2D DECAYING TURBULENCE DOES NOT HAVE EXPONENTIAL SEPARATION OF TRAJECTORIES

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Abstract. In $2d$ decaying turbulence, all solutions eventually $\to 0$ as $t \to \infty$ after a long period of approximately constant kinetic energy. This report proves that during this dynamically interesting, long period the separation between two trajectories can have at most linear growth in time (saturating at $O(\nu^{-1})$):

$$||u_1(t) - u_2(t)|| \leq \exp(-\nu t)\{||u_1(0) - u_2(0)|| + C\min\{t, \nu^{-1}\}\}.$$

Key words. chaos, predictability, decaying turbulence, Lyapunov exponent

AMS subject classification.

1. Introduction.

This is a supplementary version, containing extreme detail, of a report with similar title.

Even if the fundamental problem (of uniqueness of weak solutions) is resolved positively, the critical issue of *predictability* remains. It is thought that severe limits to predictability are due to chaotic dynamics (the NSE having both an absorbing ball and exponential local separation of trajectories, [11]). The possibility of chaotic dynamics is still a topic of interest and impact for $2d$ flows as these are often taken as a model for flow in thin layers, e.g. [4]. We show herein that, in the long period of dynamic interest[1], for $2d$ decaying turbulence trajectories *separate at most linearly with rate independent of ν or Re.* Proposition 1.1 below (whose proof is particularly simple) reveals a gap between chaotic dynamics and predictability in that the latter issue remains open.

Let u_1, u_2 denote two solutions of the $2d$, unforced NSE for small ν on $\Omega = (0, 2\pi) \times (0, 2\pi)$ with periodic boundary conditions (and zero mean) corresponding to different initial conditions

$$u_{j,t} + u_j \cdot \nabla u_j - \nu \triangle u_j + \nabla p_j = 0, \text{ and } \nabla \cdot u_j = 0 \tag{1.1}$$
$$\text{where } \varepsilon := ||u_1(x,0) - u_2(x,0)|| > 0,$$

with $|| \cdot ||$ the $L^2(\Omega)$ norm. Due to the triangle inequality and the basic energy estimate, the deviation is bounded and after a long initial period $\to 0$

$$||u_1(t) - u_2(t)|| \leq e^{-\nu C_{PF}^{-2} t}\{||u_1(0)|| + ||u_2(0)||\}.$$

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[1] The standard estimate is

$$||u(t)||^2 \leq \exp(-2\nu C_{PF}^{-2} t)||u(0)||^2$$

which predictes a half-life of the kinetic energy of

$$T_{1/2} = \frac{\ln(2) C_{PF}^2}{2}\nu^{-1}.$$

If, for example, $\nu = 10^{-4}$ this forecasts a very long period of approximately constant energy.

The nonlinear term vanishes (and is not put on the RHS) in its derivation. Thus, this basic estimate gives a saturation level that in many cases is smaller than that derivable by other methods (in which the nonlinearity is fought). Nevertheless, there is still interest in estimates of deviations that improve the above for small ε and time intervals of dynamic interest. For these estimates, Gronwall's inequality becomes a basic tool and exponential (local) separation seems to be the inevitable prediction. However, due to self-organization of $2d$ decaying turbulence (e.g., [4], [6], [7], [5], [3]), it is plausible that the rate of separation of velocities may not be exponential and predictability increased. We prove that *linear separation until saturation is worst possible.*

Let C_{PF}[2] denote the constant occurring in the Poincaré-Friedrichs inequality, C_1 the constant in the embedding inequality[3] $||v||_{L^4}^2 \leq C_1 ||v|| ||\nabla v||$, $H^1_\#(\Omega)$ the closure under the norm $||\nabla v||$ of the smooth, zero mean, divergence free, 2π periodic functions

$$C' = 2C_1 \left(||\nabla u_1(0)|| + ||\nabla u_2(0)||\right) ||\nabla u_2(0)|| \text{ and}$$
$$m(t) = \min\{t, C_{PF}^2 \nu^{-1}\}.$$

Proposition 1.1 (Linear Separation). *For $u_1(0), u_2(0), \nabla u_1(0), \nabla u_2(0) \in H^1_\#(\Omega)$*

$$||u_1(t) - u_2(t)|| \leq \exp(-\nu C_{PF}^{-2} t)\{\varepsilon + \frac{C'}{2} m(t)\}. \tag{1.2}$$

In (1.2) the separation of two trajectories is not controlled by the initial separation ε and *thus the question of predictability remains open.* The standard estimate from Gronwall's inequality is as follows.

