ANALYSIS OF THE VARIABLE STEP METHOD OF DAHLQUIST, LINIGER AND NEVANLINNA FOR FLUID FLOW

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Abstract. The two-step time discretization proposed by Dahlquist, Liniger and Nevanlinna is variable step *G*-stable. (In contrast, for increasing time steps, the BDF2 method loses A-stability and suffers non-physical energy growth in the approximate solution.) While unexplored, it is thus ideal for time accurate approximation of the Navier-Stokes equations. This report presents an analysis, for variable time-steps, of the method's stability and convergence rates when applied to the NSE. It is proven that the method is variable step, unconditionally, long time stable and second order accurate. Variable step error estimates are also proven. The results are supported by several numerical tests.

1. Introduction. The accurate numerical simulation of flows of an incompressible, viscous fluid, with the accompanying complexities occurring in practical settings, is a problem where speed, memory and accuracy never seem sufficient. For time discretization (considered herein), many simulations use the constant step, first order, fully implicit method, e.g., Chen and Mclaughlin [9], Jiang [25], Jiang and Tran [27], and (with few exceptions noted in Section 1.1) the remainder use the constant timestep trapezoid / implicit midpoint scheme, e.g. Baker [3], Baker, Dougalis and Karakashian [4], Ingram [24], Labovsky, Manica and Neda [29], Simo, Armero and Taylor [35], (often combined with fractional steps, Bristeau, Glowinski and Périaux [7] or with ad hoc fixes to correct for oscillations due to lack of *L*-stability, Østerby [33]) or the BDF2 method (e.g., Akbas, Kaya and Rebholz [1], Ascher and Petzold [2], Grigorieff [20], Mays and Neda [31], Rong and Fiordilino [34]). Time *accuracy* requires time step *adaptivity* within the computational, space and cognitive complexity limitations of CFD. Beyond accuracy, adaptivity has the secondary benefit (depending on implementation) of reducing memory requirements and decreasing the number of floating point operations.

The richness of scales of higher Reynolds number flows and the cost per step of their solution suggests a preference for *A*-stable (or even *L*-stable) multi-step methods, called *Smart Integrators* in Gresho, Sani and Engelman [19, Section 3.16.4]. For *constant* time steps, a complete analysis of the general (2 parameter family of) 2-step, *A*-stable linear multi-step method is performed in the 1979 book Girault and Raviart [18] but there is no analogous stability or convergence analysis for the important case of *variable* timesteps. As an example of the challenges involved in variable steps, BDF2 (a popular member of that *A*-stable family) loses *A*-stability for increasing time steps, allowing non-physical energy growth. The instability is weak since 0-stability is preserved for smoothly varying timesteps, Boutelje and Hill [5], Söderlind, Fekete and Faragó [36]. Similarly, the (2-leg) trapezoidal method can exhibit energy growth, when used with variable steps (Dahlquist, Liniger and Nevanlinna [14], page 1073). Liniger [32] presents a 2-step method that is non-autonomous *A*-stable (applied to $y' = \lambda(t)y$). Dahlquist, Liniger and Nevanlinna [14] give one that is *G*-stable (nonlinearly, energetically stable, e.g., Dahlquist [11–13], Hairer, Nørsett and Wanner [21]) for *any* sequence of increasing or decreasing time-steps. Herein we give an analysis of this method of Dahlquist, Liniger and Nevanlinna [14] (the DLN method henceforth) for the Navier-Stokes Equations (NSE) with variable timesteps.

Let Ω be the flow domain in $\mathbb{R}^d$ ($d = 2$ or 3). The fluid velocity is denoted $u(x,t)$, pressure $p(x,t)$ and body force $f(x,t)$. We analyze the variable step, DLN time discretization for the NSE

$$
\begin{aligned}
& u_t + u\cdot\nabla u - \nu\Delta u + \nabla p = f, \;\; x\in\Omega, \;\; 0 < t \le T, \\
& \nabla\cdot u = 0, \;\; x\in\Omega \;\text{ for }\; 0 < t \le T, \;\; u(x,0) = u_0(x), \;\; x\in\Omega, \\
& u = 0 \;\text{ on }\; \partial\Omega, \;\; \int_\Omega p\,dx = 0 \;\text{ for }\; 0 < t \le T.
\end{aligned}
$$

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Section 2 recalls the DLN method. Applied to the NSE, it takes the form

$$\left(\frac{\alpha_2 u_{n+1}^h + \alpha_1 u_n^h + \alpha_0 u_{n-1}^h}{\widehat{k}_n}, v^h\right) + \nu(\nabla u_{n,*}^h, \nabla v^h) + b^*(u_{n,*}^h, u_{n,*}^h, v^h)$$
$$-(p_{n,*}^h, \nabla\cdot v^h) = (f(t_{n,*}), v^h),$$
$$(\nabla\cdot u_{n+1}^h, q^h) = 0, \text{ where } u_{n,*} = \sum_{\ell=0}^{2}\beta_\ell^{(n)} u_{n-1+\ell}.$$

Here $\widehat{k}_n$ is a similar average of the variable time steps k_{n-1} and k_n, and the multi-step method's coefficients $\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0$ are given in Section 2. The DLN method is a one-parameter family (with parameter denoted θ) and A-stable. Thus the constant time step case (not considered herein) is a subset of the analysis in Girault and Raviart [18]. Section 2 also presents its critical property of variable step stability of G-stability with the *G-matrix independent of the time step ratio*. Notations and preliminaries are presented in Section 3. Section 4 gives a proof of variable timestep, unconditional, long time, nonlinear stability of the one-leg DLN method for NSE. Let $\|\cdot\|$ denote the L^2-norm. This analysis shows that the natural kinetic energy, $\mathscr{E}(t_n)$, and numerical dissipation rate, $\mathscr{D}(t_n)$, of the DLN approximation are

$$\mathscr{E}(t_n) = \frac{1}{4}(1+\theta)\|u_n^h\|^2 + \frac{1}{4}(1-\theta)\|u_{n-1}^h\|^2, \quad \theta = \text{method parameter},$$
$$\mathscr{D}(t_n) = \frac{1}{\widehat{k}_n}\left\|\sum_{\ell=0}^{2} a_\ell^{(n)} u_{n-1+\ell}^h\right\|^2, \text{ where the coefficients } a_\ell^{(n)} \text{ are given in (2.2).}$$

Section 5 provides the variable step error analysis. The DLN method is proven second-order for any sequence of time steps. Numerical tests are presented in Section 6. The first example confirms the theoretical prediction of second order accuracy. The second test shows that DLN has stability advantages over BDF2 for variable timesteps. There is a recent idea of Capuano, Sanderse, Angelis and Coppola [8] to adapt the time step to control the ratio of numerical to physical dissipation. Rather than test a standard approach to error estimation and adaptivity, we also test this idea in Section 6.

1.1. Related work. The number of papers studying timestepping methods for flow problems is very large. The general (2 parameter) 2-step A-stable method was analyzed for the NSE for *constant* time steps in Girault and Raviart [18], and developed further by Jiang, Mohebujjaman and Rebholz [26]. Time adaptive discretizations of the NSE have been limited by the Dahlquist barrier, storage limitations and the cognitive complexity of extending to the NSE many of the standard methods for systems of ordinary differential equations. One early and important work is that of Kay, Gresho, Griffiths and Silvester [28]. It presents an adaptive algorithm based on the trapezoid scheme / linearized midpoint rule (with error estimation done using an explicit AB2 type method) that is memory and computation efficient. It is well known for systems of ODEs that variable step, variable order (VSVO) methods are the ones of choice. These have only been considered for the NSE in three recent works, Hay, Etienne, Pelletier and Garon [22] (based on the BDF family), Decaria, Guzel and Li [15], Decaria and Zhao [16] (based on time filters). The methods based on time filters are promising but relatively unexplored. For example, their variable step G-stability is unknown.

2. The variable step DLN method. The DLN method is a 1-parameter ($0 \le \theta \le 1$) family of A-stable, 2-step, G-stable methods. If $\theta = 1$ it reduces to the one-step, one-leg trapezoid (midpoint) scheme. Its key property is that *the G-stability matrix depends on the parameter θ but not on the timestep ratio* in Lemma 2.2 below. Let $y : [0,T] \to \mathbb{R}^d$, $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$. Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(0) = y_0.$$

Let partition P on $[0,T]$ be $\{t_n\}_{n=0}^{M}$ ($M \in \mathbb{N}$) where

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T.$$

We recall the following notation from Dahlquist, Liniger and Nevanlinna [14] for the local step size k_n, the stepsize variability $\varepsilon_n \in (-1,1)$:

$$k_n = t_{n+1} - t_n, \qquad \varepsilon_n = \frac{k_n - k_{n-1}}{k_n + k_{n-1}},$$

and the coefficients $\{\alpha_\ell, \beta_\ell\}_{\ell=0:2}$ are

$$\begin{pmatrix} \alpha_2 & \beta_2^{(n)} \\ \alpha_1 & \beta_1^{(n)} \\ \alpha_0 & \beta_0^{(n)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\theta+1) & \frac{1}{4}\Big(1 + \frac{1-\theta^2}{(1+\varepsilon_n\theta)^2} + {\varepsilon_n}^2 \frac{\theta(1-\theta^2)}{(1+\varepsilon_n\theta)^2} + \theta\Big) \\ -\theta & \frac{1}{2}\Big(1 - \frac{1-\theta^2}{(1+\varepsilon_n\theta)^2}\Big) \\ \frac{1}{2}(\theta-1) & \frac{1}{4}\Big(1 + \frac{1-\theta^2}{(1+\varepsilon_n\theta)^2} - {\varepsilon_n}^2 \frac{\theta(1-\theta^2)}{(1+\varepsilon_n\theta)^2} - \theta\Big) \end{pmatrix}. \tag{2.1}$$

For constant time steps, the DLN stability region boundary with $\theta = \frac{1}{2}$ and that of BDF2 for comparison plotted by the root locus are given in Figure 2.1. We also recall the definitions of the DLN's averaged timestep

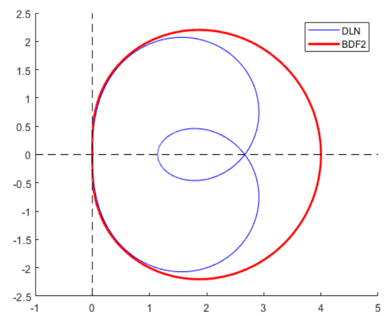


FIG. 2.1. *Boundaries of Stability Region for constant DLN and BDF2.*

$\widehat{k}_n$:

$$\widehat{k}_n = \alpha_2 k_n - \alpha_0 k_{n-1} = \frac{1}{2}(1+\theta)k_n + \frac{1}{2}(1-\theta)k_{n-1} = \theta\frac{k_n - k_{n-1}}{2} + \frac{k_n + k_{n-1}}{2},$$

and the coefficients $a_\ell^{(n)}$:

$$a_1^{(n)} = -\frac{\sqrt{\theta\left(1-\theta^2\right)}}{\sqrt{2}(1+\varepsilon_n\theta)}, \quad a_2^{(n)} = -\frac{1-\varepsilon_n}{2}a_1^{(n)}, \quad a_0^{(n)} = -\frac{1+\varepsilon_n}{2}a_1^{(n)}, \tag{2.2}$$

which are used in the expression of the numerical dissipation.

