Preliminary Exam May 2019

Problem 1. Let $\sigma > 0$. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions $f_k : \mathbb{R} \to \mathbb{R}$ with $f_k(0) = 0$. Moreover let $(A_k)_{k\in\mathbb{N}} \subset [0,\infty)$ be a *bounded* sequence of real numbers such that

$$|f_k(x) - f_k(y)| \le A_k|x-y|^\sigma \quad \text{for all } x, y \in \mathbb{R}.$$

(a) Show that there exists $f : \mathbb{R} \to \mathbb{R}$ such that a subsequence f_{k_i} converges uniformly to f in every interval $[-a, a]$, $a > 0$.
(b) Show that f satisfies

$$|f(x) - f(y)| \le A|x-y|^\sigma$$

where $A = \liminf_{k\to\infty} A_k$.

Problem 2. Prove that if X is a metric space and $f : X \times [0,1] \to \mathbb{R}$ is a continuous function, then $g : X \to \mathbb{R}$, defined by $g(x) = \sup_{t\in[0,1]} f(x,t)$, is continuous.

Problem 3. Prove (using only the material covered in the course) that there is no continuous and one-to-one function $f : \mathbb{R}^2 \to \mathbb{R}$. **Hint:** *Assume that such a function exists and then restrict the function to the unit circle in* $\mathbb{R}^2$.

Problem 4. Suppose $f : \mathbb{R}^2_+ \to \mathbb{R}$ is a continuous function defined on

$$\mathbb{R}^2_+ = \{(x,y) : x \in \mathbb{R},\ y > 0\}.$$

Assume also that the limits

$$g(u,v) = \lim_{t\to 0} \frac{f((u+t)\cos v, (u+t)\sin v) - f(u\cos v, u\sin v)}{t},$$

and

$$h(u,v) = \lim_{t\to 0} \frac{f(u\cos(v+t)), u\sin(v+t)) - f(u\cos v, u\sin v)}{t},$$

exist and define continuous functions g, h on the domain

$$D = \{(u,v) : u > 0,\ 0 < v < \pi\}.$$

Prove that the function f is differentiable on $\mathbb{R}^2_+$.

Problem 5. Let $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open and bounded. Let $\Delta f = \sum_{i=1}^n \partial^2 f/\partial x_i^2$ be the Laplace operator.

(a) Show that if for some $\varepsilon > 0$ and $x_0 \in \Omega$ we have $\Delta f(x_0) \ge \varepsilon$, then f has no local maximum at x_0.
(b) Conclude that if $\Delta f(x) \ge \varepsilon$ for some $\varepsilon > 0$ and all $x \in \Omega$, then we have $\sup_\Omega f = \sup_{\partial\Omega} f$.
(c) Conclude that if $\Delta f(x) \ge 0$ for all $x \in \Omega$, then we have $\sup_\Omega f = \sup_{\partial\Omega} f$.

Hint for part (c): *Observe that* $\Delta|x|^2 = 2n$. *Use it to modify a function* f *in (c) so that you can apply part (b).*

Problem 6. For $x = (x_1, x_2) \in \mathbb{R}^2$, let $|x| = \sqrt{x_1^2 + x_2^2}$. Let $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ and let $f : \overline{D} \to \mathbb{R}$ be continuous on $\overline{D}$. Prove that

$$\lim_{n\to\infty} \iint_D (n+2)|x|^n f(x)\, dA = \int_0^{2\pi} f(\cos t, \sin t)\, dt.$$