A Test of a Modular, Wall Adapted Nonlinear Filter Model for Underresolved Flows

W. Layton and A. Takhirov

ABSTRACT. Stabilization using filters is intended to model and extract the energy lost to resolved scales due to nonlinearity breaking down resolved scales to unresolved scales. This process is highly nonlinear. We consider nonlinear filters which select eddies for damping (simulating breakdown) based on knowledge of how nonlinearity acts in real flow problems. The particular form of the nonlinear filter allows for easy incorporation of more knowledge into the filter process and its computational complexity is comparable to calculating a linear filter of similar form. Herein we show how to adapt nonlinear filters to the near wall region, adapting an idea of Nicoud and Ducros, give a convergence result for the wall-adapted method and give a test which shows eddy viscosity can be highly localized and produce an excellent result.

1. Introduction

Nonlinear filtering (NLF), recently introduced in [**20**], gives an approach to modeling and simulation of turbulent flows that

- has a strong mathematical foundation,
- yields new turbulence models that can be evolved to greater accuracy and reliability and
- provides a modular implementation of any selected turbulence models within legacy codes, laminar flow codes and complex application codes.

The physical idea behind nonlinear filtering is that a turbulence model should act as a proxy for the action of nonlinearity upon marginally resolved structures. Nonlinearity does not break down scales uniformly. Intermittence, nonuniformity, locality and backscatter occur. If nonlinearity breaks down a local structure, the model should strongly and locally damp the structure while if nonlinearity allows such a structure to persist the modeling terms should be negligible locally. If this physical idea is realized correctly, the NLF model immediately corrects through indicator functions the over damping of persistent, transitional, recirculation and other flows in eddy viscosity models. The examples given in [**20**] of indicator functions and their associated eddy viscosity models are all adapted to free turbulence (away from walls). Herein we show in Section 2.2 how to use the WALE eddy viscosity of Nicoud and Ducros [**24**] to construct an indicator function for the nonlinear filter model that is adapted to the important near wall region.

2000 *Mathematics Subject Classification.* Primary 76F65; Secondary 65M12.

Key words and phrases. turbulence, eddy viscosity, nonlinear filter.

The work of WL and AT was partially supported by NSF grant DMS-0810385.

This paper is an expanded version of one with a similar title. It is not for publication. The extra material and reformatting is done to make this version easier to read by providing background material and more explanation.

Consider the NSE in a domain $\Omega \subset \mathbb{R}^{2 \text{ or } 3}$

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = f(x,t), \tag{1.1}$$
$$\nabla \cdot u = 0\ ,\ u = 0 \text{ on } \partial\Omega \times (0,T], \text{ and } \ u(x,0) = u^0(x) \text{ in } \Omega.$$

Given a method for the NSE at low or moderate Reynolds numbers, e.g. (LegacyStep) below, the method (Modular NLFilter) adapts it to high Reynolds number flows by adding the modular steps (NLfilter) and (Relax) below. Suppressing the spacial discretization: we advance $u^n \simeq u(t^n)$ to u^{n+1} by:

ALGORITHM 1 (Modular NLFilter). *Pick* $\chi \in [0,1]$ *and* $\triangle t > 0$, *given* u^n, p^n
Step 1: *Find* w^{n+1}

$$\frac{w^{n+1} - u^n}{\triangle t} + w^{n+1} \cdot \nabla w^{n+1} - \nu \triangle w^{n+1} + \nabla p^{n+1} = f^{\,n+1} \tag{LegacyStep}$$
$$\nabla \cdot w^{n+1} = 0 \quad \textit{and} \quad w^{n+1} = 0\ ,\ \textit{on}\ \partial\Omega.$$

Step 2: *Nonlinear filter:* $\quad w^{n+1} \to \overline{w^{n+1}}$ *by*

$$-\delta^2 \nabla \cdot \left(a(w^{n+1}) \nabla \overline{w^{n+1}} \right) + \overline{w^{n+1}} + \nabla \lambda = w^{n+1}, \tag{NLfilter}$$
$$\nabla \cdot \overline{w^{n+1}} = 0, \quad \textit{and} \quad \overline{w^{n+1}} = w^{n+1}\ ,\ \textit{on}\ \partial\Omega.$$

Step 3: *Relax:*

$$u^{n+1} := (1-\chi) w^{n+1} + \chi \overline{w^{n+1}}. \tag{Relax}$$

The nonlinear filtering in Step 2 requires solving one linear system. With common FEM discretizations of (NLfilter) and $\delta = O(\triangle x)$ the condition number of the $1,1$ block in the associated mixed linear system is $O(1)$. To specify (NLfilter), requires selecting the function $a(u, \nabla u)$ (abbreviated $a(u)$) which we call an *indicator function.*

