Long-time H1-stability of Cauchy's one-leg θ-method for the Navier-Stokes equations

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Abstract

In this paper we study the long-time stability of the Cauchy one-leg θ-methods for the two-dimensional Navier-Stokes equations. We establish the uniform dissipativity in H^1, in the sense that the semi-discrete-in-time approximations possess a global attractor for a small enough time step, using the discrete Grönwall lemma and the discrete uniform Grönwall lemma.

1 Introduction and background

The incompressible Navier-Stokes equations (NSE) are the fundamental mathematical model for viscous fluids flow, having a wide range of applications such as aircraft aerodynamics, weather prediction, and blood flow simulation [12–15, 17, 26, 27, 30, 34, 35]. We consider the two-dimensional case

$$\begin{aligned} &u_t + u\cdot\nabla u - \nu\Delta u + \nabla p = f, \\ &\nabla\cdot u = 0, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a domain with boundary $\partial\Omega$ of class C^2, $u(t,x) : [0,\infty)\times\Omega \to \mathbb{R}^2$ denotes the velocity, $p(t,x) : [0,\infty)\times\Omega \to \mathbb{R}$ is the pressure, ν is the kinematic viscosity, and $f \in L^\infty(\mathbb{R}_+; L^2(\Omega)^2)$ represents the body forces applied to the fluid. Furthermore, we consider an initial condition $u_0 = u(0,x) \in H_0^1(\Omega)$ and a no-slip boundary condition $u|_{\partial\Omega} = 0$.

An understanding of the behavior of (1.1) for long times is essential for many applications, especially the ones concerning climate modeling. As shown in [12, 25, 27], in the continuous, two-dimensional case, the $L^2(\Omega)$ norm of the solution is uniformly bounded in time

$$\|u(t)\|^2_{L^2(\Omega)} \le \|u_0\|^2 e^{-\nu\lambda_1 t} + \frac{1}{\nu^2\lambda_1^2}(1-e^{-\nu\lambda_1 t})\|f\|^2_{L^\infty(\mathbb{R}_+;L^2(\Omega)^2)}, \tag{1.2}$$

where λ_1 is the first eigenvalue of the Stokes operator. Moreover, as stated in [27, 29], u can be uniformly bounded in $H_0^1(\Omega)$ by a function which depends on the initial condition

$$\|\nabla u(t)\|^2 \le K(\|\nabla u_0\|, \|f\|_{L^\infty(\mathbb{R}_+;L^2(\Omega)^2)}). \tag{1.3}$$

The dependence on the initial condition is transient, i.e., we have that for sufficiently large times

$$\|\nabla u(t)\|^2 \le K(\|f\|_{L^\infty(\mathbb{R}_+;L^2(\Omega)^2)}), \tag{1.4}$$

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demonstrating the existence of a continuous global attractor. Indeed, the uniform bound (1.2) in $L^2(\Omega)$ provides the existence of an absorbing set for the dynamical system, while the additional $H^1(\Omega)$ bounds (1.3)-(1.4) yield the compactness needed to guarantee the existence of a global attractor.

For a numerical model to be reliable, it is essential that it captures the behavior of its continuous counterpart. Therefore, it is important to develop numerical methods that exhibit long-time stability. Starting from the work of Tone and Wirosoetisno [29] for the Backward Euler (BE) semi-discrete in time approximation, numerous studies have proved this result for a variety of numerical schemes, both for the semi-discretization in time [16, 18, 24, 29, 31–33] and fully discrete formulations [1, 2, 4, 9–11, 16, 19, 25, 28, 36, 37]. Most of the results that employ a full space-time discretization require a time-step restriction depending on the mesh size, namely a Courant-Friedrichs-Lewy (CFL) condition. Our goal is to extend the long-time stability result to the class of Cauchy one-leg θ-methods, using only a semi-discretization in time.

The Cauchy one-leg θ-method discretizes an initial value problem of the form $y'(t) = f(t,y)$ by computing

$$\frac{y^{n+1}-y^n}{\tau} = f(t_{n+\theta}, y^{n+\theta}), \tag{1.5}$$

where $t_{n+\theta} = t_n + \theta\tau$, $y^{n+\theta} = \theta y^{n+1} + (1-\theta)y^n$, and $\theta \in [0,1]$.

For more information and properties of this scheme, we refer the reader to [3, 5–8, 23]. Note that if $\theta = 1$, this method corresponds to the BE method, for which long-time stability of the semi-discretization in time was proved in [29]. Whenever $\theta = \frac{1}{2}$, this method becomes the symplectic midpoint, or Crank-Nicolson (CN) method, which is a second-order scheme. We note that the long-time stability of the full space-time discretization of CN was proved in [28] under a CFL condition. For $\theta \in \left(\frac{1}{2}, 1\right)$, (1.5) is an implicit method that interpolates between CN and BE, providing some numerical dissipation. This method is second-order accurate whenever $\theta = \frac{1}{2} + \mathscr{O}(\tau)$. In [4], the long-time stability of a second-order, fully discrete, linearly extrapolated method, which interpolates between CN and BDF2, is proved also under a CFL condition. The paper [4] extends the results for the cases $\theta = 1$ [1] and $\theta = \frac{1}{2}$ [2]. As in our analysis, that stability proof applies only to $\theta > \frac{1}{2}$. In [3], where the authors use the one-leg θ-method to construct discrete space-time approximations which converge to weak solutions to NSE which are suitable in the sense of Scheffer and Caffarelli-Kohn-Nirenberg. Their result also requires that $\theta > \frac{1}{2}$.

Following [8], we can rewrite the Cauchy one-leg θ-method (1.5) as a two-step scheme consisting of a BE step and a linear extrapolation (which is equivalent to Forward Euler (FE)). This allows us to leverage some of the techniques developed for BE in [29], so part of our argument for the H^1 bound will mimic that proof. The method (1.5) applied to (1.1) reads: given initial condition u_0, compute u^{n+1} for all $n \geq 0$ as follows

$$\begin{aligned} \frac{1}{\theta\tau}(u^{n+\theta} - u^n) - \nu\Delta u^{n+\theta} + u^{n+\theta}\cdot\nabla u^{n+\theta} + \nabla p^{n+\theta} &= f^{n+\theta}, \\ \nabla\cdot u^{n+\theta} &= 0, \end{aligned} \tag{1.6}$$

$$u^{n+1} = \frac{1}{\theta}u^{n+\theta} - \left(\frac{1}{\theta} - 1\right)u^n. \tag{1.7}$$

In this paper, we establish the uniform dissipativity of u^n, or the long-time stability of (1.6)-(1.7) in $L^2(\Omega)$ and $H^1(\Omega)$, hence proving that a discrete global attractor exists. In Section 2, we introduce the mathematical background necessary for our analysis. Then we proceed to prove the $L^2(\Omega)$ and $H^1(\Omega)$ stability in sections 3 and 4 respectively.

2 Mathematical preliminaries

In this section we present some of the mathematical tools and notations used in the analysis.

We consider the following solenoidal Sobolev spaces

$$V = \{v \in H_0^1(\Omega)^2, \nabla\cdot v = 0\} \tag{2.1}$$

$$H = \{v \in L^2(\Omega)^2, \nabla\cdot v = 0, v\cdot\hat{n} = 0 \text{ on } \partial\Omega\}, \tag{2.2}$$

where $\hat{n}$ is the outward unit normal on $\partial\Omega$. The space H is endowed with the norm and inner product of $L^2(\Omega)^2$, which we denote by $\|\cdot\|$ and $(\cdot,\cdot)$. The space V is endowed with the $H^1(\Omega)$ semi-inner product and norm, which define a norm and inner product on V. We assume that $f \in L^\infty(\mathbb{R}_+, H)$, and

$$\|f\|_\infty = \|f\|_{L^\infty(\mathbb{R}_+,H)}. \tag{2.3}$$

We denote by A the Stokes operator, a linear, unbounded, positive definite operator from V to V' such that

$$\langle Au, v\rangle_{V',V} = (\nabla u, \nabla v) \qquad \forall u, v \in V.$$

The domain of A, denoted by $D(A)$, satisfies

$$D(A) = H^2(\Omega)^2 \cap V, \qquad D(A) \subset V \subset H.$$

For more information on operator A, we refer the reader to [27]. Denoting the first eigenvalue of A as $\lambda_1 > 0$, we recall the Poincaré-Friedrichs inequality

$$\|u\| \le \frac{1}{\sqrt{\lambda_1}}\|\nabla u\| \qquad \forall u \in V. \tag{2.4}$$

We also denote $b(u,v,w) = (u\cdot\nabla v, w)$, which is a continuous, trilinear, and skew-symmetric form

$$b(u,v,w) = -b(u,w,v) \qquad \forall u,v,w \in V. \tag{2.5}$$

We define a bilinear operator B from $V \times V$ to V' by

$$\langle B(u,v), w\rangle_{V',V} = b(u,v,w) \qquad \forall u,v,w \in V.$$

Then (1.1) can be rewritten as [22]

$$\frac{du}{dt} + \nu Au + B(u,u) = f, \qquad u(0) = 0,$$

and the discrete Cauchy one-leg method (1.6)-(1.7) becomes

$$\frac{1}{\tau}(u^{n+1} - u^n) + \nu Au^{n+\theta} + B(u^{n+\theta}, u^{n+\theta}) = f^{n+\theta}. \tag{2.6}$$

Finally, we recall the Ladyzhenskaya inequality in two spatial dimensions [15, 20, 21]

$$\|u\|_{L^4(\Omega)} \le 2^{-1/4}\|u\|^{1/2}\|\nabla u\|^{1/2}. \tag{2.7}$$

3 H-Stability

To prove the long-time stability of u^n in V, we first establish a bound in H, which is a discrete analogue of (1.2). We also collect several auxiliary results essential for the V stability analysis.