Proposition 1.2 (The Standard Estimate). *Under the above assumptions*

$$||u_1(t) - u_2(t)|| \leq \exp(-\frac{\nu}{2} C_{PF}^{-2} t) \exp\left\{\frac{C_1 ||\nabla u_2(0)||^2}{2\nu} \min\{t, \frac{1}{2} C_{PF}^{-2} \nu^{-1}\}\right\} \varepsilon. \tag{1.3}$$

In comparison to Proposition 1.1, (1.3) predicts a long period of exponential separation before decay and that if $u_1(0) = u_2(0)$ then $u_1(t) = u_2(t)$ for all t thereafter. Thus, the interest in Proposition 1.1 is that separation is not exponential under periodic boundary conditions. This result may be due to non-creation of vorticity at the boundary under periodic boundary conditions since the proof uses a uniform in ν estimate on ∇u (or equivalently $\nabla \times u$) not valid under, e.g., no slip boundary conditions. This difference may explain why the (very interesting) experiments in [1] show exponential separation[4] for $2d$ decaying turbulence. A positive effective averaged Lyapunov exponent was also observed for no-slip boundary conditions in [8].

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[2] For $u, \nabla u \in H^1_\#(\Omega)$ there holds:

$$||u|| \leq C_{PF} ||\nabla u|| \text{ and } ||\nabla u|| \leq C_{PF} ||\triangle u||,$$

Here $C_{PF} = \lambda_{\min}^{-2}$ where λ is the smallest eigenvalue of the Stokes operator under the given boundary conditions. For the given domain $\Omega = (0, 2\pi)^2$ and periodic with zero mean boundary conditions $C_{PF} = \lambda_{\min} = 1$.

[3] $C_1 < \infty$ since Ω is bounded and $2d$. Alternately, C_1 could be defined by

$$C_1 = \sup\left\{\frac{\int_\Omega v \cdot \nabla w \cdot v dx}{||\nabla w|| ||v|| ||\nabla v||} : 0 \neq v, w \in H^1_\#(\Omega)\right\}.$$

[4] While Figure 1, page 727 resembles $m(t)$, the scale is logarithmic showing clearly exponential separation.

Remark 1.3. *Linear separation also holds if (linear) terms modelling any of the following effects are added to the NSE: hyperviscosity, Coriolis force and bottom friction* $\propto \left(\nu\pi^2\right)/\left(4H^2\right)$ *due to flow of a thin layer over a no-slip surface.*

2. Proof of Proposition 1.1. Let $\phi(t) := u_1(t) - u_2(t)$. Subtracting two realizations of (1.1), multiplying by ϕ, integrating and rearranging the nonlinear terms, gives

$$\frac{d}{dt}||\phi||^2 + 2\nu||\nabla\phi||^2 = -2\int_\Omega \phi \cdot \nabla u_2 \cdot \phi dx.$$

For the RHS we apply the standard estimate

$$|2\int_\Omega \phi \cdot \nabla u_2 \cdot \phi dx| \leq 2||\nabla u_2||||\phi||^2_{L^4} \leq 2C_1||\nabla u_2||||\phi||||\nabla\phi||.$$

Remark 2.1. *The usual step (leading to Proposition 1.2) is to apply Gronwall's inequality after*

$$|\int_\Omega \phi \cdot \nabla u_2 \cdot \phi dx| \leq \frac{\nu}{2}||\nabla\phi||^2 + \frac{C}{\nu}||\nabla u_2||^2||\phi||^2$$

leading to exponential growth $\exp(\frac{C}{\nu}\int_0^t ||\nabla u_2||dt')$ *with rate constant* $\to\infty$ *as* $\nu\to 0$.