DEFINITION 1.1 (Indicator function). $a = a(u, \nabla u)$ is a function with $0 \le a(\cdot) \le 1$, and

$$a(u(x)) \doteq 0 \text{ for laminar or persistent flow structures}$$
$$a(u(x)) \doteq 1 \text{ for rapidly decaying flow structures.}$$

Quite generally, Proposition 1, (Modular NLFilter) is stable and convergent:

$$\text{Global Error} = O(\triangle t + \frac{\chi}{\triangle t}\delta^2 + \text{Spacial Error}).$$

1.1. Related work. Linear filter stabilization was developed by Boyd [**4**] and Fischer and Mullen [**12**], [**26**] (who introduced relaxation in Step 3), used by Dunca [**9**] and Mathew et al [**25**], Garnier, Adams and Sagaut [**7**] and Visbal and Rizzetta [**6**] and analyzed in [**11**]. One clear description for explicit methods (where the whole plan is simplest) was in the paper [**25**]; consider an explicit method plus filtering

$$\frac{w^{n+1} - u^n}{\triangle t} + NSE(u,p)^n = f^{\,n}$$
$$u^{n+1} = \overline{w^{n+1}}.$$

Eliminating w^{n+1} gives

$$\frac{u^{n+1} - u^n}{\triangle t} + NSE(u,p)^n + \frac{1}{\triangle t}\left[w^{n+1} - \overline{u^{n+1}} \right] = f^{\,n}.$$

The extra term $\frac{1}{\triangle t}\left[w^{n+1} - \overline{u^{n+1}} \right]$ is a diffusion operator but it grows as $\triangle t \to 0$. The latter effect is why evolve then filter is often abandoned as over diffusive. Fischer and Mullen [**12**], [**26**] contributed the algorithmically simple but very elegant idea of adding a relaxation step and choosing the relaxation parameter to eliminate this growth[1]. Two salient issues remained.

- The error induces by Step 2 is normally low: $O(\delta^2)$.
- Diffusive operators needed for turbulent flows need to adapt themselves to the local flow structures.

Numerical analysis of modular postprocessing algorithms was begun in [**11**]. In the numerical analysis of modular stabilizations, this paper was the first step. It studied algorithms with linear filtering and deconvolution to address the issue of accuracy:

ALGORITHM 2 (Linear Filter stabilization). *Pick* $\chi \in [0,1]$, *typically* $\chi = \triangle t$, *and* $\triangle t > 0$, *given* u^n, p^n

Step 1: *Find* w^{n+1}

$$\frac{w^{n+1}-u^n}{\triangle t} + w^{n+1}\cdot\nabla w^{n+1} - \nu\triangle w^{n+1} + \nabla p^{n+1} = f^{\,n+1}$$
$$\nabla\cdot w^{n+1} = 0 \quad and \quad w^{n+1} = 0 \ , \ on\ \partial\Omega.$$

Step 2a: *Linear filter:* $w^{n+1} \to \overline{w^{n+1}}$ *by*

(linear differential filter) $$-\delta^2\triangle\overline{w^{n+1}} + \overline{w^{n+1}} + \nabla\lambda = w^{n+1},$$
$$\nabla\cdot\overline{w^{n+1}} = 0, \quad and \quad \overline{w^{n+1}} = w^{n+1} \ , \ on\ \partial\Omega.$$

Step 2b: *Deconvolve:*

$$\overline{w^{n+1}} \to D\left(\overline{w^{n+1}}\right).$$

Step 3: *Relax:*

(Relax) $$u^{n+1} := (1-\chi)w^{n+1} + \chi D\left(\overline{w^{n+1}}\right).$$

The deconvolution operator for Step 2b analyzed in [**11**] was van Cittert deconvolution. Its implementation requires several steps of repeated filtering. The analytical strategy developed for (Linear Filter Stabilization) carried through as a general approach for nonlinear filtering. However, almost none of the methods of proof (the tactics) carrier through because linearity was used throughout the analysis of (Linear Filter Stabilization) in [**11**]. Extension to modular implementation of Variational Multiscale Methods was performed in [**21**].

The idea of modular VMS in its simplest form (the idea of modularity in its simplest form) is as follows. Suppose one wants to solve the NSE with a stabilization operator we denote by A (in bold below) included, as in:

$$\frac{u^{n+1}-u^n}{\triangle t} + u^{n+1}\cdot\nabla u^{n+1} - \nu\triangle u^{n+1} + \nabla p^{n+1} + \mathbf{A}\mathbf{u}^{n+1} = f^{\,n+1}.$$

Let Π denote a postprocessing operator and consider

$$\frac{w^{n+1}-u^n}{\triangle t} + w^{n+1}\cdot\nabla w^{n+1} - \nu\triangle w^{n+1} + \nabla p^{n+1} = f^{\,n+1}$$
$$u^{n+1} = \Pi w^{n+1}.$$