This section is organized as follows. In Lemma 1, we derive an energy estimate for the BE step (1.6). In Lemma 2, we use (1.7) to reformulate the result in Lemma 1 in terms of u^{n+1} and u^n instead of $u^{n+\theta}$. We then include a remark explaining why the result of this paper does not hold for $\theta = \frac{1}{2}$. In Proposition 1, we establish an H-bound on u^n, and in Corollary 1, we prove the existence of an absorbing set in H. Finally, in Lemmas 3 and 4, we obtain $L^2(H^1(\Omega))$ bounds on the fractional time values $u^{n+\theta}$ and the integer values u^n, respectively.

In the first result, we use the kinematic dissipation and the numerical dissipation of the BE step (2.6) to bound $\|u^{n+\theta}\|$ in terms of $\|u^n\|$ and $\|f\|_\infty$.

Lemma 1. *For every $n \ge 0$ and $\theta \in [0,1]$ we have*

$$(1 + \lambda_1\nu\theta\tau)\,\|u^{n+\theta}\|^2 - \|u^n\|^2 + \|u^{n+\theta} - u^n\|^2 \le \frac{1}{\nu}\theta\tau\frac{1}{\lambda_1}\|f\|_\infty^2. \tag{3.1}$$

Proof. Testing (2.6) with $\theta\tau u^{n+\theta}$ in H and using (2.5), the divergence theorem, the polarization identity, and the Hölder, Young, and Poincaré-Friedrichs (2.4) inequalities we obtain

$$\|u^{n+\theta}\|^2 - \|u^n\|^2 + \|u^{n+\theta} - u^n\|^2 + \nu\theta\tau\|\nabla u^{n+\theta}\|^2 \leq \frac{\theta\tau}{\nu}\frac{1}{\lambda_1}\|f^{n+\theta}\|^2. \tag{3.2}$$

Using the Poincaré-Friedrichs inequality (2.4) gives the relation (3.1). □

In the rest of the paper we assume the following time-step restriction

$$\tau \leq \frac{1}{\lambda_1 \nu} =: \kappa_1. \tag{3.3}$$

The next step is to write the inequality (3.1) in terms of u^{n+1} and u^n using the extrapolation (1.7). To simplify the presentation, we use the following notations

$$\alpha_\theta := \theta - \frac{1}{2}(2\theta-1)(1-\theta)\lambda_1\nu\tau + \frac{1}{2}\Big[(2\theta-1)(1-\theta)\Big(4\theta-(1-\theta)(2\theta+1)\lambda_1\nu\tau\Big)\lambda_1\nu\tau\Big]^{1/2}, \tag{3.4}$$

$$\varepsilon_\theta := \lambda_1\nu(2\theta-1)\tau, \tag{3.5}$$

$$a_\theta := \frac{1}{2}\Big\{(2\theta-1)\Big(4\theta-\lambda_1\nu(2\theta+1)(1-\theta)\tau\Big)^{1/2} - \Big(\lambda_1\nu(1-\theta)\tau\Big)^{1/2}\Big\}, \tag{3.6}$$

$$b_\theta := \frac{1}{2}\Big\{(2\theta-1)\Big(4\theta-\lambda_1\nu(2\theta+1)(1-\theta)\tau\Big)^{1/2} + \Big(\lambda_1\nu(1-\theta)\tau\Big)^{1/2}\Big\}. \tag{3.7}$$

We note that, under the assumption (3.3) on the size of the time step, for all $\theta \in \big(\frac{1}{2}, 1\big)$, we have that

$$\alpha_\theta \in (\tfrac{1}{2}, \tfrac{3}{2}], \qquad \varepsilon_\theta > 0. \tag{3.8}$$

The next result rearranges the left-hand-side of (3.1) in order to extend the dissipation result for the fractional time $\|u^{n+\theta}\|$ to the integer time $\|u^n\|$.

Lemma 2. *Under the time-step assumption (3.3), for every $\theta \in \big(\frac{1}{2}, 1\big)$, there exist constants (depending on θ) $\alpha_\theta, \varepsilon_\theta > 0$ and $a_\theta, b_\theta \in \mathbb{R}$ given in (3.4)-(3.7) such that*

$$(1+\lambda_1\nu\theta\tau)\|u^{n+\theta}\|^2 - \|u^n\|^2 + \|u^{n+\theta} - u^n\|^2 = (\alpha_\theta + \varepsilon_\theta)\|u^{n+1}\|^2 - \alpha_\theta\|u^n\|^2 + \|a_\theta u^{n+1} - b_\theta u^n\|^2 \quad \forall n \geq 0. \tag{3.9}$$

Proof. The conclusion follows by direct calculation. First, using the extrapolation (1.7) into the left hand side of (3.1) gives

$$\begin{aligned}
&(1+\lambda_1\nu\theta\tau)\,\|u^{n+\theta}\|^2 - \|u^n\|^2 + \|u^{n+\theta} - u^n\|^2 \\
&= \theta^2\,(2+\lambda_1\nu\theta\tau)\,\|u^{n+1}\|^2 + 2\theta\,[(1-\theta)\,(1+\lambda_1\nu\theta\tau) - \theta]\,(u^{n+1}, u^n) \\
&\quad + (\theta-1)\,[(1+\lambda_1\nu\theta\tau)\,(\theta-1) + (\theta+1)]\,\|u^n\|^2.
\end{aligned}$$

Next, we note that by (3.8) the constants $\alpha_\theta, \varepsilon_\theta > 0$, $a_\theta, b_\theta \in \mathbb{R}$ from (3.4)-(3.7) satisfy

$$\alpha_\theta + \varepsilon_\theta + a_\theta^2 = \theta^2\,(2+\lambda_1\nu\theta\tau)\,, \tag{3.10}$$

$$b_\theta^2 - \alpha_\theta = (\theta-1)\,[(1+\lambda_1\nu\theta\tau)\,(\theta-1) + (\theta+1)]\,, \tag{3.11}$$

$$2a_\theta b_\theta = 2\theta\,[\theta - (1-\theta)\,(1+\lambda_1\nu\theta\tau)]\,, \tag{3.12}$$

and therefore

$$\begin{aligned}
&(\alpha_\theta + \varepsilon_\theta)\|u^{n+1}\|^2 - \alpha_\theta\|u^n\|^2 + \|a_\theta u^{n+1} - b_\theta u^n\|^2 \\
&= (\alpha_\theta + \varepsilon_\theta + a_\theta^2)\|u^{n+1}\|^2 + (b_\theta^2 - \alpha_\theta)\|u^n\|^2 - 2a_\theta b_\theta(u^{n+1}, u^n) \\
&= \theta^2\,(2+\lambda_1\nu\theta\tau)\,\|u^{n+1}\|^2 + (\theta-1)\,[(1+\lambda_1\nu\theta\tau)\,(\theta-1) + (\theta+1)]\,\|u^n\|^2 \\
&\quad + 2\theta\,[(1-\theta)\,(1+\lambda_1\nu\theta\tau) - \theta]\,(u^{n+1}, u^n),
\end{aligned} \tag{3.13}$$

which finally gives (3.9). □

Remark. We emphasize that, when $\theta = \frac{1}{2}$, there is no $\alpha_\theta, \varepsilon_\theta > 0$ such that (3.9) holds. We first note that for (3.9) to hold, the relations (3.10)-(3.12) are necessary and sufficient. Adding these expressions we have that

$$(a_\theta + b_\theta)^2 + \varepsilon_\theta = 4\theta^2 - 1 + (2\theta - 1)^2 (1 + \lambda_1 \nu \theta \tau),$$

which, for $\theta = \frac{1}{2}$, shows there is no $\varepsilon_\theta > 0$ satisfying (3.9).

The next proposition is the main result of this section: a uniform bound of the semidiscrete-in-time velocity u^n in H in terms of the initial data u_0 and the forcing term f.

Proposition 1. *Assume that the time-step condition (3.3) holds. Then*

$$\|u^n\|^2 \le e^{-\frac{1}{15}\lambda_1 \nu (2\theta - 1) n\tau} \|u_0\|^2 + 3 \frac{1}{\lambda_1 \nu^2} \frac{1}{(2\theta - 1)} \|f\|_\infty^2 \qquad \forall n \ge 0, \tag{3.14}$$

and there is a constant

$$K_1(\|u_0\|, \|f\|_\infty) := \|u_0\|^2 + 3 \frac{1}{\lambda_1 \nu^2} \frac{1}{(2\theta - 1)} \|f\|_\infty^2$$

such that

$$\|u^n\|^2 \le K_1 \qquad \forall n \ge 0. \tag{3.15}$$

Proof. From (3.1) and (3.9) we have

$$\|u^{n+1}\|^2 \le \frac{1}{1 + \varepsilon_\theta / \alpha_\theta} \|u^n\|^2 + \frac{1}{\nu} \frac{1}{\alpha_\theta + \varepsilon_\theta} \theta \tau \|f\|_\infty^2,$$

and inductively,

$$\|u^{n+1}\|^2 \le \left(\frac{1}{1 + \varepsilon_\theta / \alpha_\theta} \right)^{n+1} \|u_0\|^2 + \sum_{j=1}^{n+1} \left(\frac{1}{1 + \varepsilon_\theta / \alpha_\theta} \right)^{n-j+1} \frac{1}{\nu} \frac{1}{\alpha_\theta + \varepsilon_\theta} \theta \tau \|f\|_\infty^2.$$

Now let $g(x) := e^{-\frac{1}{10}x} - \frac{1}{1+x}$, with $g'(x) \ge 0$ for all $x \in [0, 2]$, $g(0) = 0$, and $g(x) \ge 0$ for all $x \in [0, 2]$. Then by (3.8) we have

$$\frac{1}{1 + \varepsilon_\theta / \alpha_\theta} \le e^{-\frac{1}{10}\frac{\varepsilon_\theta}{\alpha_\theta}} \le e^{-\frac{1}{10}\frac{2}{3}\varepsilon_\theta} = e^{-\frac{1}{15}\varepsilon_\theta}.$$

