## Preliminary Exam in Analysis, May 2021

**Problem 1.** Let $M \subset \mathbb{R}^n$ be a compact set and let $m_0, m_1 \in M$ be two points. Let $X \subset C^0([0,1], \mathbb{R}^n)$ be the set of all maps $\gamma \in C^0([0,1], \mathbb{R}^n)$ that satisfy

- $\gamma(0) = m_0$, $\gamma(1) = m_1$,
- $|\gamma(x) - \gamma(y)| \leq |x - y|$ for all $x \in [0,1]$, and
- $\gamma(x) \in M$ for all $x \in [0,1]$.

Assume that $X \neq \emptyset$.

(1) Show that $X$ is a compact space with respect to the usual $C^0$-metric.
(2) Conclude that if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then there exists a map $\bar{\gamma} \in X$ such that

$$\int_{[0,1]} f(\bar{\gamma}(t))dt = \inf_{\gamma \in X} \int_{[0,1]} f(\gamma(t))dt$$

**Problem 2.** Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping of class $C^\infty$. Suppose that $f(0) = 0$ and that for some $1 \leq k \leq n$ we have

$$\det \left[ \frac{\partial f_i}{\partial x_j}(0) \right]_{1 \leq i,j \leq k} \neq 0.$$

Prove that there is a diffeomorphism $\Phi : U \to \Phi(U) \subset \mathbb{R}^n$ defined in a neighborhood of $0 \in U \subset \mathbb{R}^n$ such that

$$(f \circ \Phi)(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_k, *, * \ldots *) \quad \text{for all } x \in U,$$

i.e., the first $k$ components of $f \circ \Phi$ are $x_1, \ldots, x_k$, and we do not have information about the remaining components.
In the proof you are allowed to use the inverse function theorem only. If you want to use more complicated arguments like implicit function theorem, you have to prove suitable statements from the inverse function theorem.

**Problem 3.** Suppose that $f \in C^\infty(\mathbb{R})$ is such that $f(0) = f'(0) = f''(0) = f'''(0) = \ldots = 0$, there is $x \in (0,1)$ such that $f(x) \neq 0$, and a series $\sum_{n=1}^\infty a_n f^{(n)}(x)$ converges uniformly on $\mathbb{R}$ ($f^{(n)}$ denotes the $n$-th derivative). Prove that $\lim_{n \to \infty} a_n n! = 0$.
(**Hint:** If $\sum_{n=1}^\infty f_n$ converges uniformly on $\mathbb{R}$, what can you say about $\sup_x |f_n(x)|$?)

**Problem 4.** Suppose that $f \in C^1(\mathbb{R}^2)$ is such that $|f(x,y)| \leq 1$ on the unit disc $D = \{(x,y) : x^2 + y^2 \leq 1\}$. Prove that there is $(x_o, y_o)$ in the interior of $D$ such that $|\nabla f(x_o, y_o)| < 4$.
(**Hint:** Consider the function $g(x,y) = f(x,y) + 2(x^2 + y^2)$).

**Problem 5.** Suppose that $f : X \to Y$, $f(X) = Y$, is a continuous and surjective map between compact metric spaces. Prove that if $Y$ is connected and for every $y \in Y$, the preimage $f^{-1}(y)$ is connected, then $X$ is connected.

**Problem 6.** Suppose that given functions $f, g, h : \mathbb{R}^n \to \mathbb{R}$ and $x_o \in \mathbb{R}^n$ satisfy

(a) $g \leq f \leq h$ on $\mathbb{R}^n$,
(b) $g(x_o) = f(x_o) = h(x_o)$,
(c) functions $g$ and $h$ are differentiable at $x_o$.

Prove that $f$ is differentiable at $x_o$.