The α_ℓ-coefficients do not depend on the time-step ratio. The β_ℓ- and a_ℓ-coefficients depend on the time-step ratios through the variability coefficients ε_n.

The one-leg DLN method is then

$$\sum_{\ell=0}^{2} \alpha_\ell y_{n-1+\ell} = \widehat{k}_n f\Big(\sum_{\ell=0}^{2} \beta_\ell^{(n)} t_{n-1+\ell}, \sum_{\ell=0}^{2} \beta_\ell^{(n)} y_{n-1+\ell}\Big). \tag{DLN}$$

Let $\|\cdot\|$ and $(\cdot,\cdot)_{\mathbb{R}^d}$ denote in this section the usual norm and inner product on Euclidean space $\mathbb{R}^d$.

DEFINITION 2.1. *For $\theta \in [0,1]$, define the symmetric semi-positive definite $G(\theta)$ matrix*

$$G(\theta) = \begin{pmatrix} \frac{1}{4}(1+\theta)\mathbb{I}_d & 0 \\ 0 & \frac{1}{4}(1-\theta)\mathbb{I}_d \end{pmatrix}, \tag{2.3}$$

with the corresponding G-norm

$$\left\|\begin{matrix} u \\ v \end{matrix}\right\|_{G(\theta)}^{2} := \frac{1}{4}(1+\theta)\|u\|^{2} + \frac{1}{4}(1-\theta)\|v\|^{2} \quad \textit{for } u,v \in \mathbb{R}^{d}. \tag{2.4}$$

Recall the following result, from Dahlquist, Liniger and Nevanlinna [14], related to the G-stability of the DLN method (DLN), which will be used in proving main theorems herein.

LEMMA 2.2. *Let $0 \leq \theta \leq 1$. The variable step, one-leg DLN method (DLN) is G-stable, i.e. for any $n = 1,2,\cdots M-1$, with $a_{\ell}^{(n)}$ $(\ell = 0,1,2)$ given above (2.2), we have*

$$\Big(\sum_{\ell=0}^{2} \alpha_{\ell} y_{n-1+\ell}, \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} y_{n-1+\ell}\Big)_{\mathbb{R}^{d}} = \left\|\begin{matrix} y_{n+1} \\ y_{n} \end{matrix}\right\|_{G(\theta)}^{2} - \left\|\begin{matrix} y_{n} \\ y_{n-1} \end{matrix}\right\|_{G(\theta)}^{2} + \Big\|\sum_{\ell=0}^{2} a_{\ell}^{(n)} y_{n-1+\ell}\Big\|^{2}. \tag{2.5}$$

Proof. The proof (implicit in Dahlquist, Liniger and Nevanlinna [14]) is an algebraic calculation. □

3. Notation and Preliminaries. Let Ω be any domain in $\mathbb{R}^{d}$ ($d = 2$ or 3). For $1 \leq p < \infty$, $\|\cdot\|_{L^{p}}$ and $\|\cdot\|_{W_{p}^{k}}$ are norms on function spaces $L^{p}(\Omega)$ and $W_{p}^{k}(\Omega)$ respectively. There is a special case: if $p = 2$, we denote $\|\cdot\|$ be L^{2}-norm with inner product $(\cdot,\cdot)$. $H^{k}(\Omega)$ is the Sobolev space $W_{2}^{k}(\Omega)$ with norm $\|\cdot\|_{k}$. The velocity and pressure (u,p) are in the spaces (X,Q) given by

$$X = \Big\{v : \Omega \to \mathbb{R}^{d} : v \in L^{2}(\Omega), \nabla v \in L^{2}(\Omega) \text{ and } v = 0 \text{ on } \partial\Omega\Big\},$$
$$Q = \Big\{q : \Omega \to \mathbb{R} : v \in L^{2}(\Omega) \text{ and } \int_{\Omega} q\, dx = 0\Big\}.$$

The spaces of divergence free functions is denoted

$$V = \Big\{v \in X : (\nabla \cdot v, q) = 0, \ \forall q \in Q\Big\}.$$

The space X^{*} and V^{*} are the dual space of X and V with norms given by

$$\|f\|_{-1} := \sup_{0 \neq v \in X} \frac{(f,v)}{\|\nabla v\|}, \quad \|f\|_{*} := \sup_{0 \neq v \in V} \frac{(f,v)}{\|\nabla v\|},$$

respectively. For functions $v(x,t)$ and $1 \leq p < \infty$, define

$$\|v\|_{\infty,k} := \operatorname*{ess\,sup}_{0<t<T} \|v(t,\cdot)\|_{k} \quad \text{and} \quad \|v\|_{p,k} := \Big(\int_{0}^{T} \|v(t,\cdot)\|_{k}^{p} dt\Big)^{1/p}.$$

For $u,v,w \in X$, define the explicitly skew symmetrized trilinear form

$$b^{*}(u,v,w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v).$$

$b^{*}(u,v,w)$ satisfies the bound, [30, p.11 Lemma 3],

$$b^{*}(u,v,w) \leq C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\|,$$
$$b^{*}(u,v,w) \leq C(\Omega)\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|\nabla w\|. \tag{3.1}$$

We recall the following standard lemma for b^{*}

LEMMA 3.1. *For any $u,v,w \in X$*

$$b^{*}(u,v,v) = 0, \tag{3.2}$$

and

$$b^{*}(u,v,w) = (u \cdot \nabla v, w), \tag{3.3}$$

for all $u \in V$ and $v, w \in X$.

Proof. By definition of b^*, we have $b^*(u,v,v) = 0,\ \forall u,v \in X$. For second part, integrate by parts then use $\nabla \cdot u = 0$ and $u|_{\partial\Omega} = 0$. $\square$

We base our analysis on the finite element method (FEM) for the spatial discretization. The approximate solutions for the velocity and pressure are in the finite element spaces, based on an edge to edge triangulation Ω (with maximum triangle diameter h) denoted by

$$X_h \subset X, \quad Q_h \subset Q.$$

We assume that X_h and Q_h satisfy the usual discrete inf-sup condition (LBB^h condition). The Taylor-Hood elements, which satisfy the condition, are used in the numerical tests. The discretely divergence-free subspace of X_h is

$$V_h := \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0,\ \forall q_h \in Q_h\}.$$

We also need the following interpolation error estimate for the velocity u and pressure p: for $k, s \in \mathbb{N}$,

$$\begin{aligned} \|u - I^h u\|_r &\le C h^{k+1-r} \|u\|_{k+1}, \quad u \in H^{k+1}(\Omega)^d, \quad 0 \le r \le k \\ \|p - I^h p\|_r &\le C h^{s+1-r} \|p\|_{s+1}, \quad p \in H^{s+1}(\Omega), \quad 0 \le r \le s \end{aligned} \tag{3.4}$$

where $I^h u$ and $I^h p$ are the L^2 projection of u and p onto X^h and Q^h respectively, see e.g. Brenner and Scott [6].

Let $[0,T]$ be a time interval, $P_0 = \{t_n\}_{n=0}^{M}$ a partition on $[0,T]$, and $\{k_n\}_{n=0}^{M-1}$ denote the set of time-step sizes.

DEFINITION 3.2. *For any given sequence $\{z_n\}_{n\ge 1}$, we denote by*

$$z_{n,*} = \sum_{\ell=0}^{2} \beta_\ell^{(n)} z_{n-1+\ell}$$

the convex combination of the three adjacent terms in the sequence.

As examples, $\{t_{n,*}\}$ is the set of time-values and $u_{n,*}$ are the implicit values where the equation is evaluated

$$\begin{aligned} t_{n,*} &= \beta_2^{(n)} t_{n+1} + \beta_1^{(n)} t_n + \beta_0^{(n)} t_{n-1}, \\ u_{n,*} &= \beta_2^{(n)} u_{n+1} + \beta_1^{(n)} u_n + \beta_0^{(n)} u_{n-1}. \end{aligned}$$

The variational formulation of the one-leg DLN method for the NSE is as follows. With the DLN coefficients (2.1), given $u_n^h, u_{n-1}^h \in X_h$ and $p_n^h, p_{n-1}^h \in Q_h$, find u_{n+1}^h and p_{n+1}^h satisfying

$$\begin{aligned} \Big(\frac{\alpha_2 u_{n+1}^h + \alpha_1 u_n^h + \alpha_0 u_{n-1}^h}{\widehat{k}_n}, v^h\Big) + \nu(\nabla u_{n,*}^h, \nabla v^h) + b^*(u_{n,*}^h, u_{n,*}^h, v^h) - (p_{n,*}^h, \nabla \cdot v^h) \\ = (f(t_{n,*}), v^h) \qquad \forall v^h \in X^h, \\ (\nabla \cdot u_{n+1}^h, q^h) = 0 \qquad \forall q^h \in Q^h. \end{aligned} \tag{3.5}$$

Under the discrete inf-sup condition, (3.5) is equivalent to

$$\Big(\frac{\alpha_2 u_{n+1}^h + \alpha_1 u_n^h + \alpha_0 u_{n-1}^h}{\widehat{k}_n}, v^h\Big) + \nu(\nabla u_{n,*}^h, \nabla v^h) + b^*(u_{n,*}^h, u_{n,*}^h, v^h) = (f_{n,*}, v^h), \quad \forall v^h \in V^h. \tag{3.6}$$

Furthermore, we need the following variable timestep, discrete Gronwall inequality (see Heywood and Rannacher [23] for the proof).

LEMMA 3.3. *Let a_n, b_n, c_n, d_n, k, B be nonnegative numbers, for integers $n \ge 0$ such that*

$$a_\ell + k \sum_{n=0}^{\ell} b_n \le k \sum_{n=0}^{\ell} d_n a_n + k \sum_{n=0}^{\ell} c_n + B \quad \textit{for} \ \ \ell \ge 0.$$

Suppose that $k d_n < 1$ for all n, then

$$a_\ell + k \sum_{n=0}^{\ell} b_n \le \exp\left(\sum_{n=0}^{\ell} \frac{k d_n}{1 - k d_n}\right)\left(k \sum_{n=0}^{\ell} c_n + B\right) \quad \textit{for} \ \ \ell \ge 0.$$

4. Stability of DLN for the NSE. In this section, we prove the unconditional, long time, variable timestep energy-stability of (3.5), using the G-stability property (2.5) of the method.