Therefore,

$$\begin{aligned}
\|u^{n+1}\|^2 &\le \left(\frac{1}{1 + \varepsilon_\theta / \alpha_\theta} \right)^{n+1} \|u_0\|^2 + \sum_{j=1}^{n+1} \left(\frac{1}{1 + \varepsilon_\theta / \alpha_\theta} \right)^{n-j+1} \frac{1}{\nu} \frac{1}{\alpha_\theta + \varepsilon_\theta} \theta \tau \|f\|_\infty^2 \\
&\le e^{-\frac{1}{15}(n+1)\varepsilon_\theta} \|u_0\|^2 + \frac{1}{\nu} \tau \|f\|_\infty^2 \left(\sum_{j=0}^{n} \left(\frac{1}{1 + 2/3\varepsilon_\theta} \right)^j \right) \frac{1}{1/2 + 2/3\varepsilon_\theta} \\
&= e^{-\frac{1}{15}(n+1)\varepsilon_\theta} \|u_0\|^2 \\
&\quad + \frac{1}{\nu} \tau \|f\|_\infty^2 \frac{1}{1/2 + 2/3\varepsilon_\theta} \left(1 - \left(\frac{1}{1 + 2/3\varepsilon_\theta} \right)^{n+1} \right) \left(\frac{1}{1 - \frac{1}{1 + 2/3\varepsilon_\theta}} \right) \\
&\le e^{-\frac{1}{15}(n+1)\varepsilon_\theta} \|u_0\|^2 + \frac{1}{\nu} \tau \|f\|_\infty^2 \left(1 - \left(\frac{1}{1 + 2/3\varepsilon_\theta} \right)^{n+1} \right) \left(\frac{2}{2/3\varepsilon_\theta} \right) \\
&\le e^{-\frac{1}{15}(n+1)\tau\nu\lambda_1(2\theta - 1)} \|u_0\|^2 + 3 \frac{1}{\lambda_1 \nu^2} \frac{1}{2\theta - 1} \|f\|_\infty^2,
\end{aligned}$$

which yields (3.14) and (3.15). □

The following bound, which follows by direct computation from (3.14), proves the existence of an absorbing set in H.

Corollary 1. *Under the small time-step assumption* (3.3)*, provided the computational time interval $n\tau$ is sufficiently large*

$$n\tau \geq 15\frac{1}{\lambda_1}\frac{1}{\nu}\frac{1}{(2\theta-1)}\ln\left(\nu^2\lambda_1(2\theta-1)\frac{\|u_0\|^2}{\|f\|_\infty^2}\right),$$

we have that

$$\|u^n\|^2 \leq 4\frac{1}{\lambda_1\nu^2}\frac{1}{(2\theta-1)}\|f\|_\infty^2. \tag{3.16}$$

The next two lemmas are needed in Section 4 to prove the existence of an absorbing set in V. First, we derive an $L^2(H^1(\Omega))$ bound on the fractional time values $u^{n+\theta}$.

Lemma 3. *Assume that the time-step restriction* (3.3) *holds. Then we have the following bounds*

$$\nu\tau\sum_{i=0}^{n}\|\nabla u^{i+\theta}\|^2 \leq \|u_0\|^2 + \frac{1}{\nu}(n+1)\tau\frac{1}{\lambda_1}\|f\|_\infty^2 \qquad \forall n\geq 0, \tag{3.17}$$

and for any integer $p\geq 0$

$$\nu\tau\sum_{i=n}^{n+p}\|\nabla u^{i+\theta}\|^2 \leq \|u^n\|^2 + \frac{1}{\nu}(p+1)\tau\frac{1}{\lambda_1}\|f\|_\infty^2 \qquad \forall n\geq 0. \tag{3.18}$$

Proof. Testing (1.6) with $\theta\tau u^{n+\theta}$, and (1.7) with $(1-\theta)\tau u^{n+\theta}$ in H, using the polarization identity, and adding both equations yields

$$\frac{1}{2}\left(\|u^{n+1}\|^2-\|u^n\|^2\right)+\frac{1}{2}\left(\|u^{n+\theta}-u^n\|^2-\|u^{n+1}-u^{n+\theta}\|^2\right)+\nu\tau\|\nabla u^{n+\theta}\|^2=\tau(f^{n+\theta},u^{n+\theta}).$$

The following is a key step to prove the long-time stability of the Cauchy one-leg θ-method (2.6). We recall that, by (1.7), $u^{n+\theta}=\theta u^{n+1}+(1-\theta)u^n$. Therefore,

$$\begin{aligned}\|u^{n+\theta}-u^n\|^2-\|u^{n+1}-u^{n+\theta}\|^2 &= \|\theta u^{n+1}+(1-\theta)u^n-u^n\|^2-\|u^{n+1}-\theta u^{n+1}-(1-\theta)u^n\|^2\\ &= \theta^2\|u^{n+1}-u^n\|^2-(1-\theta)^2\|u^{n+1}-u^n\|^2\\ &= (2\theta-1)\|u^{n+1}-u^n\|^2,\end{aligned}$$

which, by using the Hölder, Young, and Poincaré-Friedrichs inequalities yields

$$\|u^{n+1}\|^2-\|u^n\|^2+(2\theta-1)\|u^{n+1}-u^n\|^2+\nu\tau\|\nabla u^{n+\theta}\|^2 \leq \frac{1}{\nu}\tau\frac{1}{\lambda_1}\|f^{n+\theta}\|^2 \leq \frac{1}{\nu}\tau\frac{1}{\lambda_1}\|f\|_\infty^2. \tag{3.19}$$

Summation in n implies (3.17), and similarly, summation for $i=n:n+p$, gives (3.18). □

Secondly, we derive an $L^2(H^1(\Omega))$ bound on the integer time values u^n. We introduce the following notation

$$\begin{aligned}
C_1 &:= \left(2K_1^2(2\theta^2-2\theta+1)\right)^{1/4},\\
K_2 &:= \left(\frac{1}{2}+\left|\frac{C_1^4(1-\theta)^4\theta^3}{\nu^3(2\theta-1)^3}-\frac{\nu}{2\theta}\right|\frac{1}{\nu}\right)\|u_0\|^2+\frac{\tau\nu}{8\theta}(4\theta^2-6\theta+3)\|\nabla u_0\|^2,\\
K_3 &:= \left(\frac{4\theta}{\nu(2\theta-1)}+\left|\frac{C_1^4(1-\theta)^4\theta^3}{\nu^3(2\theta-1)^3}-\frac{\nu}{2\theta}\right|\frac{1}{\nu^2}\right)\frac{1}{\lambda_1}\|f\|_\infty^2,\\
K_4 &:= \frac{1}{2}+\left|\frac{C_1^4(1-\theta)^4\theta^3}{\nu^3(2\theta-1)^3}-\frac{\nu}{2\theta}\right|\frac{1}{\nu}.
\end{aligned}$$

Lemma 4. *Assume that the time-step restriction (3.3) holds. Then we have*

$$\frac{\nu(2\theta-1)}{16\theta}\tau\sum_{i=0}^{n}\|\nabla u^{i+1}\|^2 \le K_2+(n+1)\tau K_3 \qquad \forall n\ge 0, \tag{3.20}$$

and for any integer $p\ge 0$

$$\frac{\nu(2\theta-1)}{16\theta}\tau\sum_{i=n}^{n+p}\|\nabla u^{i+1}\|^2 \le 4K_4K_1+K_3(p+1)\tau+\frac{\tau\nu}{8\theta}(4\theta^2-6\theta+3)\|\nabla u^n\|^2 \qquad \forall n\ge 0. \tag{3.21}$$

Proof. Multiplying (1.6) by $\theta\tau$ and (1.7) by $(1-\theta)\tau$ and adding yields

$$u^{n+1}-u^n-\tau\nu\Delta u^{n+\theta}+\tau u^{n+\theta}\cdot\nabla u^{n+\theta}+\tau\nabla p^{n+\theta}=\tau f^{n+\theta}. \tag{3.22}$$

Testing with u^{n+1} in H and using the polarization identity we have

$$\begin{aligned}\frac12\|u^{n+1}\|^2-\frac12\|u^n\|^2+\frac12\|u^{n+1}-u^n\|^2+\nu\tau(\nabla u^{n+\theta},\nabla u^{n+1})&\\ =-\tau b(u^{n+\theta},u^{n+\theta},u^{n+1})+\tau(f^{n+\theta},u^{n+1}).&\end{aligned} \tag{3.23}$$

We now explain how to handle each of the last three terms of the previous expression.

The first key step in the proof deals with the kinematic dissipation term $\nu\tau(\nabla u^{n+\theta},\nabla u^{n+1})$, which we rewrite

$$(1-\theta)(\nabla u^n,\nabla u^{n+1})=\frac{1}{2\theta}\left(-\theta^2\|\nabla u^{n+1}\|^2-(1-\theta)^2\|\nabla u^n\|^2+\|\theta\nabla u^{n+1}+(1-\theta)\nabla u^n\|^2\right),$$

and therefore,

$$\begin{aligned}(\nabla u^{n+\theta},\nabla u^{n+1})&=(\nabla(\theta u^{n+1}+(1-\theta)u^n),\nabla u^{n+1})\\ &=\theta\|\nabla u^{n+1}\|^2+(1-\theta)(\nabla u^n,\nabla u^{n+1})\\ &=\theta\|\nabla u^{n+1}\|^2+\frac{1}{2\theta}\left(-\theta^2\|\nabla u^{n+1}\|^2-(1-\theta)^2\|\nabla u^n\|^2+\|\theta\nabla u^{n+1}+(1-\theta)\nabla u^n\|^2\right)\\ &=\frac{\theta}{2}\|\nabla u^{n+1}\|^2-\frac{(1-\theta)^2}{2\theta}\|\nabla u^n\|^2+\frac{1}{2\theta}\|\nabla u^{n+\theta}\|^2.\end{aligned} \tag{3.24}$$

Next, we bound the body force term using the Hölder, Young, and Poincaré-Friedrichs inequalities. This yields

$$\tau(f^{n+\theta},u^{n+1})\le\tau\|f^{n+\theta}\|\|u^{n+1}\|\le\frac{\tau}{2\delta}\frac{1}{\lambda_1}\|f^{n+\theta}\|^2+\tau\frac{\delta}{2}\|\nabla u^{n+1}\|^2 \tag{3.25}$$

for any $\delta>0$, to be chosen later.