Eliminating the intermediate variable in one term shows this is

$$\frac{u^{n+1}-u^n}{\triangle t} + w^{n+1}\cdot\nabla w^{n+1} - \nu\triangle w^{n+1} + \nabla p^{n+1} + \frac{1}{\triangle t}\left[w^{n+1}-u^{n+1}\right] = f^{\,n+1}.$$

The extra term is $\frac{1}{\triangle t}\left[w^{n+1}-u^{n+1}\right] = \frac{1}{\triangle t}[w^{n+1}-\Pi w^{n+1}]$. Thus the postprocessing operator is determined by the equation

$$\frac{1}{\triangle t}[w-\Pi w] = A\left(\Pi w\right).$$

[1]They also used higher order filters with pseudospectral methods. These are all very interesting papers with many contributions glossed over in this short summary.

The difference between $w^{n+1}\cdot\nabla w^{n+1}$ and $u^{n+1}\cdot\nabla u^{n+1}$ is a higher order term (but must still be accounted for in the stability and convergence analysis).

The development of modular filter based stabilization culminated in using nonlinear filters to adapt models in [**20**]. The three indicator functions tested in worked well away from walls. Thus, the main question (considered herein) was: *How to treat the near wall region*? Nonlinear filters have since been applied to regularizations in [**2**] and Olshanskii and Xiong [**23**] have recently given a precise elaboration and analysis of the connection of (Modular NLFilter) with an eddy viscosity model.

2. Three Examples of Indicator Functions and Nonlinear Filters

The method (NLfilter) is no better than the indicator function selected. This limitation has been one intractable difficulty in previous eddy viscosity models: the EV used is over-sensitive to some persistent structures and the calculation is over-damped as a result. The idea of nonlinear filters is that different indicator functions with different sensitivities can be combined to advantage: since the *geometric average* of indicator functions is again an indicator function, given indicator functions $a_i(u), i=1,\cdots,M$, we can choose in (NLfilter)

$$a(u) := \sqrt[M]{\Pi_{i=1}^{M} a_i(u)} \tag{2.1}$$

addressing the over-sensitivity problem. Thus, modularity has the added advantage that

evolving the accuracy and reliability of the induced turbulence model means adding one function subroutine.

These $a_i(u)$ are obtained from theories of intermittence and eduction. We review three indicator functions in [**20**] then construct and test a wall adapted indicator from the WALE model of [**24**], the main contribution herein.

2.1. Three indicators tested in [20]. The Q criterion indicator. The Q criterion [**18**] marks persistent, coherent vortices where $Q>0$ or local rigid body rotation dominates deformation:

$$\begin{aligned} Q(u,u) \quad &: \quad = \frac{1}{2}\left(\nabla^{ss}u : \nabla^{ss}u - \nabla^{s}u : \nabla^{s}u\right) \text{ where} \\ \nabla^{s}u \quad &: \quad = \frac{1}{2}\left(\nabla u + \nabla u^{tr}\right) \quad\text{and}\quad \nabla^{ss}u := \frac{1}{2}\left(\nabla u - \nabla u^{tr}\right). \end{aligned}$$

Rescale so $Q>0$ or $Q<0$ corresponds to $a(u)$ close to 0 or 1, respectively.

Definition 2.1. A Q-criterion indicator function is

$$a_Q(u) := \frac{1}{2} + \frac{1}{\pi}\arctan\left(-Q(u,u)\right).$$

The functional form chosen is not unique. The Q indicator seems to be sensitive to the exact functional form chosen to construct the indicator function so further testing is needed for this indicator.

Vreman [**29**] constructs an eddy viscosity model based on a function $B(u)$ below. $B(u)$ is constructed to vanish for many coherent (non turbulent) flows:

$$\begin{aligned} |\nabla w|_F^2 \quad &= \quad \sum_{i,j=1,2,3}\left(\frac{\partial u_j}{\partial x_i}\right)^2, \beta_{ij} := \sum_{m=1,2,3}\frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_m}, \text{ and} \\ B(u) \quad &: \quad = \beta_{11}\beta_{22} - \beta_{12}^2 + \beta_{11}\beta_{33} - \beta_{13}^2 + \beta_{22}\beta_{33} - \beta_{23}^2. \end{aligned}$$

Let $|\nabla u|_F$ denote the Frobenius norm of ∇u

$$|\nabla u|_F := \sqrt{\sum_{i,j=1,2,3}\left|\frac{\partial u_j}{\partial x_i}\right|^2}.$$

The construction of the Vreman indicator below uses that

$$0 \le \frac{B(u)}{|\nabla u|_F^4} \le 1.$$

Definition 2.2. The Vreman indicator function is

$$a_V(u) = \sqrt{B(u)/|\nabla u|_F^4}.$$

Relative helicity density. Let $\omega = \nabla \times u$. Helicity is the total streamwise vorticity. It is an integral invariant of the 3d Euler equations in the absence of boundaries.