THEOREM 4.1 (Unconditional, Long Time Stability). *The one-leg DLN method by* (3.5) *or* (3.6) *is unconditionally, long-time stable: for any integer $M > 1$,*

$$\frac{1}{4}(1+\theta)\|u_M^h\|^2 + \frac{1}{4}(1-\theta)\|u_{M-1}^h\|^2 + \sum_{n=1}^{M-1}\left\|\sum_{\ell=0}^{2} a_\ell^{(n)} u_{n-1+\ell}^h\right\|^2 + \frac{\nu}{2}\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla u_{n,*}^h\|^2$$
$$\leq \frac{1}{2\nu}\sum_{n=1}^{M-1}\widehat{k}_n\|f(t_{n,*})\|_*^2 + \frac{1}{4}(1+\theta)\|u_1^h\|^2 + \frac{1}{4}(1-\theta)\|u_0^h\|^2,$$

where $a_i^{(n)}, i = 0,1,2$, given previously by (2.2)*, are*

$$a_1^{(n)} = -\frac{\sqrt{\theta\left(1-\theta^2\right)}}{\sqrt{2}\left(1+\varepsilon_n\theta\right)},\ a_2^{(n)} = -\frac{1-\varepsilon_n}{2}a_1^{(n)},\ a_0^{(n)} = -\frac{1+\varepsilon_n}{2}a_1^{(n)}.$$

Proof. For $n = 1,\cdots,M-1$, set $v^h = u_{n,*}^h$ in (3.6). Then, using the skew-symmetry relation (3.2) and the Cauchy-Schwarz inequality, we obtain

$$\Big(\sum_{\ell=0}^{2}\alpha_\ell u_{n-1+\ell}^h\,,u_{n,*}^h\Big) + \frac{\nu}{2}\widehat{k}_n\|\nabla u_{n,*}^h\|^2 \leq \frac{1}{2\nu}\widehat{k}_n\|f(t_{n,*})\|_*^2.$$

The G-stability relation (2.5) implies

$$\left\|\begin{matrix} u_{n+1}^h \\ u_n^h \end{matrix}\right\|_{G(\theta)}^2 - \left\|\begin{matrix} u_n^h \\ u_{n-1}^h \end{matrix}\right\|_{G(\theta)}^2 + \left\|\sum_{\ell=0}^{2} a_\ell^{(n)} u_{n-1+\ell}^h\right\|^2 + \frac{\nu}{2}\widehat{k}_n\|\nabla u_{n,*}^h\|^2 \leq \frac{1}{2\nu}\widehat{k}_n\|f(t_{n,*})\|_*^2.$$

Summation over n from 1 to $M-1$, and the definition (2.4) yields the conclusion. □

The above stability result identifies the DLN method's kinetic energy and numerical energy dissipation rates:

$$\mathscr{E}_n = \frac{1}{4}(1+\theta)\|u_n^h\|^2 + \frac{1}{4}(1-\theta)\|u_{n-1}^h\|^2,\ \ \mathscr{D}_n = \frac{1}{\widehat{k}_n}\Big\|\sum_{\ell=0}^{2} a_\ell^{(n)} u_{n-1+\ell}^h\Big\|^2.$$

5. Variable Time-step Error Analysis. In this section, we analyze the error in the approximate solutions by the one-leg DLN method for variable time steps. The discrete time error analysis requires norms that are discrete time analogues of the norms used in the continuous time case. As before, let $[0,T]$ denote the whole time interval, $P_0 = \{t_n\}_{n=0}^M$ be a partition on $[0,T]$ and $\{k_n\}_{n=0}^{M-1}$ be the set of time-step sizes. For a function $v(x,t)$ and $1 \leq p < \infty$, we define

$$|||v|||_{\infty,k} = \max_{0\leq n\leq M}\|v_n\|_k,\ |||v|||_{p,k}^{P_0,L} = \Big(\sum_{n=0}^{M-1}k_n\|v_n\|_k^p\Big)^{1/p},\ |||v|||_{p,k}^{P_0,R} = \Big(\sum_{n=1}^{M}k_{n-1}\|v_n\|_k^p\Big)^{1/p}.$$

In the above definitions, the last two terms are forms of Riemann sums in which the function v is evaluated at the left endpoint or right endpoint of each small time interval $[t_n,t_{n+1}]$. P is the given partition on $[0,T]$ and L,R means that the sum involves the value of the function at the left endpoint or right endpoint of each time interval $[t_n,t_{n+1}]$ respectively.

Then we define two new partitions related to partition P_0: If M is odd

$$P_1 := \left\{s_\ell : 0 = s_0 < s_1 < \cdots < s_{\frac{M+1}{2}} = T \text{ and } s_\ell = t_{2\ell} \text{ for } 1 \leq \ell \leq \frac{M-1}{2}\right\},$$

$$P_2 := \left\{ s_\ell : 0 = s_0 < s_1 < \cdots < s_{\frac{M+1}{2}} = T \text{ and } s_\ell = t_{2\ell-1} \text{ for } 1 \le \ell \le \frac{M-1}{2} \right\},$$

and if M is even

$$P_1 := \left\{ s_\ell : 0 = s_0 < s_1 < \cdots < s_{\frac{M}{2}} = T \text{ and } s_\ell = t_{2\ell} \text{ for } 1 \le \ell \le \frac{M}{2} - 1 \right\},$$
$$P_2 := \left\{ s_\ell : 0 = s_0 < s_1 < \cdots < s_{\frac{M}{2}+1} = T \text{ and } s_\ell = t_{2\ell-1} \text{ for } 1 \le \ell \le \frac{M}{2} \right\}.$$

Based on the partitions above, define

$$|||v|||_{p,k} := \left(\sum_{\ell=0}^{2} \left(|||v|||_{p,k}^{P_\ell,R} \right)^p + \sum_{\ell=0}^{2} \left(|||v|||_{p,k}^{P_\ell,L} \right)^p \right)^{1/p}.$$

Furthermore given the partitions $\{P_\ell\}_{\ell=0}^2$ above, define the new partitions $\widetilde{P}_\ell$ ($\ell = 1,2$): if M is odd,

$$\widetilde{P}_1 := \left\{ s_\ell : 0 = s_0 < s_1 < \cdots < s_{\frac{M-1}{2}} = t_{M-1} \text{ and } s_\ell = t_{2\ell} \text{ for } 1 \le \ell \le \frac{M-3}{2} \right\},$$
$$\widetilde{P}_2 := \left\{ s_\ell : t_1 = s_0 < s_1 < \cdots < s_{\frac{M-1}{2}} = T \text{ and } s_\ell = t_{2\ell+1} \text{ for } 1 \le \ell \le \frac{M-3}{2} \right\},$$

if M is even,

$$\widetilde{P}_1 := \left\{ s_\ell : 0 = s_0 < s_1 < \cdots < s_{\frac{M}{2}} = T \text{ and } s_\ell = t_{2\ell} \text{ for } 1 \le \ell \le \frac{M}{2} - 1 \right\},$$
$$\widetilde{P}_2 := \left\{ s_\ell : t_1 = s_0 < s_1 < \cdots < s_{\frac{M}{2}-1} = t_{M-1} \text{ and } s_\ell = t_{2\ell+1} \text{ for } 1 \le \ell \le \frac{M}{2} - 2 \right\}.$$

For $\widetilde{P}_1$, we have $t_{2\ell-1} \in [t_{2\ell-2}, t_{2\ell}] = [s_{\ell-1}, s_\ell]$ and let $\bar{s}_\ell := t_{2\ell-1,*}$. Similarly for $\widetilde{P}_2$, $t_{2\ell} \in [t_{2\ell-1}, t_{2\ell+1}] = [s_{\ell-1}, s_\ell]$, $\bar{s}_\ell := t_{2\ell,*}$. For the function $v(x,t)$ above, define

$$|||v_*|||_{p,k}^{\widetilde{P}_i} := \left(\sum_{\ell=1}^{\#\widetilde{P}_i - 1} (s_\ell - s_{\ell-1}) \, \|v(\bar{s}_\ell)\|_k^p \right)^{1/p}, \quad i = 1,2,$$

where $\#\widetilde{P}_i$ is number of set $\widetilde{P}_i$, and

$$|||v_*|||_{p,k} := \left(\left(|||v_*|||_{p,k}^{\widetilde{P}_1} \right)^p + \left(|||v_*|||_{p,k}^{\widetilde{P}_2} \right)^p \right)^{1/p}.$$

Now we introduce the following lemma to be used often in error analysis.

LEMMA 5.1. *Let v be a continuous function on interval $[0,T] \times \Omega$ and $\{P_\ell\}_{\ell=0}^2, \{\widetilde{P}_\ell\}_{\ell=1}^2$ be the partitions on $[0,T]$ same as stated above. Then for any $1 \le p < \infty$, we have*

$$\sum_{n=1}^{M-1} (k_n + k_{n-1}) \sum_{\ell=0}^{2} \|v_{n-1+\ell}\|_{p,k+1}^p \le |||v|||_{p,k}{}^p,$$

and

$$\sum_{n=1}^{M-1} (k_n + k_{n-1}) \, \|v(t_{n,*})\|_k^p \le \left(|||v_*|||_{p,k} \right)^p.$$

We can also define the discrete norm of functions with respect to the dual norm $\|\cdot\|_*$, and derive a related lemma similar to Lemma 5.1. Moreover we need the following lemma dealing with consistency error.

LEMMA 5.2 (consistency errors). *Let $u(t)$ be any continuous function on $[0,T]$. If $u_{tt} \in L^2(\Omega \times (t_{n-1}, t_n))$, then*

$$\left\| \sum_{\ell=0}^{2} \beta_\ell^{(n)} u(t_{n-1+\ell}) - u(t_{n,*}) \right\|^2 \leq C(k_n + k_{n-1})^3 \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|^2 \, dt.$$

For $\theta \in [0,1)$, if $u_{ttt} \in L^2(\Omega \times (t_{n-1}, t_n))$, then

$$\left\| \frac{\alpha_2 u(t_{n+1}) + \alpha_1 u(t_n) + \alpha_0 u(t_{n-1})}{\widehat{k}_n} - u_t(t_{n,*}) \right\|^2 \leq C(\theta)(k_n + k_{n-1})^3 \int_{t_{n-1}}^{t_{n+1}} \|u_{ttt}\|^2 dt.$$

Proof. The proof for smooth functions is simply Taylor expansion with integral reminder after expanding function $u(t_{n+1}), u(t_{n-1})$ and $u(t_{n,*})$ at t_n. For less smooth functions it then follows by a density argument. □

Now we introduce the main theorem about error analysis under the following timestep condition:

$$C(\theta) \sum_{\ell=0}^{2} \left(\nu^{-3} \|\nabla u_{n-1+\ell,*}\|^4 + 1 \right) \widehat{k}_{n-1+\ell} < 1 \tag{5.1}$$

for $2 \leq n \leq M-2$.