The third key step deals with the non-linear convective term. Using the skew-symmetry property (2.5), we have

$$\begin{aligned}-\tau b(u^{n+\theta},u^{n+\theta},u^{n+1})&=-\tau b(u^{n+\theta},\theta u^{n+1}+(1-\theta)u^n,u^{n+1})\\ &=-\tau\theta b(u^{n+\theta},u^{n+1},u^{n+1})-\tau(1-\theta)b(u^{n+\theta},u^n,u^{n+1})\\ &=-\tau(1-\theta)b(u^{n+\theta},u^n,u^{n+1}).\end{aligned}$$

Using Hölder's inequality with $p=2$, $q=4$, $r=4$ and the Ladyzhenskaya inequality (2.7) we obtain

$$-\tau(1-\theta)b(u^{n+\theta},u^n,u^{n+1})\le(1-\theta)\tau 2^{-1/2}\|\nabla u^n\|\|u^{n+\theta}\|^{1/2}\|\nabla u^{n+\theta}\|^{1/2}\|u^{n+1}\|^{1/2}\|\nabla u^{n+1}\|^{1/2}.$$

Now due to (3.15), we have

$$\|u^{n+\theta}\|^2=\|\theta u^{n+1}+(1-\theta)u^n\|^2\le 2\theta^2\|u^{n+1}\|^2+2(1-\theta)^2\|u^n\|^2\le 2(2\theta^2-2\theta+1)K_1,$$

hence

$$\|u^{n+1}\|^{1/2}\|u^{n+\theta}\|^{1/2} \leq C_1.$$

Therefore, applying Young's inequality twice, we get that for arbitrary $\varepsilon, x > 0$ (to be made precise later),

$$\begin{aligned} -\tau(1-\theta)b(u^{n+\theta}, u^n, u^{n+1}) &\leq (1-\theta)\tau 2^{-1/2} C_1 \|\nabla u^n\| \|\nabla u^{n+\theta}\|^{1/2} \|\nabla u^{n+1}\|^{1/2} \\ &\leq \frac{(1-\theta)\tau}{\sqrt{2}} C_1 \frac{\varepsilon}{2} \|\nabla u^n\|^2 + \frac{(1-\theta)\tau}{\sqrt{2}} C_1 \frac{x}{4\varepsilon} \|\nabla u^{n+1}\|^2 + \frac{(1-\theta)\tau}{\sqrt{2}} C_1 \frac{1}{4x\varepsilon} \|\nabla u^{n+\theta}\|^2. \end{aligned} \tag{3.26}$$

Plugging (3.24), (3.25), and (3.26) back into (3.23) we obtain

$$\begin{aligned} &\frac{1}{2}\|u^{n+1}\|^2 - \frac{1}{2}\|u^n\|^2 + \frac{1}{2}\|u^{n+1} - u^n\|^2 \\ &\quad + \left(\frac{\tau\nu\theta}{2} - \frac{(1-\theta)\tau}{\sqrt{2}} C_1 \frac{x}{4\varepsilon} - \frac{\delta\tau}{2}\right)\|\nabla u^{n+1}\|^2 - \left(\frac{\tau\nu(1-\theta)^2}{2\theta} + \frac{(1-\theta)\tau}{\sqrt{2}} C_1 \frac{\varepsilon}{2}\right)\|\nabla u^n\|^2 \\ &\quad + \left(\frac{\nu\tau}{2\theta} - \frac{(1-\theta)\tau}{\sqrt{2}} C_1 \frac{1}{4\varepsilon x}\right)\|\nabla u^{n+\theta}\|^2 \\ &\leq \frac{\tau}{2\delta}\frac{1}{\lambda_1}\|f^{n+\theta}\|^2. \end{aligned} \tag{3.27}$$

With the values $\varepsilon = \frac{\nu(2\theta-1)}{2\theta(1-\theta)C_1\sqrt{2}}$, $x = \frac{\nu^2(2\theta-1)^2}{2\theta^2(1-\theta)^2C_1^2}$, and $\delta = \frac{\nu(2\theta-1)}{8\theta}$, we see that (3.27) writes as

$$\begin{aligned} &\frac{1}{2}\|u^{n+1}\|^2 - \frac{1}{2}\|u^n\|^2 + \frac{1}{2}\|u^{n+1} - u^n\|^2 \\ &\quad + \frac{\tau\nu(2\theta-1)}{16\theta}\|\nabla u^{n+1}\|^2 + \frac{\tau\nu}{8\theta}(4\theta^2 - 6\theta + 3)\|\nabla u^{n+1}\|^2 - \frac{\tau\nu}{8\theta}(4\theta^2 - 6\theta + 3)\|\nabla u^n\|^2 \\ &\leq \frac{\tau 4\theta}{\nu(2\theta-1)}\frac{1}{\lambda_1}\|f^{n+\theta}\|^2 + \tau\left(\frac{C_1^4(1-\theta)^4\theta^3}{\nu^3(2\theta-1)^3} - \frac{\nu}{2\theta}\right)\|\nabla u^{n+\theta}\|^2. \end{aligned} \tag{3.28}$$

Adding from $i = 0$ to n, and using the $L^2(H^1(\Omega))$ bound (3.17) on the fractional time values yields

$$\begin{aligned} &\frac{1}{2}\|u^{n+1}\|^2 + \frac{1}{2}\sum_{i=0}^{n}\|u^{i+1} - u^i\|^2 + \frac{\tau\nu}{8\theta}(4\theta^2 - 6\theta + 3)\|\nabla u^{n+1}\|^2 + \frac{\tau\nu(2\theta-1)}{16\theta}\sum_{i=0}^{n}\|\nabla u^{i+1}\|^2 \\ &\leq \left(\frac{1}{2} + \left|\frac{C_1^4(1-\theta)^4\theta^3}{\nu^3(2\theta-1)^3} - \frac{\nu}{2\theta}\right|\frac{1}{\nu}\right)\|u_0\|^2 + \frac{\tau\nu}{8\theta}(4\theta^2 - 6\theta + 3)\|\nabla u_0\|^2 \\ &\quad + \left(\frac{4\theta}{\nu(2\theta-1)} + \left|\frac{C_1^4(1-\theta)^4\theta^3}{\nu^3(2\theta-1)^3} - \frac{\nu}{2\theta}\right|\frac{}{\nu^2}\right)(n+1)\tau\frac{1}{\lambda_1}\|f\|_\infty^2, \end{aligned}$$

which implies (3.20). Finally, from (3.28), adding from $i = n$ to $n+p$, we obtain

$$\begin{aligned} &\frac{1}{2}\|u^{n+p+1}\|^2 + \frac{1}{2}\sum_{i=n}^{n+p}\|u^{i+1} - u^i\|^2 + \frac{\tau\nu}{8\theta}(4\theta^2 - 6\theta + 3)\|\nabla u^{n+p+1}\|^2 + \frac{\tau\nu(2\theta-1)}{16\theta}\sum_{i=n}^{n+p}\|\nabla u^{i+1}\|^2 \\ &\leq \frac{1}{2}\|u^n\|^2 + \frac{\tau\nu}{8\theta}(4\theta^2 - 6\theta + 3)\|\nabla u^n\|^2 + \frac{\tau 4\theta}{\nu(2\theta-1)}(p+1)\frac{1}{\lambda_1}\|f\|_\infty^2 \\ &\quad + \left|\frac{C_1^4(1-\theta)^4\theta^3}{\nu^3(2\theta-1)^3} - \frac{\nu}{2\theta}\right|\frac{1}{\nu}\|u^n\|^2 + \left|\frac{C_1^4(1-\theta)^4\theta^3}{\nu^3(2\theta-1)^3} - \frac{\nu}{2\theta}\right|\frac{1}{\nu^2}(p+1)\tau\frac{1}{\lambda_1}\|f\|_\infty^2 \\ &\leq 4K_4K_1 + K_3(p+1)\tau + \frac{\tau\nu}{8\theta}(4\theta^2 - 6\theta + 3)\|\nabla u^n\|^2, \end{aligned}$$

which, in particular, yields (3.21). □

4 V-Stability

This section establishes our main result, namely that the Cauchy one-leg θ-method (2.6) is V-stable. Our goal is to prove bounds analogous to the continuous-in-time bounds (1.3) and (1.4). We follow the approach of [29], and briefly outline the key steps here.

In Lemma 5, we show that, provided a suitable time-step restriction holds at step n, then there is a bound on $\|\nabla u^{n+1}\|$ in terms of $\|\nabla u^n\|$. In Lemma 6, we recall the Discrete Grönwall Lemma. In Proposition 2, we establish finite-time stability using Lemma 5 and Lemma 6. In Lemma 7, we recall the discrete Uniform Grönwall Lemma. Finally, in Theorem 1, we prove the long-time V-stability.

