Preliminary Exam January 2021

Problem 1. Suppose that \( a_{ij} \leq 1 \) for all \( i, j \in \mathbb{N} \). Suppose also that
\[
\forall i \quad a_{i1} \leq a_{i2} \leq a_{i3} \leq \ldots \quad \text{and} \quad \forall j \quad a_{1j} \leq a_{2j} \leq a_{3j} \leq \ldots
\]
Prove that the following iterated limits exist and they are equal
\[
\lim_{i \to \infty} \lim_{j \to \infty} a_{ij} = \lim_{j \to \infty} \lim_{i \to \infty} a_{ij}.
\]

Problem 2. Let \( A \in \mathbb{R}^{n \times n} \) be an invertible matrix.

(a) Show that there exists \( \Lambda > 0 \) such that \( |x| \leq \Lambda |Ax| \) for all \( x \in \mathbb{R}^n \).
(b) Let \( 0 \leq \lambda < \Lambda^{-1} \) and let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a Lipschitz map with \( |f(x) - f(y)| \leq \lambda|x - y| \) for all \( x, y \in \mathbb{R}^n \). Show that for any \( y \in \mathbb{R}^n \), there exists exactly one \( x \in \mathbb{R}^n \) satisfying
\[
Ax + f(x) = y.
\]

Problem 3. Let \( n \geq 2 \).

(a) Show that there is a non-constant map \( F : \mathbb{R}^n \to \mathbb{R}^n \) of class \( C^\infty \) such that
\[
(F(x) - F(y), x - y) = 0 \quad \text{for all} \quad x, y \in \mathbb{R}^n.
\]
(b) Show that if \( f \in C^\infty(\mathbb{R}^n) \) and \( F = \nabla f : \mathbb{R}^n \to \mathbb{R}^n \) satisfies (1), then \( F \) is constant.

Hint: To prove (b), use the definition of the directional derivative, apply it to \( F = \nabla f \) and use (1) to obtain some identity for \( D^2 f \). Then prove that \( D^2 f = 0 \) and conclude (with a short proof) that \( F = \nabla f \) is constant. At some point you might need some facts from linear algebra about symmetric matrices.

Problem 4. Let \( \{f_n\}_{n=1}^{\infty} \) be a uniformly bounded equicontinuous sequence of real-valued functions on a compact metric space \( (X, d) \).

(a) Prove that the family \( g_n(x) = \max\{f_1(x), \ldots, f_n(x)\} \), \( n \in \mathbb{N} \) is uniformly bounded and equicontinuous.
(b) Prove that the sequence \( \{g_n\}_{n=1}^{\infty} \) converges uniformly on \( X \).

Problem 5. For \( a = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1} \), \( a_n \neq 0 \) let \( P_a(x) = a_nx^n + \ldots + a_1x + a_0 \). Suppose that for \( a^0 = (a_0^0, \ldots, a_n^0) \), \( a_n^0 \neq 0 \) the polynomial \( P_{a^0}(x) \) has \( n \) distinct real roots. Prove that there is \( \varepsilon > 0 \) and \( C^\infty \) smooth functions
\[
\lambda_1, \ldots, \lambda_n : B^{n+1}(a^0, \varepsilon) \to \mathbb{R}
\]
such that for any \( a \in B^{n+1}(a^0, \varepsilon) \), \( \lambda_1(a), \ldots, \lambda_n(a) \) are distinct roots of the polynomial \( P_a(x) \). In other words, prove that in a small neighborhood of \( a^0 \), roots of the polynomial \( P_a \) depend smoothly on the coefficients \( a_0, a_1, \ldots, a_n \).

Problem 6. Let \( T := [0, 1]^2 \) and assume that \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous and \( \mathbb{Z}^2 \)-periodic, that is
\[
f(x + k) = f(x) \quad \text{for all} \quad x \in \mathbb{R}^2 \quad \text{and all} \quad k \in \mathbb{Z}^2.
\]
Prove that
\[
\int_T f(x) \, dx = \int_T f(x + z) \, dx \quad \text{for any} \quad z \in \mathbb{R}^2.
\]