Definition 2.3. The helicity is

$$H(t) := \frac{1}{|\Omega|}\int_\Omega u \cdot \omega dt.$$

The helicity density is

$$HD(x,t) := u(x,t) \cdot \omega(x,t)$$

The relative helicity density is

$$RHD(x,t) = \frac{u(x,t) \cdot \omega(x,t)}{|u(x,t)||\omega(x,t)|}.$$

High helicity suppresses nonlinearity and thus breakdown by the NSE nonlinearity. (Helicity may also related to intermittence, e.g., [**28**].) Indeed, helicity, $u \cdot \omega$, and the NSE nonlinearity, $u \times \omega$, are related by

$$\frac{\text{Helicity}^2 + |\text{NSE nonlinearity}|^2}{|u|^2|\omega|^2} = 1.$$

Thus, per unit energy and enstrophy, local regions of high helicity means locally low nonlinearity and visa versa.

Definition 2.4. The relative helicity indicator function is

$$a_H(u) := 1 - \left|\frac{u(x,t) \cdot \omega(x,t)}{|u(x,t)||\omega(x,t)| + \delta^2}\right|.$$

3. A Wall Adapted Indicator

Many models, including the ones induced by the above indicator functions, are optimized for turbulence away from walls. One exception is the WALE eddy viscosity model of Nicoud and Ducros [**24**] which accounts for both strain and rotation of the smallest resolved scales and recovers the proper y^3 near-wall eddy viscosity scaling. Let $g_{ij}^2 = \partial u_j/\partial x_i$ and $g_{ij}^2 := g_{ik}g_{kj}$ (summation convention). Consider the traceless symmetric part of g_{ij}^2:

$$S_{ij}^d = \frac{1}{2}(g_{ij}^2 + g_{ji}^2) - \frac{1}{3}\delta_{ij}g_{kk}^2, \;\; \delta_{ij} = \text{Kronecker } \delta.$$

Nicoud and Ducros [**24**] construct an eddy viscosity model beginning with

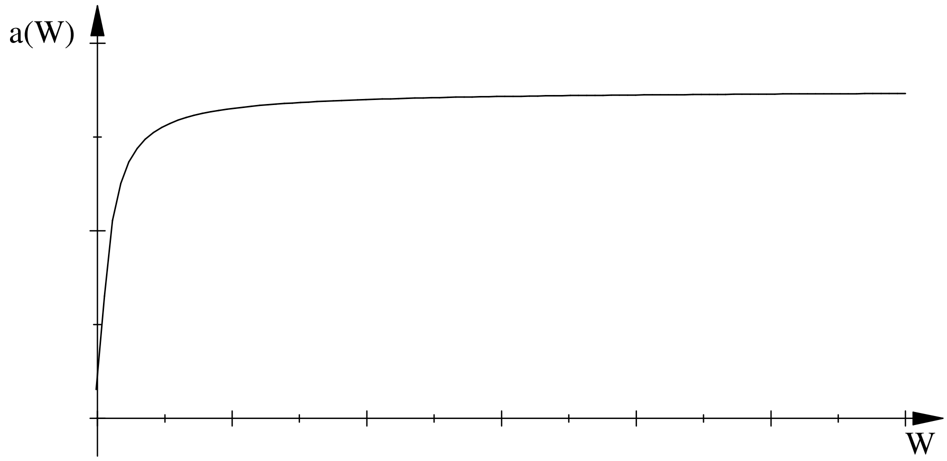
$$W(u) := \frac{(S^d(u) : S^d(u))^{3/2}}{(S^d(u) : S^d(u))^{5/4} + (D(u) : D(u))^{5/2}}. \tag{3.1}$$

We adapt $W(u)$ to an indicator function as follows.

Definition 3.1. The wall adjusted indicator function is given by

$$a_{WALE}(u) := \frac{2.0}{\pi}\arctan\left(\frac{1}{\delta}\frac{|W(u)|}{\delta^2 + |W(u)|}\right).$$

The form of the arctangent function in $a(u)$ is one of many options. Considering the plot $a = a(W)$ (below for δ moderate), $a(W)$ transitions quickly from zero where $W(u) = 0$ to near 1 at all other values of $W(u)$.



Indicator: $a(W)$ against W

4. Convergence of Nonlinear Filter Based Stabilization

There is a general stability and convergence theory for (Modular NLFilter) from [**20**] which implies convergence for the WALE indicator based model.