To set the stage for the first lemma, we argue as in the proof of (3.19) in Lemma 3. We test (1.6) with $-\theta\tau\Delta u^{n+\theta}$ and (1.7) with $-(1-\theta)\tau\Delta u^{n+\theta}$ in V, use the polarization identity, the Hölder and Young inequalities, and add to obtain

$$\frac{1}{2}\left(\|\nabla u^{n+1}\|^2-\|\nabla u^n\|^2\right)+\frac{1}{2}(2\theta-1)\|\nabla(u^{n+1}-u^n)\|^2+\frac{3}{4}\nu\tau\|\Delta u^{n+\theta}\|^2\le\frac{1}{\nu}\tau\|f\|_\infty^2-\tau b(u^{n+\theta},u^{n+\theta},-\Delta u^{n+\theta}).$$

Next, we use the Hölder inequality with $p=4$, $q=4$, $r=2$, the Ladyzhenskaya inequality (2.7), and the Young inequality to bound the non-linear term, and we get

$$\begin{aligned}&\frac{1}{2}\left(\|\nabla u^{n+1}\|^2-\|\nabla u^n\|^2\right)+\frac{1}{2}(2\theta-1)\|\nabla(u^{n+1}-u^n)\|^2+\frac{1}{2}\nu\tau\|\Delta u^{n+\theta}\|^2\\&\le\frac{1}{\nu}\tau\|f\|_\infty^2+\tau\frac{27}{16}\frac{1}{\nu^3}\|u^{n+\theta}\|^2\|\nabla u^{n+\theta}\|^4.\end{aligned}\tag{4.1}$$

Since $u^{n+\theta}=\theta u^{n+1}+(1-\theta)u^n$, we have

$$\begin{aligned}\|u^{n+\theta}\|^2&=\|\theta u^{n+1}+(1-\theta)u^n\|^2\le 2(\|u^{n+1}\|^2+\|u^n\|^2),\\\|\nabla u^{n+\theta}\|^4&=\|\theta\nabla u^{n+1}+(1-\theta)\nabla u^n\|^4\le 2^3(\|\nabla u^{n+1}\|^4+\|\nabla u^n\|^4).\end{aligned}$$

Therefore, (4.1) yields

$$\|\nabla u^{n+1}\|^2-\|\nabla u^n\|^2+(2\theta-1)\|\nabla(u^{n+1}-u^n)\|^2+\nu\tau\|\Delta u^{n+\theta}\|^2\le\frac{2}{\nu}\tau\|f\|_\infty^2+\tau\frac{54}{\nu^3}2K_1\|\nabla u^{n+1}\|^4+\tau\frac{54}{\nu^3}2K_1\|\nabla u^n\|^4,$$

and moreover

$$0\le\tau\frac{54}{\nu^3}2K_1\|\nabla u^{n+1}\|^4-\|\nabla u^{n+1}\|^2+\|\nabla u^n\|^2+\tau\frac{54}{\nu^3}2K_1\|\nabla u^n\|^4+\frac{2}{\nu}\tau\|f\|_\infty^2.\tag{4.2}$$

Before we state the first lemma of this section, we define some notations. Note that for any $\theta\in\left(\frac{1}{2},1\right]$ there exists two positive constants $\delta_1,\varepsilon_1>0$ such that

$$C_2:=2^{-1/2}\theta(1+2\theta)\sqrt{K_1},\tag{4.3}$$

$$C_3:=\nu\theta^2+\frac{\nu}{2}\theta^2-\varepsilon_1-\nu\theta(2-\theta)\frac{\delta_1}{2}>0,\tag{4.4}$$

$$C_4:=\frac{1}{\varepsilon_1}C_2^2-\nu(1-\theta)^2+\nu-\frac{\nu}{2}\theta^2+\nu\theta(2-\theta)\frac{1}{2\delta_1}>0.\tag{4.5}$$

Lemma 5. *Suppose that the time-step restriction (3.3) holds, and assume that, for some n, we have*

$$\tau\frac{108}{\nu^3}K_1\left[\left(\frac{2}{\nu C_3}\frac{1}{\lambda_1}+\frac{2}{\nu^2}\frac{1}{\lambda_1}\right)\|f\|_\infty^2+\left(1+\frac{C_4}{C_3}\right)\|\nabla u^n\|^2+\frac{108}{\nu^4}\frac{1}{\lambda_1}K_1\|\nabla u^n\|^4\right]\le\frac{1}{5}.\tag{4.6}$$

Then

$$\|\nabla u^{n+1}\|^2\le\|\nabla u^n\|^2\left(1+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2\right)\left[1+2\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4\right)\right]+\frac{18}{5\nu}\tau\|f\|_\infty^2.\tag{4.7}$$

Proof. We begin by noting that the right-hand side of (4.2) is a quadratic polynomial in $\|\nabla u^{n+1}\|^2$, hence either

$$\|\nabla u^{n+1}\|^2 \leq \frac{1-\sqrt{\Delta_n}}{2K_1\tau\frac{108}{\nu^3}}, \tag{4.8}$$

or

$$\|\nabla u^{n+1}\|^2 \geq \frac{1+\sqrt{\Delta_n}}{2K_1\tau\frac{108}{\nu^3}}, \tag{4.9}$$

where

$$\Delta_n = 1-4\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4+\frac{2}{\nu}\tau\|f\|_\infty^2\right).$$

Moreover, we note that (3.3) and (4.6) imply $\Delta_n > 0$. We now show that only (4.8) holds.

Testing (1.6) with $2\tau(u^{n+\theta}-u^n)$ in H and using the divergence theorem and polarization identity gives

$$\begin{aligned}&\frac{2}{\theta}\|u^{n+\theta}-u^n\|^2+\nu\tau\|\nabla u^{n+\theta}\|^2-\nu\tau\|\nabla u^n\|^2+\nu\tau\|\nabla u^{n+\theta}-\nabla u^n\|^2+2\tau b(u^{n+\theta},u^{n+\theta},u^{n+\theta}-u^n)\\&=2\tau(f^{n+\theta},u^{n+\theta}-u^n).\end{aligned} \tag{4.10}$$

Applying the properties of the trilinear form and the Hölder, Ladyzhenskaya, and Young inequalities yields

$$\begin{aligned}&b(u^{n+\theta},u^{n+\theta},u^{n+\theta}-u^n)=b(u^{n+\theta},u^n,u^{n+\theta})\\&=b(\theta u^{n+1}+(1-\theta)u^n,u^n,\theta u^{n+1}+(1-\theta)u^n)\\&=\theta^2 b(u^{n+1},u^n,u^{n+1})-\theta b(u^n,u^{n+1},u^n)+\theta^2 b(u^n,u^{n+1},u^n)\\&\leq 2^{-1/2}\theta^2\|u^{n+1}\|\|\nabla u^{n+1}\|\|\nabla u^n\|+2^{-1/2}\theta^2\|u^n\|\|\nabla u^n\|\|\nabla u^{n+1}\|+2^{-1/2}\theta\|u^n\|\|\nabla u^n\|\|\nabla u^{n+1}\|\\&\leq 2^{-1/2}(2\theta^2\sqrt{K_1}+\theta\sqrt{K_1})\|\nabla u^n\|\|\nabla u^{n+1}\|\\&\leq\frac{1}{2\varepsilon_1}C_2^2\|\nabla u^n\|^2+\frac{\varepsilon_1}{2}\|\nabla u^{n+1}\|^2.\end{aligned}$$

We use the Hölder, Young, and Poincaré-Friedrichs inequalities to bound the forcing term

$$2\tau(f^{n+\theta},u^{n+\theta}-u^n)\leq\frac{\nu\tau}{2}\|\nabla(u^{n+\theta}-u^n)\|^2+\tau\frac{2}{\nu}\frac{1}{\lambda_1}\|f^{n+\theta}\|^2.$$

Plugging these two inequalities into (4.10) implies

$$\begin{aligned}&\frac{2}{\theta}\|u^{n+\theta}-u^n\|^2+\nu\tau\|\nabla u^{n+\theta}\|^2-\nu\tau\|\nabla u^n\|^2+\nu\tau\|\nabla u^{n+\theta}-\nabla u^n\|^2\\&\leq\frac{2\tau}{2\varepsilon_1}C_2^2\|\nabla u^n\|^2+\frac{2\tau\varepsilon_1}{2}\|\nabla u^{n+1}\|^2+\frac{\nu\tau}{2}\|\nabla(u^{n+\theta}-u^n)\|^2+\frac{2\tau}{\nu}\frac{1}{\lambda_1}\|f^{n+\theta}\|^2.\end{aligned} \tag{4.11}$$

Using $u^{n+\theta}=\theta u^{n+1}+(1-\theta)u^n$, we write the norms of $u^{n+\theta}$ in terms of u^{n+1} and u^n

$$\begin{aligned}\|\nabla u^{n+\theta}\|^2&=\theta^2\|\nabla u^{n+1}\|^2+(1-\theta)^2\|\nabla u^n\|^2+2\theta(1-\theta)(\nabla u^{n+1},\nabla u^n),\\\|\nabla u^{n+\theta}-\nabla u^n\|^2&=\theta^2\|\nabla u^{n+1}\|^2+\theta^2\|\nabla u^n\|^2+2\theta^2(\nabla u^{n+1},\nabla u^n),\end{aligned}$$

so (4.11) becomes

$$\begin{aligned}&\frac{2}{\theta}\|u^{n+\theta}-u^n\|^2+(\nu\tau\theta^2+\frac{\nu\tau}{2}\theta^2-\tau\varepsilon)\|\nabla u^{n+1}\|^2+(\nu\tau(1-\theta)^2-\nu\tau+\frac{\nu\tau}{2}\theta^2-\frac{\tau}{\varepsilon}C_2^2)\|\nabla u^n\|^2\\&\leq\frac{2\tau}{\nu}\frac{1}{\lambda_1}\|f^{n+\theta}\|^2-\nu\tau\theta(2-\theta)(\nabla u^{n+1},\nabla u^n).\end{aligned}$$

Moreover, using the Hölder, Young, and Poincaré-Friedrichs inequalities, we obtain

$$\begin{aligned}&\left(\nu\tau\theta^2+\frac{\nu\tau}{2}\theta^2-\tau\varepsilon_1-\nu\tau\theta(2-\theta)\frac{\delta_1}{2}\right)\|\nabla u^{n+1}\|^2\\&\leq\frac{2\tau}{\nu}\frac{1}{\lambda_1}\|f\|_\infty^2+\left(\frac{\tau}{\varepsilon_1}C_2^2-\nu\tau(1-\theta)^2+\nu\tau-\frac{\nu\tau}{2}\theta^2+\nu\tau\theta(2-\theta)\frac{1}{2\delta_1}\right)\|\nabla u^n\|^2,\end{aligned}$$

so

$$\|\nabla u^{n+1}\|^2\leq\frac{2}{\nu C_3}\frac{1}{\lambda_1}\|f\|_\infty^2+\frac{C_4}{C_3}\|\nabla u^n\|^2.$$