THEOREM 5.3. *Let $(u(t), p(t))$ be a sufficiently smooth, strong solution of the NSE. When applying one-leg DLN's algorithm (3.5) or (3.6), there is a constant $C > 0$ such that under timestep condition (5.1), the following error estimates hold*

$$|||u - u^h|||_{\infty,0} \leq Ch^{k+1} |||u|||_{\infty,k+1} + F\Big(h, \max_{1 \leq n \leq M-1} (k_n + k_{n-1})\Big),$$

and

$$\begin{aligned} &\Big(\nu \sum_{n=1}^{M-1} \widehat{k}_n \|\nabla (u(t_{n,*}) - u_{n,*}^h)\|^2 \Big)^{\frac{1}{2}} \\ \leq & C\nu^{\frac{1}{2}} \max_{1 \leq n \leq M-1} \{(k_n + k_{n-1})^2\} \|\nabla u_{tt}\|_{2,0} + C\nu^{\frac{1}{2}} h^k |||u|||_{2,k+1} + F\Big(h, \max_{1 \leq n \leq M-1} (k_n + k_{n-1})\Big), \end{aligned}$$

where

$$\begin{aligned} F\Big(h, \max_{1 \leq n \leq M-1} (k_n + k_{n-1})\Big) &= C\nu^{\frac{1}{2}} h^k |||u|||_{2,k+1} \\ + C\nu^{-\frac{1}{2}} h^{k+\frac{1}{2}} & \Big(|||u|||_{4,k+1}^2 + |||\nabla u|||_{4,0}^2 \Big) + C\nu^{-\frac{1}{2}} h^{s+1} |||p_*|||_{2,s+1} \\ + C\nu^{-\frac{1}{2}} h^k & \big(|||u|||_{4,k+1}^2 + \nu^{-1} |||f|||_{2,*} + \nu^{-\frac{1}{2}} \|u_1^h\| + \nu^{-\frac{1}{2}} \|u_0^h\| \big) \\ + C \max_{1 \leq n \leq M-1} & \{(k_n + k_{n-1})^2\} \Big(\|u_{ttt}\|_{2,0} + \nu^{-\frac{1}{2}} \|p_{tt}\|_{2,0} + \|f_{tt}\|_{2,0} \\ + \nu^{\frac{1}{2}} \|\nabla u_{tt}\|_{2,0} & + \nu^{-\frac{1}{2}} \|\nabla u_{tt}\|_{4,0}^2 + \nu^{-\frac{1}{2}} |||\nabla u|||_{4,0}^2 + \nu^{-\frac{1}{2}} |||\nabla u_*|||_{4,0}^2 \Big). \end{aligned}$$

Remark: The timestep restriction (5.1) comes from discrete Gronwall inequality as it applies to the non-linearly implicit method. If a linearly implicit realization for the same method is used, the analysis can be sharpened to remove the restriction (5.1), as discussed in Ingram [24].

Proof. For $\theta = 1$, one-leg DLN method becomes one-leg trapezoid rule and the conclusions of the theorem have been proved in many places, e.g., Girault and Raviart [18]. Now we consider the case $\theta \in [0,1)$. Start with NSE at time $t_{n,*}$ $(1 \leq n \leq M-1)$. For any $v^h \in V^h$, the variational formulation becomes

$$(u_t(t_{n,*}), v^h) + \nu(\nabla u(t_{n,*}), \nabla v^h) + b^*(u(t_{n,*}), u(t_{n,*}), v^h) - (p(t_{n,*}), \nabla \cdot v^h) = (f(t_{n,*}), v^h).$$

Equivalently

$$\left(\frac{\alpha_2 u_{n+1}+\alpha_1 u_n+\alpha_0 u_{n-1}}{\widehat{k}_n}, v^h\right)+b^*\left(u_{n,*}, u_{n,*}, v^h\right)+\nu\left(\nabla u_{n,*}, \nabla v^h\right)-\left(p_{n,*}, \nabla \cdot v^h\right)$$
$$=\left(f_{n,*}, v^h\right)+\tau\left(u_{n,*}, p_{n,*}, v^h\right), \tag{5.2}$$

where the truncation error is

$$\tau\left(u_{n,*}, p_{n,*}, v^h\right)=\left(\frac{\alpha_2 u_{n+1}+\alpha_1 u_n+\alpha_0 u_{n-1}}{\widehat{k}_n}-u_t\left(t_{n,*}\right), v^h\right)$$
$$+\nu\left(\nabla(u_{n,*}-u\left(t_{n,*}\right)), \nabla v^h\right)+b^*\left(u_{n,*}, u_{n,*}, v^h\right)-b^*\left(u\left(t_{n,*}\right), u\left(t_{n,*}\right), v^h\right)$$
$$-\left(p_{n,*}-p\left(t_{n,*}\right), \nabla \cdot v^h\right)+\left(f\left(t_{n,*}\right)-f_{n,*}\ , v^h\right).$$

Define the finite element error $e_n := u_n - u_n^h$ and subtract (5.2) from the one-leg DLN FEM equation (3.6)

$$\left(\frac{\alpha_2 e_{n+1}+\alpha_1 e_n+\alpha_0 e_{n-1}}{\widehat{k}_n}, v^h\right)+b^*\left(u_{n,*}, u_{n,*}, v^h\right)-b^*\left(u_{n,*}^h, u_{n,*}^h, v^h\right)+\nu\left(\nabla e_{n,*}, \nabla v^h\right)$$
$$=\left(p_{n,*}, \nabla \cdot v^h\right)+\tau\left(u_{n,*}, p_{n,*}, v^h\right) \quad \forall v^h \in V^h. \tag{5.3}$$

Denote U_n to be L^2 projection of u_n onto V^h and decompose e_n as

$$e_n = u_n - U_n - \left(u_n^h - U_n\right) := \eta_n - \phi_n^h.$$

Setting $v^h = \phi_{n,*}^h$, (5.3) writes

$$\Big(\frac{\alpha_2\phi_{n+1}^h+\alpha_1\phi_n^h+\alpha_0\phi_{n-1}^h}{\widehat{k}_n}, \phi_{n,*}^h\Big)+\nu\|\nabla\phi_{n,*}\|^2+b^*(u_{n,*}^h, u_{n,*}^h, \phi_{n,*}^h)-b^*(u_{n,*}, u_{n,*}, \phi_{n,*}^h)$$
$$=\Big(\frac{\alpha_2\eta_{n+1}^h+\alpha_1\eta_n^h+\alpha_0\eta_{n-1}^h}{\widehat{k}_n}, \phi_{n,*}^h\Big)+\nu(\nabla\eta_{n,*}, \nabla\phi_{n,*}^h)-(p_{n,*}, \nabla\cdot\phi_{n,*}^h)-\tau(u_{n,*}, p_{n,*}, \phi_{n,*}^h).$$

Using $(q^h, \nabla \cdot \phi_{n,*}^h) = 0$ for any $q^h \in Q^h$ and multiplying the above equation by $\widehat{k}_n$, we obtain

$$\Big(\sum_{\ell=0}^{2}\alpha_\ell\phi_{n-1+\ell}^h\ , \phi_{n,*}^h\Big)+\nu\widehat{k}_n\|\nabla\phi_{n,*}\|^2 \tag{5.4}$$
$$=\Big(\sum_{\ell=0}^{2}\alpha_\ell\eta_{n-1+\ell}^h\ , \phi_{n,*}^h\Big)+\widehat{k}_n b^*(u_{n,*}, u_{n,*}, \phi_{n,*}^h)-\widehat{k}_n b^*(u_{n,*}^h, u_{n,*}^h, \phi_{n,*}^h)$$
$$+\nu\widehat{k}_n(\nabla\eta_{n,*}, \nabla\phi_{n,*}^h)-\widehat{k}_n(p_{n,*}-q^h, \nabla\cdot\phi_{n,*}^h)-\widehat{k}_n\tau(u_{n,*}, p_{n,*}, \phi_{n,*}^h) \quad \forall q^h \in Q^h.$$

Then we analyze the terms on the right-hand side of (5.4). By the property of projection operators and the linearity of inner products, we have

$$(\alpha_2\eta_{n+1}^h+\alpha_1\eta_n^h+\alpha_0\eta_{n-1}^h, \phi_{n,*}^h)=0.$$

Next we apply Lemma 3.1. This yields

$$\widehat{k}_n b^*(u_{n,*}, u_{n,*}, \phi_{n,*}^h)-\widehat{k}_n b^*(u_{n,*}^h, u_{n,*}^h, \phi_{n,*}^h)$$
$$=\widehat{k}_n b^*(u_{n,*}-u_{n,*}^h, u_{n,*}, \phi_{n,*}^h)+\widehat{k}_n b^*(u_{n,*}^h, u_{n,*}-u_{n,*}^h, \phi_{n,*}^h)$$
$$=\widehat{k}_n b^*(\eta_{n,*}, u_{n,*}, \phi_{n,*}^h)-\widehat{k}_n b^*(\phi_{n,*}^h, u_{n,*}, \phi_{n,*}^h)+\widehat{k}_n b^*(u_{n,*}^h, \eta_{n,*}, \phi_{n,*}^h).$$

For any $\varepsilon > 0$, using (3.1) and Young's inequality gives

$$\begin{aligned}
\widehat{k}_n b^*(\eta_{n,*}, u_{n,*}, \phi^h_{n,*}) &\le C(\Omega)\widehat{k}_n \|\eta_{n,*}\|^{\frac{1}{2}} \|\nabla \eta_{n,*}\|^{\frac{1}{2}} \|\nabla u_{n,*}\| \|\nabla \phi^h_{n,*}\| \\
&\le \varepsilon \nu \widehat{k}_n \|\nabla \phi^h_{n,*}\|^2 + C(\varepsilon,\Omega)\widehat{k}_n \nu^{-1} \|\eta_{n,*}\| \|\nabla \eta_{n,*}\| \|\nabla u_{n,*}\|^2, \\
\widehat{k}_n b^*(\phi^h_{n,*}, u_{n,*}, \phi^h_{n,*}) &\le C(\Omega)\widehat{k}_n \|\phi^h_{n,*}\|^{\frac{1}{2}} \|\nabla \phi^h_{n,*}\|^{\frac{1}{2}} \|\nabla u_{n,*}\| \|\nabla \phi^h_{n,*}\| \\
&\le \varepsilon \nu \widehat{k}_n \|\nabla \phi^h_{n,*}\|^2 + C(\varepsilon,\Omega)\widehat{k}_n \nu^{-3} \|\phi^h_{n,*}\|^2 \|\nabla u_{n,*}\|^4, \\
\widehat{k}_n b^*(u^h_{n,*}, \eta_{n,*}, \phi^h_{n,*}) &\le C(\Omega)\widehat{k}_n \|u^h_{n,*}\|^{\frac{1}{2}} \|\nabla u^h_{n,*}\|^{\frac{1}{2}} \|\nabla \eta_{n,*}\| \|\nabla \phi^h_{n,*}\| \\
&\le \varepsilon \nu \widehat{k}_n \|\nabla \phi^h_{n,*}\|^2 + C(\varepsilon,\Omega)\widehat{k}_n \nu^{-1} \|u^h_{n,*}\| \|\nabla u^h_{n,*}\| \|\nabla \eta_{n,*}\|^2.
\end{aligned}$$