Proposition 1 (Unconditional Stability and Convergence, [**20**]). *Let spacial discretization be by finite element methods with velocity-pressure finite element spaces satisfying the discrete inf-sup condition. The energy equality (implying stability) holds:*

$$\frac{1}{2}||u_h^l||^2 + \triangle t \sum_{n=0}^{l-1} \left\{ \frac{\triangle t}{2} || \frac{w_h^{n+1} - u_h^n}{\triangle t} ||^2 + \nu ||\nabla w_h^{n+1}||^2 \right\} +$$

$$\triangle t \sum_{n=0}^{l-1} \frac{\chi}{\triangle t} \left\{ \frac{2-\chi}{2} \left(w_h^{n+1} - \overline{w_h^{n+1}}^h, w_h^{n+1} \right) + \frac{\chi}{2} \left(w_h^{n+1} - \overline{w_h^{n+1}}^h, \overline{w_h^{n+1}}^h \right) \right\} =$$

$$= \frac{1}{2}||u_h^0||^2 + \triangle t \sum_{n=0}^{l-1} (f^{n+1}, w_h^{n+1}), \ \ \textit{for any } l > 0,$$

For $0 \le \chi \le 2$ the model diffusion term is non-negative:

$$\frac{\chi}{\triangle t} \left[\frac{2-\chi}{2} \left(w_h^{n+1} - \overline{w_h^{n+1}}, w^{n+1} \right) + \frac{\chi}{2} \left(w_h^{n+1} - \overline{w_h^{n+1}}, \overline{w_h^{n+1}} \right) \right] \ge 0.$$

Let $0 \le \chi \le 1$ and suppose the velocity, pressure spaces contain piecewise polynomials of degree $(k, k-1)$. For u, p, and f sufficiently regular, the errors satisfy

$$\|u(t^l) - w_h^l\| + ||u(t^l) - u_h^l\| + \left(\nu \Delta t \sum_{n=1}^{l} \|\nabla (u(t^n) - w_h^n)\|^2 \right)^{1/2} \le$$

$$\le C(u, p, data, \nu) \left[h^k + \Delta t + \frac{\chi}{\triangle t} (\gamma h^k + h^{k+1} + \delta h^k + \delta^2 \min\{\delta^{-1}, ||\nabla \cdot (a(w_h) \nabla u)||^2\}) \right].$$

The model dissipation induced by steps 2 and 3 is independent of Step 1 and is of eddy viscosity type

$$\text{Model Dissipation} = \frac{\chi}{\triangle t} \delta^2 \left(a(w) \nabla \overline{w}, \nabla w \right) + HOdissipationTs.$$

5. Numerical Experiments

Our tests of the WALE indicator in (Modular NLFilter) used *FreeFEM++* [**17**] and Taylor-Hood elements ($X_h = C^0$ piecewise quadratics, $Q_h = C^0$ piecewise. linears).

Test of the convergence rate. First we test the predicted error and convergence rates for $a = a_{WALE}(\cdot)$ for the Green-Taylor vortex [**13**], [**14**]:

$$\begin{aligned} u_1(x,y,t) &= -\cos(\omega\pi x)\sin(\omega\pi y)\exp(-2\omega^2\pi^2 t/\tau) \\ u_2(x,y,t) &= \sin(\omega\pi x)\cos(\omega\pi y)\exp(-2\omega^2\pi^2 t/\tau) \\ p(x,y,t) &= -\frac{1}{4}(\cos(2\omega\pi x)+\cos(2\omega\pi y))\exp(-2\omega^2\pi^2 t/\tau) \end{aligned} \tag{5.1}$$

When $\tau = Re$, this is a solution of NSE with $f = 0$, consisting of $\omega \times \omega$ array of oppositely signed vortices that decay as $t \to \infty$. We take $\omega = 1$, $\tau = Re = 100$, $\Omega = (0,1)^2$, $\chi = \Delta t$, $\delta = \Delta x$ and $T = 0.1$. Table 1 shows the predicted convergence rates.

(Table 1)

Δt	h, δ	$\|\|u-u_h\|\|_{2,1}$	rate
0.005	$\frac{1}{4}$	0.0955	–
$\frac{0.005}{2}$	$\frac{1}{8}$	0.0264	1.86
$\frac{0.005}{4}$	$\frac{1}{16}$	0.0048	2.46
$\frac{0.005}{8}$	$\frac{1}{32}$	0.0012	2

Errors and convergence rates: (Modular NLFilter) using $a_{WALE}(\cdot)$

Flow over a step. The domain is a 40×10 rectangular channel with a 1×1 step five units into the channel. Boundary conditions are no-slip on the top, bottom boundaries, a parabolic inflow profile $(y(10-y)/25, 0)^T$, $\Delta t = 0.01$, $\nu = 1/600$ and do-nothing outflow. The correct behavior is a smooth velocity field away from the step and for eddies to periodically form and shed behind the step. *While not turbulent, this flow is a good test if a turbulence model over damps important dynamic structures. Generally, under damping and under resolved gives nonsensical solutions and over damping gives one large attached eddy that does not break up.* First we shown below velocity streamlines over speed contours at $T = 40$ for 2D flow over a step, found by an under resolved NSE solution (Step 1 without Steps 2 and 3, no filtering), on meshes with 1762 and 3226 dof. These are nonsensical compared to fully resolved simulations in [**5**].