Now, from assumption (4.6) we have

$$2K_1\tau\frac{108}{\nu^3}\|\nabla u^{n+1}\|^2\leq 2K_1\tau\frac{108}{\nu^3}\left(\frac{2}{\nu C_3}\frac{1}{\lambda_1}\|f\|_\infty^2+\frac{C_4}{C_3}\|\nabla u^n\|^2\right)<1,$$

which contradicts (4.9). Therefore, we must have (4.8)

$$\begin{aligned}\|\nabla u^{n+1}\|^2&\leq\frac{1-\sqrt{\Delta_n}}{2K_1\tau\frac{108}{\nu^3}}\\&=\frac{1-\left(1-4\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4+\frac{2}{\nu}\tau\|f\|_\infty^2\right)\right)^{1/2}}{2K_1\tau\frac{108}{\nu^3}}\\&=\frac{2\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4+\frac{2}{\nu}\tau\|f\|_\infty^2\right)}{1+(1-x)^{1/2}},\end{aligned}\tag{4.12}$$

where

$$x=4\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4+\frac{2}{\nu}\tau\|f\|_\infty^2\right).$$

We now proceed to proving the conclusion (4.7) of the lemma. Using the bounds (3.3) and (4.6), we get

$$\begin{aligned}x&=4\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4+\frac{2}{\nu}\tau\|f\|_\infty^2\right)\\&\leq 4\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2+\frac{1}{\lambda_1\nu}\frac{108}{\nu^3}K_1\|\nabla u^n\|^4+\frac{2}{\nu}\frac{1}{\lambda_1\nu}\|f\|_\infty^2\right)\leq\frac{4}{5},\end{aligned}$$

and since

$$\frac{2}{1+(1-x)^{1/2}}\leq 1+\frac{x}{2},$$

for $0\leq x\leq\frac{4}{5}$, then by (4.12) we obtain that

$$\begin{aligned}\|\nabla u^{n+1}\|^2&\leq\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4+\frac{2}{\nu}\tau\|f\|_\infty^2\right)\\&\quad\times\left(1+2\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4+\frac{2}{\nu}\tau\|f\|_\infty^2\right)\right)\\&=\|\nabla u^n\|^2\left(1+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2\right)+\frac{2}{\nu}\tau\|f\|_\infty^2\\&\quad+2\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4\left(1+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2\right)^2\\&\quad+2\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2\left(1+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2\right)\frac{2}{\nu}\tau\|f\|_\infty^2\end{aligned}\tag{4.13}$$

$$+2\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2\left(1+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2\right)\frac{2}{\nu}\tau\|f\|_\infty^2$$
$$+2\tau\frac{108}{\nu^3}K_1\frac{4}{\nu^2}\tau^2\|f\|_\infty^4.$$

Using the time-step restriction (3.3) and the assumption (4.6) in the right hand side of (4.13) the last three terms equate to

$$\|f\|_\infty^2\left[2\tau\frac{108}{\nu^3}K_1\frac{2}{\nu}\tau\left(2\|\nabla u^n\|^2(1+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2)+\frac{2}{\nu}\tau\|f\|_\infty^2\right)\right]\leq\frac{8}{5\nu}\tau\|f\|_\infty^2.$$

Adding this to the second term in the right-hand side of (4.13) gives

$$\frac{2}{\nu}\tau\|f\|_\infty^2+\frac{4}{\nu}\tau\|f\|_\infty^2\cdot\frac{2}{5}=\frac{18}{5\nu}\tau\|f\|_\infty^2.$$

The rest of the terms equal

$$\|\nabla u^n\|^2\left(1+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2\right)\left[1+2\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4\right)\right].$$

Therefore, putting everything together, we obtain the bound (4.7). □

For reader's convenience we recall a result similar to the Discrete Grönwall Lemma proved in [25, 29].

Lemma 6 (Discrete Grönwall Lemma)**.** *Given $\tau>0$, positive integer $n_*>0$ and positive sequences $\xi_n,\eta_n,\zeta_n,\alpha_n$ such that*

$$\xi_n\leq\xi_{n-1}(1+\tau\alpha_{n-1})(1+\tau\eta_{n-1})+\tau\zeta_n \tag{4.14}$$

for $n=1,\dots n_*$.
We have, for any $n\in\{2,\dots,n_*\}$,

$$\xi_n\leq\xi_0\exp\left(\sum_{i=0}^{n-1}\tau\eta_i\right)\exp\left(\sum_{i=0}^{n-1}\tau\alpha_i\right)+\sum_{i=1}^{n-1}\left[\tau\zeta_i\exp\left(\sum_{j=i}^{n-1}\tau\eta_j\right)\exp\left(\sum_{j=i}^{n-1}\tau\alpha_j\right)\right]+\tau\zeta_n. \tag{4.15}$$

Proof. Using (4.14) recursively, we derive

$$\xi_n\leq\xi_0\prod_{i=0}^{n-1}\left[(1+\tau\eta_i)(1+\tau\alpha_i)\right]+\sum_{i=1}^{n-1}\tau\zeta_i\prod_{j=i}^{n-1}\left[(1+\tau\eta_j)(1+\tau\alpha_j)\right]+\tau\zeta_n$$

Using the fact that $1+x\leq e^x$ for all $x\in\mathbb{R}$, the conclusion of the lemma follows. □

We are now ready to prove a V-bound on u^n for a finite time interval $[0,T]$. For ease of notation, we define the following

$$C_5:=\frac{108K_1 16\theta}{\nu^4(2\theta-1)},\qquad C_6:=3C_5K_2+2C_5^2K_2^2,\qquad C_7:=3C_5K_3+4C_5^2K_2K_3\qquad C_8:=2C_5^2K_3^2,$$

and

$$\begin{aligned}K_5(\|\nabla u_0\|,\|f\|_\infty,n\tau):=&\|\nabla u_0\|^2\exp(C_6)\exp(C_7n\tau)\exp(C_8(n\tau)^2)\\&+n\tau\frac{18}{5\nu}\|f\|_\infty^2\exp(C_6)\exp(C_7n\tau)\exp(C_8(n\tau)^2).\end{aligned} \tag{4.16}$$

Also, we denote

$$\kappa_2=\frac{1}{15\frac{108}{\nu^3}K_1(\frac{2}{\nu C_3}\frac{1}{\lambda_1}+\frac{2}{\lambda_1\nu^2})\|f\|_\infty^2}, \tag{4.17}$$

$$\kappa_3 = \frac{1}{15\frac{108}{\nu^3}K_1(1+C_4/C_3)K_5(\|\nabla u_0\|, \|f\|_\infty, T)}, \tag{4.18}$$

$$\kappa_4 = \frac{1}{15\frac{108^2}{\nu^7}K_1^2\frac{1}{\lambda_1}K_5^2(\|\nabla u_0\|, \|f\|_\infty, T)}. \tag{4.19}$$

Proposition 2. *Let $T > 0$ and let $K_5(\cdot,\cdot,\cdot)$ be the function defined in (4.16), which is monotonically increasing in all of its arguments.*
Suppose the time step is such that

$$\tau \le \min\{\kappa_1, \kappa_2(\|f\|_\infty), \kappa_3(\|\nabla u_0\|, \|f\|_\infty, T), \kappa_4(\|\nabla u_0\|, \|f\|_\infty, T)\}. \tag{4.20}$$

Then the bound (4.7) holds for all $n+1 = 1,\ldots,N := \lfloor T/\tau \rfloor$ and

$$\|\nabla u^n\|^2 \le K_5(\|\nabla u_0\|, \|f\|_\infty, n\tau) \qquad \forall n = 0,\ldots,N := \lfloor T/\tau \rfloor. \tag{4.21}$$

Proof. Fix $T > 0$ and let τ satisfy (4.20). We will use induction on n. When $n = 0$,

$$\|\nabla u_0\|^2 \le \|\nabla u_0\|^2 \exp(C_6) = K_5(\|\nabla u_0\|, \|f\|_\infty, 0).$$

By the time-step assumption (4.20) and the fact that $K_5(\cdot,\cdot,\cdot)$ is monotonically increasing in all of its arguments, the condition (4.6) is satisfied for $n = 0$, hence by Lemma 5 we have

$$\|\nabla u^1\|^2 \le \|\nabla u^0\|^2\left(1+\tau\frac{108}{\nu^3}K_1\|\nabla u^0\|^2\right)\left[1+2\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^0\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^0\|^4\right)\right]+\frac{18}{5\nu}\tau\|f\|_\infty^2.$$

Now we assume that (4.6) holds for $n = 0,\ldots,m-1$ for some $m \le N$. Then, again by Lemma 5 we have that (4.7) holds for $n+1 = 1,\ldots,m$. Furthermore, we bound $\|\nabla u^m\|$ by using the Discrete Grönwall Lemma 6. We write (4.7) in the form (4.14) in Lemma 6, where

$$\xi_n = \|\nabla u^n\|^2, \qquad \alpha_n = \frac{108}{\nu^3}K_1\|\nabla u^n\|^2, \qquad \eta_n = 2\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4\right), \qquad \zeta_n = \frac{18}{5\nu}\|f\|_\infty^2.$$

We now compute the sums that appear in the right-hand side of (4.15).