Now using the Cauchy-Schwarz and Young inequalities gives

$$\begin{aligned}
\nu \widehat{k}_n (\nabla \eta_{n,*}, \nabla \phi^h_{n,*}) &\le \nu \widehat{k}_n \|\nabla \eta_{n,*}\| \|\nabla \phi^h_{n,*}\| \le \varepsilon \nu \widehat{k}_n \|\nabla \phi^h_{n,*}\|^2 + C(\varepsilon)\nu \widehat{k}_n \|\nabla \eta_{n,*}\|^2, \\
\widehat{k}_n (p_{n,*} - q^h, \nabla \cdot \phi^h_{n,*}) &\le \widehat{k}_n \|p_{n,*} - q^h\| \|\nabla \cdot \phi^h_{n,*}\| \le \sqrt{d}\widehat{k}_n \|p_{n,*} - q^h\| \|\nabla \phi^h_{n,*}\| \\
&\le \varepsilon \nu \widehat{k}_n \|\nabla \phi^h_{n,*}\|^2 + C(\varepsilon)\widehat{k}_n \nu^{-1} \|p_{n,*} - q^h\|^2,
\end{aligned}$$

where d is the dimension of the domain Ω. Now set $\varepsilon = 1/10$, combine the analysis above and apply the G-stability relation (2.5) to (5.4). This becomes

$$\begin{aligned}
&\left\| \begin{matrix} \phi^h_{n+1} \\ \phi^h_n \end{matrix} \right\|^2_{G(\theta)} - \left\| \begin{matrix} \phi^h_n \\ \phi^h_{n-1} \end{matrix} \right\|^2_{G(\theta)} + \frac{\nu}{2}\widehat{k}_n \|\nabla \phi_{n,*}\|^2 + \left\| \sum_{\ell=0}^{2} a^{(n)}_\ell \phi^h_{n-1+\ell} \right\|^2 \\
\le & C\frac{\widehat{k}_n}{\nu^3} \|\phi^h_{n,*}\|^2 \|\nabla u_{n,*}\|^4 + C\nu \widehat{k}_n \|\nabla \eta_{n,*}\|^2 + C\frac{\widehat{k}_n}{\nu} \|\eta_{n,*}\| \|\nabla \eta_{n,*}\| \|\nabla u_{n,*}\|^2 \\
&+ C\frac{\widehat{k}_n}{\nu} \|u^h_{n,*}\| \|\nabla u^h_{n,*}\| \|\nabla \eta_{n,*}\|^2 + C\frac{\widehat{k}_n}{\nu} \|p_{n,*} - q^h\|^2 + \widehat{k}_n \left| \tau\left(u_{n,*}, p_{n,*}, \phi^h_{n,*}\right) \right|.
\end{aligned}$$

Summing up from $n = 1$ to $n = M-1$, we have

$$\begin{aligned}
&\left\| \begin{matrix} \phi^h_M \\ \phi^h_{M-1} \end{matrix} \right\|^2_{G(\theta)} - \left\| \begin{matrix} \phi^h_1 \\ \phi^h_0 \end{matrix} \right\|^2_{G(\theta)} + \sum_{n=1}^{M-1} \left\| \sum_{\ell=0}^{2} a^{(n)}_\ell \phi^h_{n-1+\ell} \right\|^2 + \frac{\nu}{2} \sum_{n=1}^{M-1} \widehat{k}_n \|\nabla \phi_{n,*}\|^2 \\
\le & \sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu^3} \|\phi^h_{n,*}\|^2 \|\nabla u_{n,*}\|^4 + \sum_{n=1}^{M-1} C\nu \widehat{k}_n \|\nabla \eta_{n,*}\|^2 + \sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu} \|\eta_{n,*}\| \|\nabla \eta_{n,*}\| \|\nabla u_{n,*}\|^2 \\
&+ \sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu} \|u^h_{n,*}\| \|\nabla u^h_{n,*}\| \|\nabla \eta_{n,*}\|^2 + \sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu} \|p_{n,*} - q^h\|^2 + \sum_{n=1}^{M-1} \widehat{k}_n \left| \tau\left(u_{n,*}, p_{n,*}, \phi^h_{n,*}\right) \right|.
\end{aligned} \tag{5.5}$$

Set the approximate solution of u at two initial time-steps t_0 and t_1 to be L^2 projection of u into V^h. We have

$$\phi^h_i = u^h_i - U_i = 0, \; i = 0, 1.$$

Using the definition of the G-norm (2.4), the estimate (5.5) becomes

$$\begin{aligned}
&\frac{1}{4}(1+\theta)\|\phi^h_M\|^2 + \frac{1}{4}(1-\theta)\|\phi^h_{M-1}\|^2 + \sum_{n=1}^{M-1} \left\| \sum_{\ell=0}^{2} a^{(n)}_\ell \phi^h_{n-1+\ell} \right\|^2 + \frac{\nu}{2} \sum_{n=1}^{M-1} \widehat{k}_n \|\nabla \phi_{n,*}\|^2 \\
\le & \sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu^3} \|\phi^h_{n,*}\|^2 \|\nabla u_{n,*}\|^4 + \sum_{n=1}^{M-1} C\nu \widehat{k}_n \|\nabla \eta_{n,*}\|^2 + \sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu} \|\eta_{n,*}\| \|\nabla \eta_{n,*}\| \|\nabla u_{n,*}\|^2 \\
&+ \sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu} \|u^h_{n,*}\| \|\nabla u^h_{n,*}\| \|\nabla \eta_{n,*}\|^2 + \sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu} \|p_{n,*} - q^h\|^2 + \sum_{n=1}^{M-1} \widehat{k}_n \left| \tau\left(u_{n,*}, p_{n,*}, \phi^h_{n,*}\right) \right|.
\end{aligned} \tag{5.6}$$

By the uniform continuity of functions $\beta_l^{(n)}(\varepsilon_n,\theta)$ $(l=0,1,2)$, we have

$$\|\nabla\eta_{n,*}\| = \left\|\left(\nabla\sum_{\ell=0}^{2}\beta_\ell^{(n)}\eta_{n-1+\ell}\right)\right\| \le \sum_{\ell=0}^{2}\left|\beta_\ell^{(n)}\right| \|\nabla\eta_{n-1+\ell}\| \le C\sum_{\ell=0}^{2}\|\nabla\eta_{n-1+\ell}\|. \tag{5.7}$$

Using the interpolation error estimates (3.4), (5.7) yields

$$\begin{aligned}\sum_{n=1}^{M-1} C\nu\widehat{k}_n\|\nabla\eta_{n,*}\|^2 &\le C\nu\sum_{n=1}^{M-1}\widehat{k}_n\sum_{\ell=0}^{2}\|\nabla\eta_{n-1+\ell}\|^2 \le C\nu h^{2k}\sum_{n=1}^{M-1}\widehat{k}_n\sum_{\ell=0}^{2}\|u_{n-1+\ell}\|_{k+1}^2\\ &\le C(\theta)\,\nu h^{2k}\sum_{n=1}^{M-1}(k_n+k_{n-1})\sum_{\ell=0}^{2}\|u_{n-1+\ell}\|_{k+1}^2,\end{aligned}$$

for some constant $C(\theta)$. Using now Lemma 5.1, this implies

$$\sum_{n=1}^{M-1} C\nu\widehat{k}_n\|\nabla\eta_{n,*}\|^2 \le C(\theta)\,\nu h^{2k}|||u|||_{2,k+1}{}^2. \tag{5.8}$$

Using again the uniform continuity of $\{\beta_\ell^{(n)}\}_{\ell=0}^2$ and the estimates (3.4), we have

$$\begin{aligned}\|\eta_{n,*}\|\,\|\nabla\eta_{n,*}\| &= \left\|\sum_{\ell=0}^{2}\beta_\ell^{(n)}\eta_{n-1+\ell}\right\|\left\|\nabla\left(\sum_{\ell=0}^{2}\beta_\ell^{(n)}\eta_{n-1+\ell}\right)\right\|\\ &\le C\left(\sum_{0\le i,j\le 2}\|\eta_{n-1+i}\|\|\nabla\eta_{n-1+j}\|\right)\\ &\le Ch^{2k+1}\sum_{0\le i,j\le 2}\|u_{n-1+i}\|_{k+1}\|\nabla u_{n-1+j}\|_{k+1}.\end{aligned}$$

Similarly,

$$\|\nabla u_{n,*}\|^2 = \left\|\nabla\left(\sum_{\ell=0}^{2}\beta_\ell^{(n)}u_{n-1+\ell}\right)\right\|^2 \le C\sum_{\ell=0}^{2}\|\nabla u_{n-1+\ell}\|^2.$$

Thus by Young's inequality and Lemma 5.1, we have

$$\begin{aligned}&\sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu}\|\eta_{n,*}\|\|\nabla\eta_{n,*}\|\|\nabla u_{n,*}\|^2\\ \le& C\nu^{-1}h^{2k+1}\sum_{n=1}^{M-1}\widehat{k}_n\left(\sum_{0\le i,j\le 2}\|u_{n-1+i}\|_{k+1}\|\nabla u_{n-1+j}\|_{k+1}\right)\left(\sum_{\ell=0}^{2}\|\nabla u_{n-1+\ell}\|^2\right)\\ \le& C(\theta)\,\nu^{-1}h^{2k+1}\sum_{n=1}^{M-1}(k_n+k_{n-1})\left(\sum_{\ell=0}^{2}\|u_{n-1+\ell}\|_{k+1}^4+\sum_{\ell=0}^{2}\|\nabla u_{n-1+\ell}\|^4\right)\\ \le& C(\theta)\,\nu^{-1}h^{2k+1}\left(|||u|||_{4,k+1}^4+|||\nabla u|||_{4,0}^4\right).\end{aligned}$$

Recall that by Theorem 4.1, we have an priori bound for $\left\|u_n^h\right\|(n=2,3,\cdots M)$. Then combine (3.4) and Young's Inequality. This yields

$$\begin{aligned}&\sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu}\|u_{n,*}^h\|\|\nabla u_{n,*}^h\|\|\nabla\eta_{n,*}\|^2 \le C\nu^{-1}\sum_{n=1}^{M-1}\widehat{k}_n\left\|\nabla u_{n,*}^h\right\|\|\nabla\eta_{n,*}\|^2\\ \le& C(\theta)\,\nu^{-1}h^{2k}\left(\sum_{n=1}^{M-1}\widehat{k}_n\left(\sum_{\ell=0}^{2}\|u_{n-1+\ell}\|_{k+1}^4\right)+\sum_{n=1}^{M-1}\widehat{k}_n\left\|\nabla u_{n,*}^h\right\|^2\right).\end{aligned}$$