Next compare these nonsensical solutions to solutions on the two same meshes using the modular nonlinear filtering model with the WALE indicator.

Compared to fine mesh results in [**5**], both velocities are accurate in major features and the eddy behind the step is starting to break up roll downstream. Outflow boundary noise is typical behavior. We next give plots of (the scalar) $a_{WALE}(u)$ for the two meshes.

As expected, stabilization is needed around the outflow boundary. It is remarkable that, away from the outflow region, the WALE indicator localized eddy viscosity to a very small area at the leading edge of the step. The WALE indicator shows that beyond a few mesh cells there, no extra eddy viscosity or stabilization is needed to get a good, coarse mesh approximation.

6. Conclusions

With *nonlinear* filtering, $w - \overline{w}$ and thus model dissipation is small in laminar regions and regions of persistent, coherent flow structures. It reduces both numerical and modeling errors and model dissipation *more closely mimics the exact physics of the energy cascade.* Nonlinear filtering reduces implementing a complex turbulence model in a flow code (possibly a legacy code of great length) to solving one well conditioned linear system each time

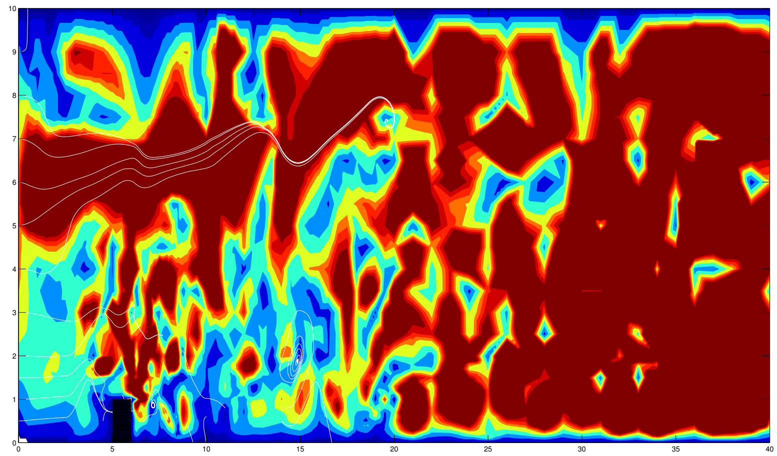


FIGURE 1. Velocity, $T = 40$, 2D step, Underresolved NSE solution, 1762 dof

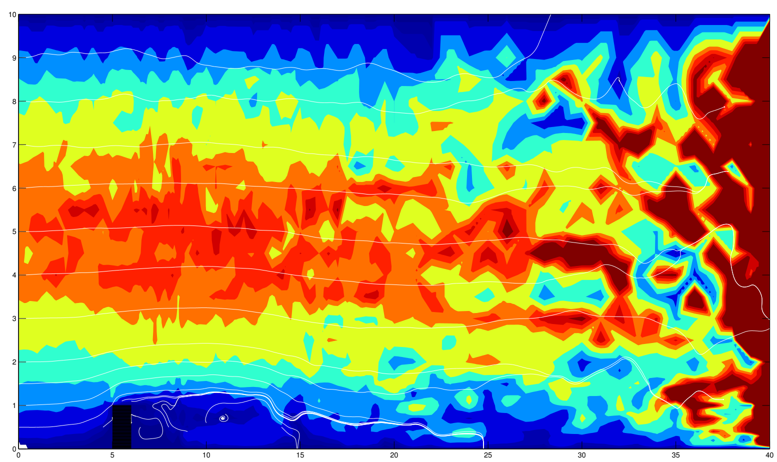


FIGURE 2. Velocity, $T = 40$, 2D step, Underresolved NSE solution, 3226 dof

step. Model accuracy is increased simply by providing additional function subroutines. The WALE indicator by itself and without combination of other indicators precise localizes the EV needed for the tested flow. We conjecture that this is because step flow is about flow-wall interactions for which WALE is an excellent tool and that for combinations of flow-wall interactions with free turbulence and coherent structures, combinations of WALE with other indicators would produce improved results over WALE alone.

References

[1] L.C. Berselli, T. Iliescu and W. Layton, Large Eddy Simulation, Springer, Berlin, 2004

[2] A. Bowers, L. Rebholz, A. Takhirov, and C. Trenchea, Improved accuracy in regularization models of incompressible flow via adaptive nonlinear filtering, IJNMF, to appear, 2012.