Using the $L^2(H^1)$ bound (3.20) in Lemma 4, we have

$$\sum_{i=0}^{n-1}\tau\alpha_i = \frac{108}{\nu^3}K_1\sum_{i=0}^{n-1}\tau\|\nabla u^i\|^2 \le \frac{108}{\nu^3}K_1\frac{16\theta}{\nu(2\theta-1)}(K_2+N\tau K_3) \tag{4.22}$$

and

$$\sum_{j=i}^{n-1}\tau\alpha_j = \frac{108}{\nu^3}K_1\sum_{j=i}^{n-1}\tau\|\nabla u^j\|^2 \le \frac{108}{\nu^3}K_1\sum_{j=0}^{n-1}\tau\|\nabla u^j\|^2 \le \frac{108}{\nu^3}K_1\frac{16\theta}{\nu(2\theta-1)}(K_2+n\tau K_3). \tag{4.23}$$

Now we use the fact that if $a_i \ge 0$, $\sum_{i=0}^{n-1}a_i^2 \le \left(\sum_{i=0}^{n-1}a_i\right)^2$ and the bounds (4.22) and (4.23) to get

$$\begin{aligned}
\sum_{i=0}^{n-1}\tau\eta_i &= \sum_{i=0}^{n-1}\tau 2\frac{108}{\nu^3}K_1\left(\|\nabla u^i\|^2+\tau\frac{108}{\nu^3}K_1\|\nabla u^i\|^4\right) \\
&= \sum_{i=0}^{n-1}\tau 2\frac{108}{\nu^3}K_1\|\nabla u^i\|^2 + 2\sum_{i=0}^{n-1}\left(\tau\frac{108}{\nu^3}K_1\|\nabla u^i\|^2\right)^2 \\
&\le 2\frac{108}{\nu^3}K_1\sum_{i=0}^{n-1}\tau\|\nabla u^i\|^2 + 2\left(\sum_{i=0}^{n-1}\tau\alpha_i\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{i=0}^{n-1}\tau\alpha_i + 2\left(\sum_{i=0}^{n-1}\tau\alpha_i\right)^2 \\
&\le 2C_5(K_2+n\tau K_3) + 2(C_5(K_2+n\tau K_3))^2 \\
&= 2C_5K_2 + 2C_5n\tau K_3 + 2C_5^2K_2^2 + 2C_5^2K_3^2(n\tau)^2 + 4C_5^2K_2K_3n\tau,
\end{aligned}
$$

and similarly,

$$
\sum_{j=i}^{n-1}\tau\eta_j \le \sum_{j=0}^{n-1}\tau\eta_i \le 2C_5K_2 + 2C_5n\tau K_3 + 2C_5^2K_2^2 + 2C_5^2K_3^2(n\tau)^2 + 4C_5^2K_2K_3n\tau.
$$

Finally, we can apply the Discrete Grönwall Lemma 6 to get the desired bound (4.21) on m

$$
\|\nabla u^m\|^2 \le K_5(\|\nabla u_0\|, \|f\|_\infty, m\tau). \tag{4.24}
$$

We note that the bound K_5 in (4.24) depends on the initial discrete value through $\|\nabla u_0\|$ and on m, but the dependence on m is only through the time $m\tau$. So as long as $m \le N = \lfloor T/\tau \rfloor$, the assumption (4.6) is satisfied, and the bound (4.7) follows from Lemma 5. □

For the reader's convenience, we now recall a result similar to Discrete Uniform Grönwall Lemma in [29].

Lemma 7 (Discrete Uniform Grönwall Lemma)**.** *Given $\tau > 0$, $n_1, n_2, n_* \in \mathbb{N}$ such that $n_1 < n_*$ and $n_1 + n_2 + 1 \le n_*$, given positive sequences $\xi_n, \eta_n, \zeta_n, \alpha_n$ such that*

$$
\xi_n \le \xi_{n-1}(1+\tau\alpha_{n-1})(1+\tau\eta_{n-1}) + \tau\zeta_n \qquad \textit{for} \quad n = n_1, \dots n_*, \tag{4.25}
$$

and given the bounds

$$
\sum_{n=n'}^{n'+n_2}\tau\eta_n \le a_1(n_1,n_*), \qquad \sum_{n=n'}^{n'+n_2}\tau\zeta_n \le a_2(n_1,n_*), \qquad \sum_{n=n'}^{n'+n_2}\tau\xi_n \le a_3(n_1,n_*), \qquad \sum_{n=n'}^{n'+n_2}\tau\alpha_n \le a_4(n_1,n_*),
$$

for any n' satisfying $n_1 \le n' \le n_* - n_2$, we have

$$
\xi_n \le \left(\frac{a_3(n_1,n_*)}{\tau n_2} + a_2(n_1,n_*)\right)e^{a_1(n_1,n_*)}e^{a_4(n_1,n_*)}
$$

for any n such that $n_1 + n_2 + 1 \le n \le n_*$.

Proof. Let $n_3, n_4 \in \mathbb{N}$ be such that $n_1 \le n_3 - 1 \le n_4 \le n_2 + n_3 - 1 \le n_* - 1$. Using (4.25) recursively, we derive

$$
\xi_{n_2+n_3} \le \xi_{n_4} e^{\sum_{i=n_4}^{n_2+n_3-1}\tau\eta_i} e^{\sum_{i=n_4}^{n_2+n_3-1}\tau\alpha_i} + \sum_{i=n_4+1}^{n_2+n_3-1}\tau\zeta_i e^{\sum_{i=n_4}^{n_2+n_3-1}\tau\eta_i} e^{\sum_{i=n_4}^{n_2+n_3-1}\tau\alpha_i} + \tau\zeta_{n_2+n_3},
$$

so

$$
\xi_{n_2+n_3} \le \xi_{n_4}e^{a_1}e^{a_4} + a_2e^{a_1}e^{a_4} = (\xi_{n_4} + a_2)e^{a_1}e^{a_4}.
$$

Multiplying this equation by τ, summing from $n_4 = n_3 - 1$ to $n_2 + n_3 - 2$, and using the assumptions concludes the argument. □

We are now ready to give the main result, which is the long-time bound on $\|\nabla u^n\|$. We define the following constants

$$
C_9 := 4K_4\frac{108}{\nu^3}\frac{16\theta}{\nu(2\theta-1)}, \qquad C_{10} := Q_4\frac{108}{\nu^3}\frac{16\theta}{\nu(2\theta-1)}, \qquad C_{11} := \frac{108}{\nu^3}\frac{2}{(2\theta-1)}(4\theta^2 - 6\theta + 3). \tag{4.26}
$$

Also,

$$\rho_0 = \frac{1}{\lambda_1 \nu^2(2\theta-1)}\|f\|_\infty^2, \qquad T_0 = 15\frac{1}{\lambda_1}\frac{1}{\nu(2\theta-1)}\ln\left(\frac{\|u_0\|^2}{\rho_0}\right), \tag{4.27}$$

and, for some fixed r, we let ρ_1 be such that

$$\begin{aligned}\rho_1^2(\|f\|_\infty;r) :=& \left(\frac{160\theta\rho_0 K_4}{\nu r(2\theta-1)} + \frac{40\theta K_3}{\nu(2\theta-1)} + \frac{\nu^3(4\theta^2-6\theta+3)}{324(2\theta-1)r\rho_0} + r\frac{18}{5\nu}\|f\|_\infty^2\right) \\ &\times \exp\left(2C_9\rho_0^2 + 2C_{10}\rho_0 r + \frac{4}{15(2\theta-1)}(4\theta^2-6\theta+3) + 8C_9^2\rho_0^4 + 8C_{10}^2\rho_0^2 r^2 + \frac{16}{15^2(2\theta-1)^2}(4\theta^2-6\theta+3)^2\right) \\ &\times \exp\left(C_9\rho_0^2 + C_{10}\rho_0 r + \frac{2}{15(2\theta-1)}(4\theta^2-6\theta+3)\right).\end{aligned} \tag{4.28}$$

We also let

$$K_6(\|f\|_\infty) := K_5(\rho_1, \|f\|_\infty, r) \tag{4.29}$$

and

$$K_7(\|\nabla u_0\|, \|f\|_\infty) := \max\{K_5(\|\nabla u_0\|, \|f\|_\infty, T_0 + r), K_6(\|f\|_\infty)\}. \tag{4.30}$$

Theorem 1. *Let $u_0 \in V$, $f \in L^\infty(\mathbb{R}_+;H)$, u^n be the solution to (1.6)-(1.7). Let $r \geq 4\kappa_1$ be arbitrarily fixed and let the time step τ be small, such that*

$$\tau \leq \min\{\kappa_1, \kappa_2(\|f\|_\infty), \kappa_3(\|\nabla u_0\|, \|f\|_\infty, T_0+r), \kappa_4(\|\nabla u_0\|, \|f\|_\infty, T_0+r), \kappa_3(\rho_1, \|f\|_\infty, r), \kappa_4(\rho_1, \|f\|_\infty, r)\}. \tag{4.31}$$

Then we have

$$\|\nabla u^n\|^2 \leq K_7(\|\nabla u_0\|, \|f\|_\infty)$$

for all $n \geq 0$, where $K_7(\cdot,\cdot)$ (defined in (4.30)) is a continuous function defined on $\mathbb{R}_+^2$, increasing in both arguments. Moreover, for $K_6(\|f\|_\infty)$ as given in (4.29) we have

$$\|\nabla u^n\|^2 \leq K_6(\|f\|_\infty)$$

for all $n \geq N_0 + N_r := \lfloor T_0/\tau \rfloor + \lfloor r/\tau \rfloor$, i.e., $\|\nabla u^n\|$ is bounded independently of u_0 beyond $N_0 + N_r$.

Proof. Let $r \geq 4\kappa_1$ and τ be such that assumption (4.31) holds.

First, we recall that from the uniform bound (3.16) and (4.27), provided $n\tau \geq T_0$, we have

$$\|u^n\|^2 \leq 4\rho_0.$$

By the small time step hypothesis (4.31), τ satisfies also (4.20) with $T = T_0 + r$. Then, by Proposition 2, we get that (4.7) holds for all $n+1 = 1, \dots, N_0 + N_r$, and

$$\|\nabla u^n\|^2 \leq K_5(\|\nabla u_0\|, \|f\|_\infty, n\tau)$$

for all $n = 1, \dots, N_0 + N_r$. So for τ satisfying (4.31),

$$\|\nabla u^{n+1}\|^2 \leq \|\nabla u^n\|^2\left(1 + \tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^2\right)\left[1 + 2\tau\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2 + \tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4\right)\right] + \frac{18}{5\nu}\tau\|f\|_\infty^2 \tag{4.32}$$

for all $n+1 = 1, \dots, N_0 + N_r$.

We now apply the discrete uniform Grönwall Lemma 7 with

$$\xi_n = \|\nabla u^n\|^2, \qquad \alpha_n = \frac{108}{\nu^3}K_1\|\nabla u^n\|^2, \qquad \eta_n = 2\frac{108}{\nu^3}K_1\left(\|\nabla u^n\|^2 + \tau\frac{108}{\nu^3}K_1\|\nabla u^n\|^4\right), \qquad \zeta_n = \frac{18}{5\nu}\|f\|_\infty^2,$$

$n_1 = N_0 + 1$, $n_2 = N_r - 2$, $n_* = N_0 + N_r$.