By the LBB^h condition, $f(t_{n,*})$ can be replace by $f_{n,*}$ in Theorem 4.1. Now we apply Theorem 4.1 to bound $\widehat{k}_n \left\|\nabla u_{n,*}^h\right\|^2$, which yields

$$\sum_{n=1}^{M-1} \widehat{k}_n \left\|\nabla u_{n,*}^h\right\|^2 \leq \sum_{n=1}^{M-1} \frac{1}{\nu^2} \widehat{k}_n \left\|f_{n,*}\right\|_*^2 + \frac{1}{\nu}\left\|u_1^h\right\|^2 + \frac{1}{\nu}\left\|u_0^h\right\|^2.$$

Applying Lemma 5.1 again, the above two inequalities imply

$$\begin{aligned}
&\sum_{n=1}^{M-1} C(\Omega) \frac{\widehat{k}_n}{\nu} \|u_{n,*}^h\| \|\nabla u_{n,*}^h\| \|\nabla \eta_{n,*}\|^2 \\
\leq & C(\theta)\nu^{-1} h^{2k}\left(|||u|||_{4,k+1}^4 + \frac{1}{\nu^2}|||f|||_{2,*}^2 + \frac{1}{\nu}\left\|u_1^h\right\|^2 + \frac{1}{\nu}\left\|u_0^h\right\|^2\right).
\end{aligned} \tag{5.9}$$

Using the interpolation error estimate for pressure p, we have

$$\begin{aligned}
\sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu}\left\|p_{n,*}-q^h\right\|^2 &\leq C\nu^{-1}\left(\sum_{n=1}^{M-1}\widehat{k}_n \left\|p_{n,*}-p(t_{n,*})\right\|^2 + \sum_{n=1}^{M-1}\widehat{k}_n\left\|p(t_{n,*})-q^h\right\|^2\right) \\
&\leq C\nu^{-1}\left(\sum_{n=1}^{M-1}\widehat{k}_n\left\|p_{n,*}-p(t_{n,*})\right\|^2 + h^{2s+2}|||p_*|||_{2,s+1}^2\right),
\end{aligned} \tag{5.10}$$

and using the consistency errors Lemma 5.2 yields

$$\begin{aligned}
\sum_{n=1}^{M-1}\widehat{k}_n\left\|p(t_{n,*})-q^h\right\|^2 &\leq C\sum_{n=1}^{M-1}\widehat{k}_n (k_n+k_{n-1})^3 \int_{t_{n-1}}^{t_{n+1}} \|p_{tt}\|^2 dt \\
&\leq C(\theta) \max_{1\leq n\leq M-1}\left\{(k_n+k_{n-1})^4\right\}\|p_{tt}\|_{2,0}^2.
\end{aligned}$$

We combine (5.9) and (5.10) to obtain

$$\sum_{n=1}^{M-1} C\frac{\widehat{k}_n}{\nu}\left\|p_{n,*}-q^h\right\|^2 \tag{5.11}$$

$$\leq C(\theta)\nu^{-1}\left(h^{2s+2}|||p_*|||_{2,s+1}{}^2 + \max_{1\leq n\leq M-1}\left\{(k_n+k_{n-1})^4\right\}\|p_{tt}\|_{2,0}^2\right). \tag{5.12}$$

Let us now treat the truncation error $\left|\tau\left(u_{n,*}, p_{n,*}, \phi_{n,*}^h\right)\right|$. Using the Cauchy-Schwarz inequality, we have

$$\left(\frac{\sum_{\ell=0}^2 \alpha_\ell u_{n-1+\ell}}{\widehat{k}_n} - u_t(t_{n,*}), \phi_{n,*}^h\right) \leq \frac{1}{2}\left\|\phi_{n,*}^h\right\|^2 + \frac{1}{2}\left\|\frac{\sum_{\ell=0}^2 \alpha_\ell u_{n-1+\ell}}{\widehat{k}_n} - u_t(t_{n,*})\right\|^2,$$

and applying again Lemma 5.2, for $\theta \in [0,1)$ to the last term above

$$\sum_{n=1}^{M-1}\widehat{k}_n\left\|\frac{\sum_{\ell=0}^2 \alpha_\ell u_{n-1+\ell}}{\widehat{k}_n} - u_t(t_{n,*})\right\|^2 \leq C(\theta)\max_{1\leq n\leq M-1}\left\{(k_n+k_{n-1})^4\right\}\|u_{ttt}\|_{2,0}^2,$$

we have

$$\begin{aligned}
&\sum_{n=1}^{M-1}\widehat{k}_n\left(\frac{\sum_{\ell=0}^2 \alpha_\ell u_{n-1+\ell}}{\widehat{k}_n} - u_t(t_{n,*}), \phi_{n,*}^h\right) \\
\leq &\frac{1}{2}\sum_{n=1}^{M-1}\widehat{k}_n\left\|\phi_{n,*}^h\right\|^2 + C(\theta)\max_{1\leq n\leq M-1}\left\{(k_n+k_{n-1})^4\right\}\|u_{ttt}\|_{2,0}^2.
\end{aligned}$$

Similarly,

$$\sum_{n=1}^{M-1}\widehat{k}_n\left(f(t_{n,*})-f_{n,*}\,,\phi^h_{n,*}\right)\le\frac{1}{2}\sum_{n=1}^{M-1}\widehat{k}_n\|\phi^h_{n,*}\|^2+C(\theta)\max_{1\le n\le M-1}\left\{(k_n+k_{n-1})^4\right\}\|f_{tt}\|^2_{2,0},$$

and also

$$\begin{aligned}&\sum_{n=1}^{M-1}\widehat{k}_n\left(p_{n,*}-p(t_{n,*}),\nabla\cdot\phi^h_{n,*}\right)\\ \le&\varepsilon\nu\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla\phi^h_{n,*}\|^2+C(\varepsilon,\theta)\nu^{-1}\max_{1\le n\le M-1}\left((k_n+k_{n-1})^4\right)\|p_{tt}\|^2_{2,0},\\ &\sum_{n=1}^{M-1}\nu\widehat{k}_n\left(\nabla(u_{n,*}-u(t_{n,*})),\nabla\phi^h_{n,*}\right)\\ \le&\varepsilon\nu\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla\phi^h_{n,*}\|^2+C(\varepsilon,\theta)\nu\max_{1\le n\le M-1}\left\{(k_n+k_{n-1})^4\right\}\|\nabla u_{tt}\|^2_{2,0}.\end{aligned}$$

Moreover

$$\begin{aligned}&b^*\left(u_{n,*},u_{n,*},\phi^h_{n,*}\right)-b^*\left(u(t_{n,*}),u(t_{n,*}),\phi^h_{n,*}\right)\\ =&b^*\left(u_{n,*}-u(t_{n,*}),u_{n,*},\phi^h_{n,*}\right)+b^*\left(u(t_{n,*}),u_{n,*}-u(t_{n,*}),\phi^h_{n,*}\right)\\ \le&C\|\nabla(u_{n,*}-u(t_{n,*}))\|\|\nabla\phi^h_{n,*}\|(\|\nabla u_{n,*}\|+\|\nabla u(t_{n,*})\|)\\ \le&\varepsilon\nu\|\nabla\phi^h_{n,*}\|^2+C(\varepsilon)\nu^{-1}\|\nabla(u_{n,*}-u(t_{n,*}))\|^2\left(\|\nabla u_{n,*}\|^2+\|\nabla u(t_{n,*})\|^2\right),\\ &\|\nabla(u_{n,*}-u(t_{n,*}))\|^2\left(\|\nabla u_{n,*}\|^2+\|\nabla u(t_{n,*})\|^2\right)\\ \le&C\left(\|\nabla u_{n,*}\|^2+\|\nabla u(t_{n,*})\|^2\right)(k_n+k_{n-1})^3\int_{t_{n-1}}^{t_{n+1}}\|\nabla u_{tt}\|^2dt\\ \le&C(k_n+k_{n-1})^3\int_{t_{n-1}}^{t_{n+1}}\left(\|\nabla u_{n,*}\|^2+\|\nabla u(t_{n,*})\|^2\right)\|\nabla u_{tt}\|^2dt\\ \le&C(k_n+k_{n-1})^3\int_{t_{n-1}}^{t_{n+1}}\left(\|\nabla u_{n,*}\|^4+\|\nabla u(t_{n,*})\|^4+\|\nabla u_{tt}\|^4\right)dt\\ \le&C(k_n+k_{n-1})^4\left(\|\nabla u_{n,*}\|^4+\|\nabla u(t_{n,*})\|^4\right)+C(k_n+k_{n-1})^3\int_{t_{n-1}}^{t_{n+1}}\|\nabla u_{tt}\|^4dt.\end{aligned}$$

Now combine Lemma 5.1 and Lemma 5.2. This yields

$$\begin{aligned}&\sum_{n=1}^{M-1}\widehat{k}_n\left(b^*\left(u_{n,*},u_{n,*},\phi^h_{n,*}\right)-b^*\left(u(t_{n,*}),u(t_{n,*}),\phi^h_{n,*}\right)\right)\\ \le&\varepsilon\nu\sum_{n=1}^{M-1}\widehat{k}_n\left\|\nabla\phi^h_{n,*}\right\|^2+\frac{C(\varepsilon,\theta)}{\nu}\max_{1\le n\le M-1}(k_n+k_{n-1})^4\|\nabla u_{tt}\|^4_{4,0}\\ &+\frac{C(\varepsilon,\theta)}{\nu}\max_{1\le n\le M-1}\left\{(k_n+k_{n-1})^4\right\}\left(\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla u_{n,*}\|^4+\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla u(t_{n,*})\|^4\right)\\ \le&\varepsilon\nu\sum_{n=1}^{M-1}\widehat{k}_n\left\|\nabla\phi^h_{n,*}\right\|^2+\frac{C(\varepsilon,\theta)}{\nu}\max_{1\le n\le M-1}(k_n+k_{n-1})^4\|\nabla u_{tt}\|^4_{4,0}\\ &+\frac{C(\varepsilon,\theta)}{\nu}\max_{1\le n\le M-1}\left\{(k_n+k_{n-1})^4\right\}\left(|||\nabla u|||^4_{4,0}+|||\nabla u_*|||^4_{4,0}\right).\end{aligned}$$

Setting $\varepsilon=1/12$ and obtain the following estimate for the truncation error term

$$\sum_{n=1}^{M-1}\widehat{k}_n\left|\tau\left(u_{n,*},p_{n,*},\phi^h_{n,*}\right)\right|\le\sum_{n=1}^{M-1}\widehat{k}_n\|\phi^h_{n,*}\|^2+\frac{1}{4}\nu\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla\phi^h_{n,*}\|^2 \tag{5.13}$$

$$+C(\theta)\max_{1\le n\le M-1}(k_n+k_{n-1})^4\Big[\|u_{ttt}\|_{2,0}^2+\nu^{-1}\|p_{tt}\|_{2,0}^2+\|f_{tt}\|_{2,0}^2+\nu\|\nabla u_{tt}\|_{2,0}^2$$
$$+\frac{1}{\nu}\|\nabla u_{tt}\|_{4,0}^4+\frac{1}{\nu}\left(|||\nabla u|||_{4,0}{}^4+|||\nabla u_*|||_{4,0}{}^4\right)\Big].$$