A non-exponential growth rate thus requires not subsuming $||\nabla\phi||$ in the viscous term and exploiting features of 2d decaying turbulence. Instead we bound $||\nabla\phi|| \leq ||\nabla u_1|| + ||\nabla u_2||$ which $\to 0$ as $t\to\infty$. Recall that in the 2d, periodic case $\int_\Omega u \cdot \nabla u \cdot \triangle u dx = 0$. Multiplying (1.1) by $-\triangle u$, using $||\nabla u|| \leq C_{PF}||\triangle u||$ and integrating leads to another (well known) estimate for $u = u_1, u_2$:

$$||\nabla u(t)|| \leq \exp(-\nu C_{PF}^{-2}t)||\nabla u(0)|| \tag{2.1}$$

and thus $||\nabla\phi|| \leq \exp(-\nu C_{PF}^{-2}t)\left(||\nabla u_1(0)|| + ||\nabla u_2(0)||\right)$. Using both estimates

$$\frac{d}{dt}||\phi||^2 + 2\nu C_{PF}^{-2}||\phi||^2 \leq \left[2C_1\exp(-2\nu C_{PF}^{-2}t)\left(||\nabla u_1(0)|| + ||\nabla u_2(0)||\right)||\nabla u_2(0)||\right]||\phi||.$$

Multiplying by $\exp(+2\nu C_{PF}^{-2}t)$ we have (denoting $k(t) := C'\exp(-\nu C_{PF}^{-2}t)$ and $x(t) = \exp(+\nu C_{PF}^{-2}t)||\phi(t)||$)

$$\frac{d}{dt}\left(x^2\right) \leq k(t)x(t) \text{ and } x(0) = \varepsilon > 0.$$

At this point we may apply Bihari's lemma[5], p. 16 in [2]. The simpler path is to

[5] Bihari's lemma, p. 16 in [2], applies to:

$$x(t) \leq M + \int_0^t k(s)g(x(s))ds,$$
$$k(t) \geq 0, M > 0,$$
$$g(\cdot) \text{ non-decreasing and positive on a half-axis.}$$

It states that if $G(\cdot)$ is a primitive of $1/g(\cdot)$ it states that

$$x(t) \leq G^{-1}(G(M) + \int_0^t k(s)ds).$$

differentiate and cancel giving

$$\frac{d}{dt}x(t) \leq \frac{1}{2}k(t) \text{ and } x(0) = \varepsilon > 0, \text{ thus}$$
$$x(t) \leq \varepsilon + \frac{1}{2}\int_0^t k(s)ds.$$

We obtain[6]

$$\exp(+\nu C_{PF}^{-2}t)||\phi(t)|| \leq \varepsilon + \frac{C'}{2}\left[\frac{1-\exp(-\nu C_{PF}^{-2}t)}{\nu C_{PF}^{-2}}\right].$$

The result follows since[7][8]

$$\frac{1-\exp(-at)}{a} \leq \min\{t, 1/a\} = m(t),\, a = \nu C_{PF}^{-2}.$$

2.1. Supplement: Proof of the standard estimate. We give a short proof of Proposition 1.2 in this supplementary subsection.

Step 1: An estimate for ∇u: Taking the inner product of the NSE with $-\triangle u$ and performing the standard steps gives $\frac{d}{dt}||\nabla u(t)||^2 + 2\nu C_{PF}^{-2}||\nabla u||^2 \leq 0$. Using an integrating factor

$$||\nabla u(t)||^2 \leq \exp\{-2\nu C_{PF}^{-2}t\}||\nabla u(0)||^2.$$

Step 2: Subtracting the two realizations of the NSE, adding and subtracting in the nonlinearity and taking the inner product with $\phi(t)$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\phi(t)||^2 + \nu||\nabla\phi(t)||^2 &= -\int_\Omega \phi\cdot\nabla u\cdot\phi dx\\ &\leq ||\nabla u||||\phi||_{L^4}^2 \leq C_1||\nabla u||||\phi||||\nabla\phi||\\ &\leq \frac{\nu}{2}||\nabla\phi||^2 + \frac{1}{2\nu}C_1^2||\nabla u||^2||\phi||^2.\end{aligned}$$

To apply Bihari's lemma, drop the second term. Set $g(x) = \sqrt{x}, G(y) = 2\sqrt{x}, G^{-1}(x) = (x/2)^2, k(t)$ as in the text, and $M = y(0)$. Then

$$x(t) \leq M + \int_0^t k(s)g(x(s))ds.$$

Bihari's lemma then concludes that

$$x(t) \leq G^{-1}\left(G(x(0)) + \int_0^t k(s)ds\right).$$

After simplification, this is the claimed result.