[3] J. Borggaard, T. Iliescu and J.P. Roop, A Bounded Artificial Viscosity Large Eddy Simulation Model, SINUM, 47(2009), 622-645.

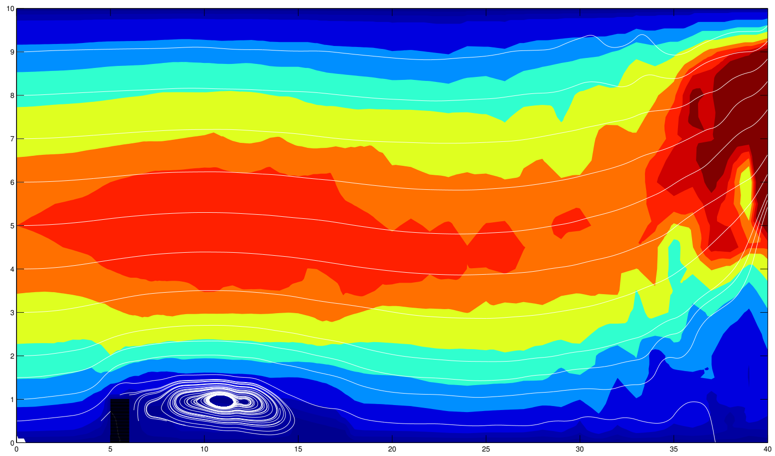


FIGURE 3. NLFiltering velocity, $T = 40$ using WALE indicator, 1762 dof

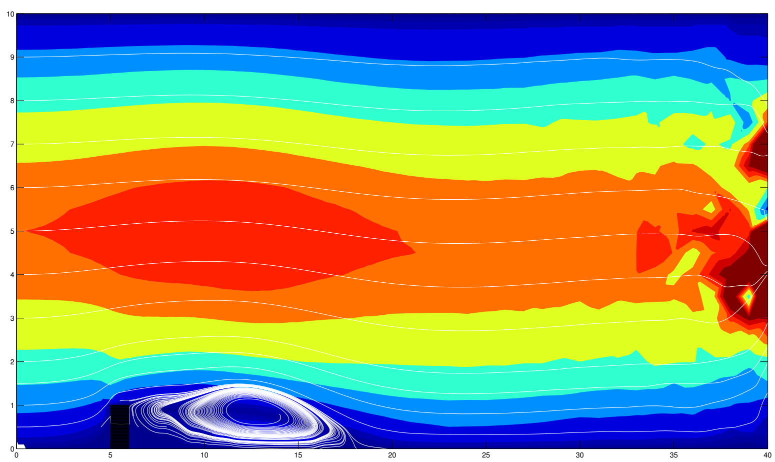


FIGURE 4. NLFiltering velocity, $T = 40$ using WALE indicator, 3226 dof

[4] J.P. Boyd, Two comments on filtering for Chebyshev and Legendre spectral and spectral element methods : Preserving the boundary conditions and interpretation of the filter as a diffusion, JCP, 143 (1998), 283-288.

[5] W. Layton, C. Manica, M. Neda, L. Rebholz, Numerical analysis and computational testing of a high accuracy Leray-Deconvolution model of turbulence. NMPDEs 2008; 24(2):555–582.

[6] M.R. Visbal and D.P. Rizzetta, Large eddy simulation on curvilinear grids using compact differencing and filtering schemes, J. Fluids Eng. 124 (2002) 836-847.

[7] E. Garnier, N. Adams and P. Sagaut, Large eddy simulation for compressible flows, Springer, Berlin, 2009.

[8] J. Connors and W. Layton, On the accuracy of the finite element method plus time relaxation, Math. Comp., 79 (2010), 619-648.

[9] A. Dunca, Investigation of a shape optimization algorithm for turbulent flows, report ANL/MCS-P1101-1003, Argonne National Lab, 2002.

[10] V. Ervin, W. Layton and M. Neda , Numerical analysis of a higher order time relaxation model of fluids, IJNAM, 4 (2007), 648-670.

