Preliminary Exam January 2021

Problem 1. Suppose that $a_{ij} \leq 1$ for all $i, j \in \mathbb{N}$. Suppose also that

$$\forall i \quad a_{i1} \leq a_{i2} \leq a_{i3} \leq \dots \quad \text{and} \quad \forall j \quad a_{1j} \leq a_{2j} \leq a_{3j} \leq \dots$$

Prove that the following iterated limits exist and they are equal

$$\lim_{i\to\infty} \lim_{j\to\infty} a_{ij} = \lim_{j\to\infty} \lim_{i\to\infty} a_{ij}.$$

Problem 2. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix.

(a) Show that there exists $\Lambda > 0$ such that $|x| \leq \Lambda |Ax|$ for all $x \in \mathbb{R}^n$.
(b) Let $0 \leq \lambda < \Lambda^{-1}$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with $|f(x) - f(y)| \leq \lambda |x - y|$ for all $x, y \in \mathbb{R}^n$. Show that for any $y \in \mathbb{R}^n$, there exists exactly one $x \in \mathbb{R}^n$ satisfying

$$Ax + f(x) = y.$$

Problem 3. Let $n \geq 2$.

(a) Show that there is a non-constant map $F : \mathbb{R}^n \to \mathbb{R}^n$ of class C^∞ such that

$$\langle F(x) - F(y), x - y \rangle = 0 \quad \text{for all } x, y \in \mathbb{R}^n. \tag{1}$$

(b) Show that if $f \in C^\infty(\mathbb{R}^n)$ and $F = \nabla f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1), then F is constant.

Hint: *To prove (b), use the definition of the directional derivative, apply it to $F = \nabla f$ and use (1) to obtain some identity for $D^2 f$. Then prove that $D^2 f = 0$ and conclude (with a short proof) that $F = \nabla f$ is constant. At some point you might need some facts from linear algebra about symmetric matrices.*

Problem 4. Let $\{f_n\}_{n=1}^\infty$ be a uniformly bounded equicontinuous sequence of real-valued functions on a compact metric space (X, d).

(a) Prove that the family $g_n(x) = \max\{f_1(x), \dots, f_n(x)\}$, $n \in \mathbb{N}$ is uniformly bounded and equicontinuous.
(b) Prove that the sequence $\{g_n\}_{n=1}^\infty$ converges uniformly on X.

Problem 5. For $a = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$, $a_n \neq 0$ let $P_a(x) = a_n x^n + \dots + a_1 x + a_0$. Suppose that for $a^o = (a^o_0, \dots, a^o_n)$, $a^o_n \neq 0$ the polynomial $P_{a^o}(x)$ has n distinct real roots. Prove that there is $\varepsilon > 0$ and C^∞ smooth functions

$$\lambda_1, \dots, \lambda_n : B^{n+1}(a^o, \varepsilon) \to \mathbb{R}$$

such that for any $a \in B^{n+1}(a^o, \varepsilon)$, $\lambda_1(a), \dots, \lambda_n(a)$ are distinct roots of the polynomial $P_a(x)$. In other words, prove that in a small neighborhood of a^o, roots of the polynomial P_a depend smoothly on the coefficients $a_0, a_1, \dots, a_n$.

Problem 6. Let $\mathbb{T} := [0, 1]^2$ and assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\mathbb{Z}^2$-periodic, that is

$$f(x + k) = f(x) \quad \text{for all } x \in \mathbb{R}^2 \text{ and all } k \in \mathbb{Z}^2.$$

Prove that

$$\int_{\mathbb{T}} f(x)\, dx = \int_{\mathbb{T}} f(x + z)\, dx \quad \text{for any } z \in \mathbb{R}^2.$$