Now we collect the terms from (5.6), (5.8), (5.9), (5.11), (5.13) and define

$$\begin{aligned}
&\widetilde{F}\left(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\right)\\
&=C(\theta)\left(\nu h^{2k}|||u|||_{2,k+1}^2+\frac{h^{2k+1}}{\nu}\left(|||u|||_{4,k+1}^4+|||\nabla u|||_{4,0}^4\right)+\frac{h^{2s+2}}{\nu}|||p_*|||_{2,s+1}^2\right)\\
&+C(\theta)\frac{h^{2k}}{\nu}\left(|||u|||_{4,k+1}^4+\frac{1}{\nu^2}|||f|||_{2,*}^2+\frac{1}{\nu}\|u_1^h\|^2+\frac{1}{\nu}\|u_0^h\|^2\right)\\
&+C(\theta)\max_{1\le n\le M-1}(k_n+k_{n-1})^4\left(\|u_{ttt}\|_{2,0}^2+\frac{1}{\nu}\|p_{tt}\|_{2,0}^2+\|f_{tt}\|_{2,0}^2+\nu\|\nabla u_{tt}\|_{2,0}^2\right.\\
&\left.+\frac{1}{\nu}\|\nabla u_{tt}\|_{4,0}^4+\frac{1}{\nu}|||\nabla u|||_{4,0}^4+\frac{1}{\nu}|||\nabla u_*|||_{4,0}^4\right).
\end{aligned}$$

Thus (5.6) becomes

$$\begin{aligned}
\frac{1}{4}\|\phi_M^h\|^2+\frac{\nu}{4}\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla\phi_{n,*}^h\|^2\le&\sum_{n=1}^{M-1}\left(C(\theta)\nu^{-3}\|\nabla u_{n,*}\|^4+1\right)\widehat{k}_n\|\phi_{n,*}^h\|^2\\
&+\widetilde{F}\big(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\big).
\end{aligned}\tag{5.14}$$

For convenience, we define the sequence $\{D_n\}_{n=1}^{M-1}$

$$D_n:=\left(C(\theta)\nu^{-3}\|\nabla u_{n,*}\|^4+1\right)\widehat{k}_n,\quad n=1,\cdots,M-1,$$

and the sequence $\{d_n\}_{n=0}^{M}$

$$\begin{aligned}
&d_0:=D_1,\ d_1:=D_1+D_2,\ d_{M-1}:=D_{M-2}+D_{M-1},\ d_M:=D_{M-1},\\
&d_n:=\sum_{\ell=0}^{2}D_{n-1+\ell}(2\le n\le M-2),\quad 2\le n\le M-2.
\end{aligned}$$

We use the triangle inequality in (5.14) to obtain

$$\left\|\phi_M^h\right\|^2+\nu\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla\phi_{n,*}\|^2\le C(\theta)\sum_{n=0}^{M}d_n\|\phi_n^h\|^2+\widetilde{F}\big(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\big),$$

then apply the discrete Gronwall inequality (Lemma 3.3) under the timestep condition (5.1)

$$\left\|\phi_M^h\right\|^2+\nu\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla\phi_{n,*}\|^2\le\exp\left(\sum_{n=1}^{M-1}C(\theta)\frac{d_n}{1-d_n}\right)\widetilde{F}\big(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\big).\tag{5.15}$$

Define

$$\begin{aligned}
F\big(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\big)&=C(\theta)\nu^{\frac12}h^k|||u|||_{2,k+1}\\
&+C(\theta)\nu^{-\frac12}h^{k+\frac12}\left(|||u|||_{4,k+1}^2+|||\nabla u|||_{4,0}^2\right)+C(\theta)\nu^{-\frac12}h^{s+1}|||p_*|||_{2,s+1}\\
&+C(\theta)\nu^{-\frac12}h^k\left(|||u|||_{4,k+1}^2+\nu^{-1}|||f|||_{2,*}+\nu^{-\frac12}\|u_1^h\|+\nu^{-\frac12}\|u_0^h\|\right)
\end{aligned}$$

$$+C(\theta)\max_{1\le n\le M-1}\{(k_n+k_{n-1})^2\}\left(\|u_{ttt}\|_{2,0}+\nu^{-\frac{1}{2}}\|p_{tt}\|_{2,0}+\|f_{tt}\|_{2,0}\right.$$
$$\left.+\nu^{\frac{1}{2}}\|\nabla u_{tt}\|_{2,0}+\nu^{-\frac{1}{2}}\|\nabla u_{tt}\|_{4,0}^2+\nu^{-\frac{1}{2}}|||\nabla u|||_{4,0}^2+\nu^{-\frac{1}{2}}|||\nabla u_*|||_{4,0}^2\right).$$

Then from (5.15) we have

$$\left\|\phi_M^h\right\|\le F\left(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\right). \tag{5.16}$$

Combining (3.4) and (5.16) yields

$$\begin{aligned}\left|\left|\left|u-u^h\right|\right|\right|_{\infty,0}&:=\max_{0\le n\le M}\left\|u_n-u_n^h\right\|\le\max_{0\le n\le M}\|\eta_n\|+\max_{0\le n\le M}\|\phi_n\|\\&\le\max_{0\le n\le M}Ch^{k+1}\|u_n\|_{k+1}+F\left(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\right)\\&=Ch^{k+1}|||u|||_{\infty,k+1}+F\left(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\right),\end{aligned}$$

where

$$F\left(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\right)=\mathscr{O}\left(h^k+h^{s+1}+\max_{1\le n\le M-1}(k_n+k_{n-1})^2\right).$$

This concludes the proof of the first part of the theorem.

For second part, we have

$$\sum_{n=1}^{M-1}\widehat{k}_n\left\|\nabla\left(u(t_{n,*})-u_{n,*}^h\right)\right\|^2\le\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla(u(t_{n,*})-u_{n,*})\|^2+\sum_{n=1}^{M-1}\widehat{k}_n\left\|\nabla\left(u_{n,*}^h-u_{n,*}\right)\right\|^2.$$

We apply Lemma 5.2 to the first term in the right hand side

$$\nu\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla(u(t_{n,*})-u_{n,*})\|^2\le C(\theta)\,\nu\max_{1\le n\le M-1}\{(k_n+k_{n-1})^4\}\|\nabla u_{tt}\|_{2,0}^2\ ,$$

and use the triangle inequality for the second term

$$\nu\sum_{n=1}^{M-1}\widehat{k}_n\left\|\nabla\left(u_{n,*}^h-u_{n,*}\right)\right\|^2\le C\nu\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla\eta_{n,*}\|^2+C\nu\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla\phi_{n,*}\|^2\ .$$

The last term inhere can be bound by (5.15), while for the first term, we use (3.4) and Lemma 5.1

$$\begin{aligned}&C\nu\sum_{n=1}^{M-1}\widehat{k}_n\|\nabla\eta_{n,*}\|^2\le C(\theta)\,\nu\sum_{n=1}^{M-1}(k_n+k_{n-1})\left(\sum_{\ell=0}^{2}\|\nabla\eta_{n-1+\ell}\|^2\right)\\&\le C(\theta)\,\nu h^{2k}\sum_{n=1}^{M-1}(k_n+k_{n-1})\left(\sum_{\ell=0}^{2}\|u_{n-1+\ell}\|_{k+1}^2\right)\le C(\theta)\,h^{2k}|||u|||_{2,k+1}^2.\end{aligned}$$

Combining the above estimates, we have

$$\nu\sum_{n=1}^{M-1}\widehat{k}_n\left\|\nabla\left(u_{n,*}^h-u_{n,*}\right)\right\|^2\le C(\theta)\,\nu h^{2k}|||u|||_{2,k+1}^2+C(\theta)\,\widetilde{F}\left(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\right).$$

Finally

$$\begin{aligned}\left(\nu\sum_{n=1}^{M-1}\widehat{k}_n\left\|\nabla\left(u(t_{n,*})-u_{n,*}^h\right)\right\|^2\right)^{\frac{1}{2}}&\le C\nu^{\frac{1}{2}}\max_{1\le n\le M-1}\{(k_n+k_{n-1})^2\}\|\nabla u_{tt}\|_{2,0}\\&+C\nu^{\frac{1}{2}}h^k|||u|||_{2,k+1}+F\left(h,\max_{1\le n\le M-1}(k_n+k_{n-1})\right),\end{aligned}$$

which concludes the proof of second part of the theorem. □

$h=\Delta t$	$\|\|e_u\|\|_{2,0}$	R	$\|\|\nabla e_u\|\|_{2,0}$	R	$\|\|e_p\|\|_{2,0}$	R
$\frac{1}{16}$	0.000740428	-	0.0610604	-	0.00169375	-
$\frac{1}{24}$	0.000228828	2.89	0.0271831	1.99	0.000687042	2.23
$\frac{1}{32}$	8.89412e-05	3.28	0.0141961	2.26	0.000359889	2.25
$\frac{1}{40}$	4.65027e-05	2.91	0.00912596	1.98	0.000220769	2.19
$\frac{1}{48}$	2.86044e-05	2.67	0.00654533	1.82	0.000152877	2.02
$\frac{1}{56}$	1.67658e-05	3.46	0.00452741	2.39	0.000107064	2.31

Table 6.1: The errors and convergence order of the DLN scheme at time $T=1$ for the velocity and pressure of L^2-norm with $\theta=0.2$.

$h=\Delta t$	$\|\|e_u\|\|_{\infty}$	R	$\|\|\nabla e_u\|\|_{\infty}$	R	$\|\|e_p\|\|_{\infty}$	R
$\frac{1}{16}$	0.00122596	-	0.101825	-	0.00254809	-
$\frac{1}{24}$	0.000399952	2.76	0.047497	1.88	0.00113562	1.99
$\frac{1}{32}$	0.000162022	3.14	0.025876	2.11	0.000638476	2.00
$\frac{1}{40}$	8.71029e-05	2.78	0.017116	1.85	0.000408904	1.99
$\frac{1}{48}$	5.43775e-05	2.58	0.0125455	1.70	0.000291014	1.86
$\frac{1}{56}$	3.24237e-05	3.35	0.00883734	2.27	0.000210233	2.11

Table 6.2: The errors and convergence order of the DLN scheme at time $T=1$ for the velocity and pressure of L^{∞}-norm with $\theta=0.2$.

6. Numerical Tests.

In this section, FreeFem++ is used for numerical tests with Taylor-Hood ($P2-P1$) finite elements. We verify the second-order convergence and stability of the DLN algorithm with variable time steps through three numerical experiments.