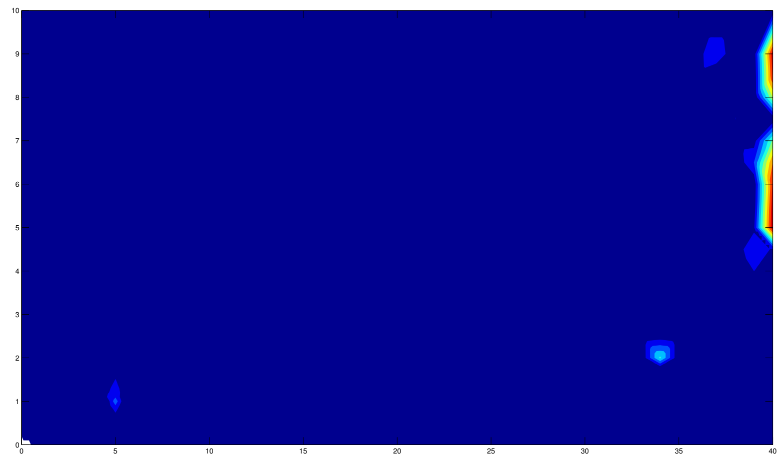


FIGURE 5. WALE indicator for 1762 dof velocity

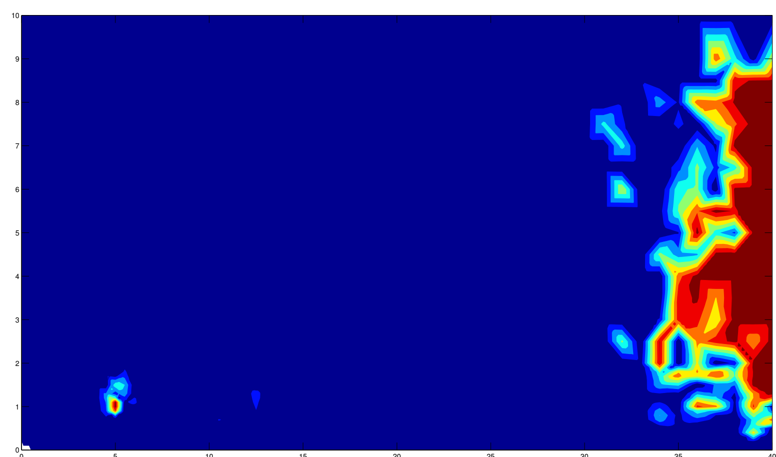


FIGURE 6. WALE indicator 3226 dof velocity

[11] V. Ervin, W. Layton and M. Neda, Numerical analysis of filter based stabilization for evolution equations, technical report, submitted to SINUM 2009.
[12] P.F. Fischer and J.S. Mullen, Filter-based stabilization of spectral element methods, CRAS, Paris, 332(1), 265 (2001).
[13] A.E. Green and G.I. Taylor, Mechanism of the production of small eddies from larger ones, Proc. Royal Soc. A, 158 (1937), 499-521.
[14] G.I. Taylor, On decay of vortices in a viscous fluid, Phil. Mag., 46, (1923), 671-674.
[15] M.D. Gunzburger, Finite Element Methods for Viscous Incompressible Flows, Academic Press, 1989.
[16] M. Germano, Differential filters of elliptic type, Phys. Fluids, 29 (1986), 1757-1758.
[17] F. Hecht and O. Pironneau, FreeFEM++ , webpage: http://www.freefem.org.
[18] J.C. Hunt, A.A. Wray and P. Moin, Eddies stream and convergence zones in turbulent flows, CTR report CTR-S88, 1988.
[19] W. Layton , Superconvergence of finite element discretization of time relaxation models of advection, BIT, 47 (2007), 565-576.
[20] W. Layton, L. Rebholz and C. Trenchea, Modular Nonlinear Filter stabilization of methods for higher Reynolds Number flow, J. Math. Fluid Mechanics to appear, 2012.

[21] W. Layton, L. Roehe and H. Tran, A modular, postprocessed induced VMS method, CMAME, to appear 2011.
[22] W. Layton and M. Neda, Truncation of scales by time relaxation, JMAA, 325 (2007), 788-807.
[23] M. Olshanskii and Xin Xiong, A connection between filter stabilization and eddy viscosity models, technical report 2012.
[24] F. Nicoud and F. Ducros, Subgrid-Scale Stress Modelling Based on the Square of the Velocity Gradient Tensor, Flow, Turbulence and Combustion, 62 (1999), 183-200.
[25] J. Mathew, R. Lechner, H. Foysi, J. Sesterhenn and R. Friedrich, An explicit filtering method for large eddy simulation of compressible flows, Physics of Fluids, 15 (2003), 2279-2289.
[26] J.S. Mullen and P.F. Fischer, Filtering techniques for complex geometry fluid flows, Comm. in Num. Meth. in Eng., 15 (1999), 9-18.
[27] P. Sagaut, Large eddy simulation for Incompressible flows, Springer, Berlin, 2001.
[28] A. Tsinober and E. Levich, On the helical nature of 3-dimensional coherent structures in turbulent flows, Physics Letters 99A (1983) 321-324.
[29] A.W. Vreman, An eddy-viscosity subgrid-scale model for turbulent shear flow: algebraic theory and applications, Phys. Fluids 16 (2004), 3670-3681.

(A. One) Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
E-mail address, A. One: `WJL@pitt.edu`
URL: `http://www.math.pitt.edu/~wjl`

Current address, A. Two: Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
E-mail address, A. Two: `AZT7@pitt.edu`
URL: `http://www.math.pitt.edu/~azt7`