[6] We calculate

$$\int_0^t k(s)ds = \int_0^t C'\exp(-\nu C_{PF}^{-2}s)ds == C'\left[\frac{1-\exp(-\nu C_{PF}^{-2}t)}{\nu C_{PF}^{-2}}\right].$$

[7] In the spirit of providing every detail in this supplementary version: Recall $t \geq 0$ and $a > 0$. Clearly then

$$\frac{1-\exp(-at)}{a} \leq \frac{1}{a}.$$

Thus, we must show only that $1-\exp(-at) \leq at$ or equivalently $f(t) := \exp(-at) - 1 + at \geq 0$. Since $f(0) = 0$ and $f'(t) = a(1-\exp(-at)) \geq 0$, we have $f(t) \geq 0$.

[8] A plot of the LHS and the RHS shows that the upper estimate does capture both asymptotics of $t \to 0$ and $t \to \infty$.

Thus, $y(t) = ||\phi(t)||^2$ satisfies

$$y'(t) + \nu C_{PF}^{-2} y(t) \leq \frac{C_1^2}{\nu} ||\nabla u(t)||^2 y(t).$$

Using Step 1 gives

$$y'(t) + \left[\nu C_{PF}^{-2} - \frac{C_1^2}{\nu} \exp\{-2\nu C_{PF}^{-2} t\} ||\nabla u(0)||^2 \right] y(t) \leq 0.$$

Let $a(t) = \nu C_{PF}^{-2} - \frac{C_1^2}{\nu} \exp\{-2\nu C_{PF}^{-2} t\} ||\nabla u(0)||^2$ and

$$A(t) = \int_0^t a(s) ds = \nu C_{PF}^{-2} t - \frac{C_1^2}{\nu} \frac{1 - \exp\{-2\nu C_{PF}^{-2} t\}}{2\nu C_{PF}^{-2}} ||\nabla u(0)||^2.$$

Recall $\left(1 - \exp\{-2\nu C_{PF}^{-2} t\}\right) / 2\nu C_{PF}^{-2} \leq \min\{t, \frac{1}{2} C_{PF}^{-2} \nu^{-1}\}$. Thus $y(t) \leq \exp(-A(t)) y(0)$, which yields the standard estimate

$$||\phi(t)||^2 \leq \exp\{-\nu C_{PF}^{-2} t + \frac{C_1^2}{\nu} \min\{t, \frac{1}{2} C_{PF}^{-2} \nu^{-1}\} ||\nabla u_2(0)||^2\} \varepsilon^2.$$

3. Conclusions. Comparing the crossover points between (1.2) and (1.3), exponential separation (e.g., [10]) is only an accurate description for very small time and very small initial separation. Linear separation of trajectories is the most accurate description through most of the dynamically interesting period. Thus, $2d$ decaying turbulence is not chaotic according to the description of a chaotic system as one with an absorbing ball and a positive Lyapunov exponent, Lorenz [11]. Further, the critical question of predictability, [9], [4], is related but not equivalent to that of chaotic dynamics.

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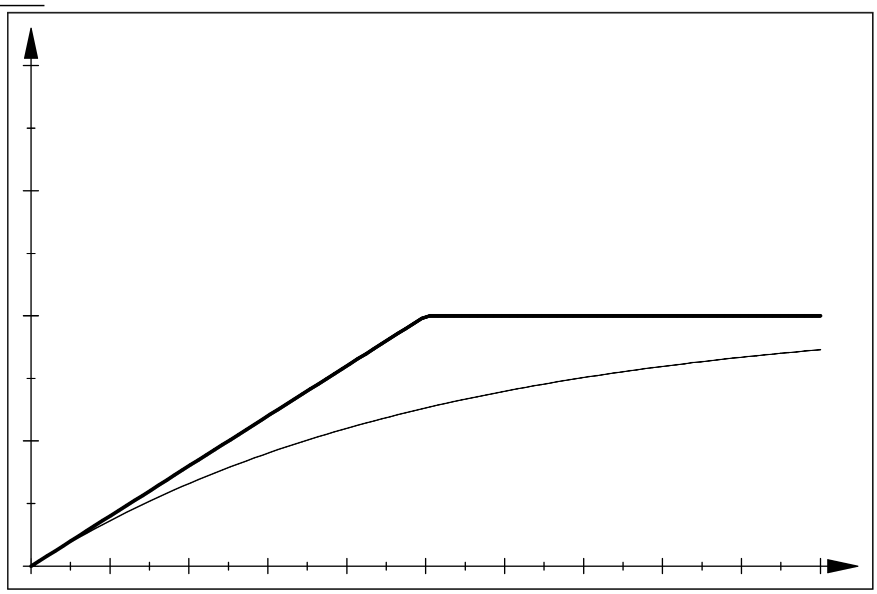

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