**Proofs and Proof Strategies** (2 weeks and throughout the semester)
p36		case analysis,
p40-41,45		proof by contrapositive,
p41ff	p529	proof by contradiction,
?	p532	without loss of generality,
p56	p73	producing examples (to prove existence) and counterexamples (to prove nonexistenc
	p525	forward chaining of (steps linked by $\Rightarrow$) vs. backward chaining (steps linked by $\Leftarrow$)
p74		breaking iff into $\Rightarrow$ and $\Leftarrow$; breaking set equality proofs into $\subseteq$ and $\supseteq$.
		(optional) proofs of *the-following-are-equivalent* statements

Sets (3 weeks) In ID, set theory is done informally in Chapter 2 and axiomatically in Appendix B.1.

TAM	ID	Topic
✓	Ch.2,p538	**Axiomatic systems** (1.3 weeks)
p72	p540	empty set,
	p539	extensionality axiom,
p73		equality,
p72		inclusion,
	p541	specification (separation) axiom,
		union, intersection, complement,
		indexed families,
		analogy between boolean operations in logic and sets,
	p104	Cartesian product,
	p540	power set,
	p541	infinite set axiom,
	p138	axiom of choice,
	Ch.3,5	**Relations and Functions** (1 week)
	p188	binary relations, directed graphs,
		order relations, min/max,
	p198	equivalence relations, equivalence classes,
	Ch.3	functions,
	p114	domain, co-domain, image, preimage,
	p126	compatibility of image/preimage with set operations,
	p122	composition, restriction,
	p130	injective, surjective, bijection, permutation, inverse, (with special treatment for finite sets)
	Ch6,Ch10	**Cardinality** (0.7 week)
	p422	cardinality of Cartesian products,
		cardinality of unions/intersections,
		cardinality of powers sets,
	p214,234	finite, countable, uncountable sets,
		injective/surjective functions between finite sets,
	p413	bijections between infinite sets, Cantor's power set theorem,
	§8.1	counting: Inclusion-exclusion, double counting
	p20,107,556	**Binary operations**

Notes: ✓ means slight coverage. ✓ means adequate coverage.

Number Systems (5 weeks)

TAM	ID	Topic
	§4.1	**Natural Numbers and Peano's axioms,** (1.5 weeks)
		arithmetic operations and order relation,
	p153	principle of recursive definition,
	p159,174	principle of mathematical induction and variations,
	p179	well ordering principle,
		unboundedness of $\mathbb{N}$,
	p157	binomial coefficients, the binomial formula,
	p7,12,530	**Integers and rational numbers** (1 week)
		integers, operations, order relation,
		rational numbers, operations, order relation,
		countability of $\mathbb{Z}$ and $\mathbb{Q}$,
		(brief review of arithmetic: divisibility, prime numbers, greatest common divisor)
	p12,Ch9,B2	**Transition to real analysis** (2.5 weeks)
		real numbers, operations, order relation,
	p357	completeness,
	p553-558	real numbers as a complete ordered field
		archimedean property
		density of $\mathbb{Q}$,

5. Sample Daily Schedule

Sample ordering of material.

(1) Propositional logic, proofs in propositional logic
(2) Boolean operations on sets (equality, inclusion, union, intersection, complement, analogy with boolean operations in logic)
(3) First order logic, proofs in first order logic
(4) Further topics in set theory and axioms, relations, and functions
(5) Proof methods
(6) Cardinality
(7) Natural numbers
(8) More on proofs, including induction and recursion
(9) Integers and rational numbers
(10) Transition to real analysis

Note: Insert sample daily schedule.

References

[ALvD] J Avigad, R Lewis, and F van Doorn. Logic and proof. This book is geared toward learning proof through computer interaction with the *Lean* theorem prover. van Doorn was a Pitt postdoc.

[AZ99] Martin Aigner and Günter M Ziegler. Proofs from the book. *Berlin. Germany*, 1(2):7, 1999.

[Bie] Thomas Bieske. An introduction to writing mathematical proofs: Shifting gears from calculus to upper-level mathematics classes.

[Con] Keith Conrad. Advice on mathematical writing. `https://kconrad.math.uconn.edu/blurbs/proofs/writingtips.pdf`.

[CPZ13] Gary Chartrand, Albert D Polimeni, and Ping Zhang. Mathematical proofs. *Kanada: Aptara Inc*, 2013.

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[Lee] Kevin Lee. A guide to writing mathematics. `https://web.cs.ucdavis.edu/~amenta/w10/writingman.pdf`.

[Lip] Seymour Lipschutz. Schaum's outline of set theory and related topics.

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[Vel19] Daniel Velleman. How to prove it: A structured approach, 2019. Cambridge University Press.