We note that, since the sums $a_1(n_1, n_*)$, $a_2(n_1, n_*)$, $a_3(n_1, n_*)$, $a_4(n_1, n_*)$ are taken for $n \geq N_0 + 1$ and since $\tau \leq \kappa_1$, we can replace K_1 by $4\rho_0$ whenever the former appears.

For every $n' = N_0 + 1, N_0 + 2$, we compute the sums required in the discrete uniform Grönwall Lemma 7. We use the fact that

$$\tau(n_2 + 1) = \tau(N_r - 1) = \tau(\lfloor r/\tau \rfloor - 1) \leq r.$$

First, we note that

$$\sum_{n=n'}^{n'+n_2} \tau\zeta_n = \sum_{n=n'}^{n'+n_2} \tau \frac{18}{5\nu} \|f\|_\infty^2 = \tau \frac{18}{5\nu} \|f\|_\infty^2 (n_2 + 1) \leq r \frac{18}{5\nu} \|f\|_\infty^2. \tag{4.33}$$

Secondly, using the $L^2(H^1)$ bound (3.21), and the finite-time bound (4.21), we get

$$\begin{aligned}
\sum_{n=n'}^{n'+n_2} \tau\xi_n &= \sum_{n=n'}^{n'+n_2} \tau \|\nabla u^n\|^2 \qquad (4.34) \\
&\leq 4K_4 \frac{16\theta}{\nu(2\theta-1)} \rho_0 + \frac{16\theta}{\nu(2\theta-1)} K_3 (n_2+1)\tau + \frac{16\theta}{\nu(2\theta-1)} \frac{\tau\nu}{8\theta} (4\theta^2 - 6\theta + 3) \|\nabla u^{n'}\|^2 \\
&\leq 4K_4 \frac{16\theta}{\nu(2\theta-1)} \rho_0 + \frac{16\theta}{\nu(2\theta-1)} K_3 r + \frac{16\theta}{\nu(2\theta-1)} \frac{\tau\nu}{8\theta} (4\theta^2 - 6\theta + 3) K_5(\|\nabla u_0\|, \|f\|_\infty, T_0 + r).
\end{aligned}$$

Similarly, using the $L^2(H^1)$ bound (3.21), and the finite-time bound (4.21), we get

$$\sum_{n=n'}^{n'+n_2} \tau\alpha_n = \sum_{n=n'}^{n'+n_2} \tau \frac{108}{\nu^3} K_1 \|\nabla u^n\|^2 \leq C_9 \rho_0^2 + C_{10} \rho_0 r + \tau C_{11} \rho_0 K_5(\|\nabla u_0\|, \|f\|_\infty, T_0 + r). \tag{4.35}$$

Finally, using the fact that for $a, b, c \geq 0$,

$$(a + b + c)^2 \leq 2(a + b)^2 + 2c^2 \leq 4a^2 + 4b^2 + 2c^2,$$

the fact that if $a_i \geq 0$, $\sum_{i=0}^{n-1} a_i^2 \leq \left(\sum_{i=0}^{n-1} a_i\right)^2$, and the $L^2(H^1)$ bound (4.35), we get

$$\begin{aligned}
\sum_{n=n'}^{n'+n_2} \tau\eta_n &= \sum_{n=n'}^{n'+n_2} \tau 2 \frac{108}{\nu^3} K_1 \|\nabla u^n\|^2 + \sum_{n=n'}^{n'+n_2} 2 \left(\tau \frac{108}{\nu^3} K_1 \|\nabla u^n\|^2 \right)^2 \\
&\leq 2 \sum_{n=n'}^{n'+n_2} \tau \frac{108}{\nu^3} K_1 \|\nabla u^n\|^2 + 2 \left(\sum_{n=n'}^{n'+n_2} \tau \frac{108}{\nu^3} K_1 \|\nabla u^n\|^2 \right)^2 \qquad (4.36) \\
&\leq 2C_9 \rho_0^2 + 2C_{10} \rho_0 r + 2\tau C_{11} \rho_0 K_5(\|\nabla u_0\|, \|f\|_\infty, T_0 + r) \\
&\qquad + 8C_9^2 \rho_0^4 + 8C_{10}^2 \rho_0^2 r^2 + 4\tau^2 C_{11}^2 \rho_0^2 K_5^2(\|\nabla u_0\|, \|f\|_\infty, T_0 + r).
\end{aligned}$$

We now note that the small time step assumption (4.31) implies

$$\tau \leq \kappa_3(\|\nabla u_0\|, \|f\|_\infty, T_0 + r), \qquad \tau \leq \kappa_4(\|\nabla u_0\|, \|f\|_\infty, T_0 + r),$$

and therefore, we have the following estimates

$$\begin{aligned}
\tau \frac{108}{\nu^3} \rho_0 K_5(\|\nabla u_0\|, \|f\|_\infty, T_0 + r) &\leq \frac{1}{15}, \\
\tau^2 \frac{108^2}{\nu^6} \rho_0^2 K_5^2(\|\nabla u_0\|, \|f\|_\infty, T_0 + r) &\leq \frac{1}{15^2}, \\
\tau K_5(\|\nabla u_0\|, \|f\|_\infty, T_0 + r) &\leq \frac{\nu^3}{15 \cdot 108 \rho_0}.
\end{aligned}$$

Using (4.33), (4.34), (4.35), and (4.36), the above estimates yield the sums defined in Lemma 7

$$a_1(n_1,n_*) = 2C_9\rho_0^2 + 2C_{10}\rho_0 r + \frac{4}{15(2\theta-1)}(4\theta^2-6\theta+3) + 8C_9^2\rho_0^4 + 8C_{10}^2\rho_0^2 r^2 + \frac{16}{15^2(2\theta-1)^2}(4\theta^2-6\theta+3)^2,$$

$$a_2(n_1,n_*) = r\frac{18}{5\nu}\|f\|_\infty^2,$$

$$a_3(n_1,n_*) = 4K_4\frac{16\theta}{\nu(2\theta-1)}\rho_0 + \frac{16\theta}{\nu(2\theta-1)}K_3 r + \frac{2}{(2\theta-1)}(4\theta^2-6\theta+3)\frac{\nu^3}{15\cdot 108\rho_0},$$

$$a_4(n_1,n_*) = C_9\rho_0^2 + C_{10}\rho_0 r + \frac{2}{15(2\theta-1)}(4\theta^2-6\theta+3).$$

By Proposition 2 regarding the V-bound on a finite time interval, we get that (4.7) holds for $n+1 = N_0+N_r$. Therefore, Lemma 7 gives

$$\|\nabla u^{N_0+N_r}\|^2 \le \left(\frac{a_3(n_1,n_*)}{\tau n_2} + a_2(n_1,n_*)\right)\exp(a_1(n_1,n_*))\exp(a_4(n_1,n_*)). \tag{4.37}$$

Since by assumption $r \ge 4\kappa_1$,

$$\frac{1}{\tau n_2} = \frac{1}{\tau(N_r-2)} \le \frac{5}{2r}.$$

So the bound (4.37) becomes

$$\|\nabla u^{N_0+N_r}\|^2 \le \rho_1^2(\|f\|_\infty; r).$$

Note that we assumed in (4.31) that $\tau \le \kappa_3(\rho_1,\|f\|_\infty,r)$ and $\tau \le \kappa_4(\rho_1,\|f\|_\infty,r)$, where κ_3,κ_4 are decreasing functions of their arguments. So we can think of $u^{N_0+N_r}$ as our initial data and apply Proposition 2 with $T = r$. We then get that (4.7) holds for all $n = N_0+N_r+1,\dots,N_0+2N_r$, and

$$\|\nabla u^n\|^2 \le K_5(\|\nabla u^{N_0+N_r}\|, \|f\|_\infty, N_r\tau)$$

for all $n = N_0+N_r+1,\dots,N_0+2N_r$.

Since $\|\nabla u^{N_0+N_r}\|^2 \le \rho_1^2(\|f\|_\infty; r)$ and $K_5(\cdot,\cdot,\cdot)$ is an increasing function of all its arguments, we get that

$$\|\nabla u^n\|^2 \le K_5(\rho_1, \|f\|_\infty, N_r\tau)$$

for all $n = N_0+N_r+1,\dots,N_0+2N_r$.

Now applying Lemma 7 with $n_1 = N_0+N_r+1, n_2 = N_r-2$, and $n_* = N_0+2N_r$,

$$\|\nabla u^{N_0+2N_r}\|^2 \le \rho_1^2(\|f\|_\infty; r).$$

Iterating Proposition 2 and Lemma 7 in this manner, we get

$$\|\nabla u^n\|^2 \le K_5(\rho_1, \|f\|_\infty, r) = K_6(\|f\|_\infty)$$

for all $n \ge N_0+N_r$, and recalling our initial bound (4.21) on a finite interval, we finally obtain

$$\|\nabla u^n\|^2 \le \max\{K_5(\|\nabla u_0\|, \|f\|_\infty, T_0+r), K_6(\|f\|_\infty)\} = K_7(\|\nabla u_0\|, \|f\|_\infty)$$

for all $n \ge 1$. □

5 Conclusions

We proved the uniform-in-time stability of the Cauchy one-leg time-stepping method for the two-dimensional Navier-Stokes equations. In particular, we established conditions under which this stability holds for the semi-discrete-in-time formulation, including a time-step restriction (4.31) and a constraint on the method parameter $\theta \in \left(\frac{1}{2}, 1\right)$.

There are two key arguments that allowed us to extend the result in [29]. First, in Lemma 4, we established an $L^2(H^1)$ bound at the integer time levels. A similar bound at the fractional time steps $u^{n+\theta}$, which arises naturally as an extension of the BE case, is not sufficient to prove the V-stability argument. The second key result is the reformulation of the energy estimate in Lemma 1 as presented in Lemma 2. This step is essential, as it shows that this type of argument cannot be applied to the case $\theta = \frac{1}{2}$ without imposing a time-step restriction. Therefore, proving long-time stability of the semi-discretization in time for the case $\theta = \frac{1}{2}$, which corresponds to the midpoint method, remains an open problem.

6 Funding

Isabel Barrio Sanchez and Catalin Trenchea were partially supported by the National Science Foundation under grant DMS-2208220.

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