6.1. Convergence Test (constant timestep size).

The second order convergence of DLN algorithm is verified on the Taylor-Green benchmark problem, Dyke [37]. In the domain $\Omega=(0,1)\times(0,1)$, the true solution is

$$\begin{aligned} u_1(x,y,t) &= -\cos(w\pi x)\sin(w\pi y)\exp(-2w^2\pi^2 t/\tau), \\ u_2(x,y,t) &= \sin(w\pi x)\cos(w\pi y)\exp(-2w^2\pi^2 t/\tau), \\ p(x,y,t) &= -\frac{1}{4}(\cos(2w\pi x)+\cos(2w\pi y))\exp(-4w^2\pi^2 t/\tau), \end{aligned}$$

and we take the final time $T=1$, $w=1$ and $\tau=Re=100$. The body force f, initial condition, and boundary condition are determined by the true solution. Setting $\Delta t=h$ to calculate the convergence order R by the error e at two successive values of Δt via

$$R=\ln(e(\Delta t_1)/e(\Delta t_2))/\ln(\Delta t_1/\Delta t_2).$$

Tables 6.1, 6.2, Tables 6.3, 6.4 and Tables 6.5, 6.6 correspond to $\theta=0.2,0.5,0.7$, respectively. The results fully verify that our DLN algorithm has second-order convergence for both velocity and pressure, and it can be seen that the convergence of velocity is better.

6.2. 2D Offset Circles Problem (with preset variable timestep size).

This is a test problem from Jiang [25] that is inspired by flow between offset cylinders. The domain is a disk with a smaller off center obstacle inside. Let $\Omega_1=\{(x,y): x^2+y^2\leq 1\}$ and $\Omega_2=\{(x,y): (x-\frac{1}{2})^2+y^2\geq 0.01\}$. The flow is driven by a rotational body force:

$$f(x,y,t)=(-4y(1-x^2-y^2),4x(1-x^2-y^2))^T.$$

with no-slip boundary conditions imposed on both circles. The body force $f=0$ on the outer circle. The flow rotates about $(0,0)$ and the inner circle induces a von Kármán vortex street which re-interacts with the immersed circle creating more complex structures. Figure 6.1 and Figure 6.2 show this situation.

$h=\Delta t$	$\|\|\|e_u\|\|\|_{2,0}$	R	$\|\|\|\nabla e_u\|\|\|_{2,0}$	R	$\|\|\|e_p\|\|\|_{2,0}$	R
$\frac{1}{16}$	0.000700594	-	0.0570129	-	0.00134003	-
$\frac{1}{24}$	0.000217831	2.88	0.0255791	1.98	0.000560912	2.11
$\frac{1}{32}$	8.53722e-05	3.26	0.0135313	2.21	0.000305539	2.16
$\frac{1}{40}$	4.50219e-05	2.87	0.00879805	1.93	0.000191838	2.08
$\frac{1}{48}$	2.78268e-05	2.64	0.00634477	1.79	0.000135402	1.91
$\frac{1}{56}$	1.63621e-05	3.44	0.00440779	2.36	9.57885e-05	2.24

Table 6.3: The errors and convergence order of the DLN scheme at time $T=1$ for the velocity and pressure of L^2-norm with $\theta=0.5$.

$h=\Delta t$	$\|\|\|e_u\|\|\|_{\infty}$	R	$\|\|\|\nabla e_u\|\|\|_{\infty}$	R	$\|\|\|e_p\|\|\|_{\infty}$	R
$\frac{1}{16}$	0.00110053	-	0.0898315	-	0.00236018	-
$\frac{1}{24}$	0.000354163	2.79	0.0434666	1.79	0.00105671	1.98
$\frac{1}{32}$	0.000147375	3.05	0.0241532	2.04	0.000595252	1.99
$\frac{1}{40}$	8.04838e-05	2.71	0.0160898	1.82	0.000381558	1.99
$\frac{1}{48}$	5.0769e-05	2.53	0.011827	1.69	0.000271851	1.86
$\frac{1}{56}$	3.04708e-05	3.31	0.00835234	2.26	0.000196439	2.11

Table 6.4: The errors and convergence order of the DLN scheme at time $T=1$ for the velocity and pressure of L^{∞}-norm with $\theta=0.5$.

$h=\Delta t$	$\|\|\|e_u\|\|\|_{2,0}$	R	$\|\|\|\nabla e_u\|\|\|_{2,0}$	R	$\|\|\|e_p\|\|\|_{2,0}$	R
$\frac{1}{16}$	0.000689478	-	0.0560293	-	0.00127634	-
$\frac{1}{24}$	0.000215154	2.87	0.025242	1.97	0.000549689	2.08
$\frac{1}{32}$	8.45301e-05	3.25	0.0133912	2.20	0.000296992	2.14
$\frac{1}{40}$	4.46583e-05	2.86	0.00872444	1.92	0.000187373	2.06
$\frac{1}{48}$	2.76364e-05	2.63	0.00629981	1.79	0.000132745	1.89
$\frac{1}{56}$	1.62635e-05	3.44	0.00438056	2.36	9.40928e-05	2.23

Table 6.5: The errors and convergence order of the DLN scheme at time $T=1$ for the velocity and pressure of L^2-norm with $\theta=0.7$.

$h=\Delta t$	$\|\|\|e_u\|\|\|_{\infty}$	R	$\|\|\|\nabla e_u\|\|\|_{\infty}$	R	$\|\|\|e_p\|\|\|_{\infty}$	R
$\frac{1}{16}$	0.00101829	-	0.0878696	-	0.00241273	-
$\frac{1}{24}$	0.000349287	2.64	0.0431141	1.76	0.00108285	1.98
$\frac{1}{32}$	0.000146272	3.03	0.0240831	2.02	0.000611728	1.99
$\frac{1}{40}$	8.01746e-05	2.69	0.0160849	1.81	0.000392496	1.99
$\frac{1}{48}$	5.06795e-05	2.52	0.0118461	1.68	0.000279776	1.86
$\frac{1}{56}$	3.05001e-05	3.29	0.00838398	2.24	0.000202406	2.10

Table 6.6: The errors and convergence order of the DLN scheme at time $T=1$ for the velocity and pressure of L^{∞}-norm with $\theta=0.7$.

For this test, we set $Re = 200$, the number of mesh points around the inner circle i and the mesh points around the outer circle o to be 10 and 40 respectively. The parameter $\theta = 0.5$ in DLN scheme, for the variable timestep size, the number of computations is $n = 1000$. We let the timestep size changes as the function used in Chen and Mclaughlin [9] to test stability a of different method:

$$k_n = \begin{cases} 0.05 & 0 \leq n \leq 10, \\ 0.05 + 0.002\sin(10t_n) & n > 10. \end{cases}$$

For comparison, we also solve this problem with a standard (Variable step) BDF2 time discretization.We calculate the energy $\frac{1}{2}\|u\|^2$ using BDF2 and DLN algorithms respectively. Here, let the number of mesh points on boundary of outside circle and inner circle be $o = 160$ and $i = 40$ respectively and timestep $k_0 = 0.05$ and $k_n = k_{n-1} + 0.001$. Figure 6.3(a) shows that when timestep k_n increases with time t, BDF2 and DLN algorithms are respectively used to calculate energy and in Figure 6.3(b), *energy of BDF2* increases with increasing timestep, while the *energy of the approximation by DLN* remains almost constant. This verifies that the DLN algorithm has greater stability.

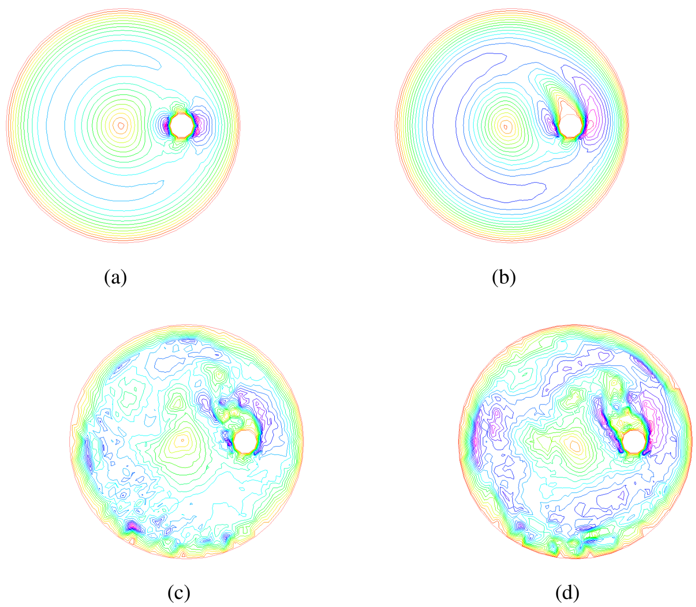


FIG. 6.1. *Spreed Contours of DLN.*

6.3. Adapting the timestep. Finally we use this example to perform a simple adaptivity experiment. For this test, we adapt the timestep using the minimum dissipation criteria of Capuano, Sanderse, De Angelis and Coppola [8]. Our goal is to test if adapting the timestep produces a significant difference in the solution. Other criteria/estimators are under study. Their idea is to adapt the timestep to keep the numerical dissipation, ε^{DLN} from the dominating physical dissipation, ε^{ν}. Thus we adapt for

$$\chi = \left| \frac{\varepsilon^{DLN}}{\varepsilon^{\nu}} \right| < \delta.$$

Here ε^{DLN} is the numerical dissipation and ε^{ν} is the viscous dissipation. These are given by:

$$\varepsilon^{DLN} = \left\| \frac{\sum_{\ell=0}^{2} a_\ell^n u_{n-1+\ell}^h}{\sqrt{\widehat{k}_n}} \right\|^2,$$

$$\varepsilon^{\nu} = \nu \left\| \nabla u_{n,*}^h \right\|^2.$$

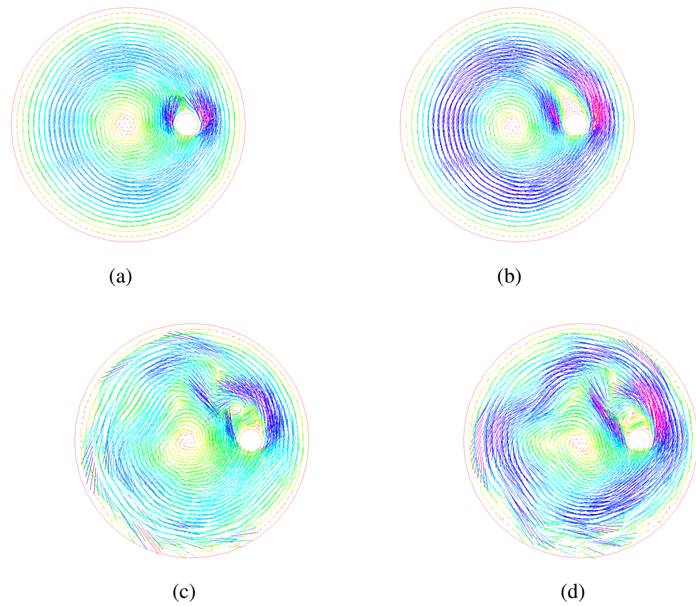


FIG. 6.2. *Velocity Streamlines of DLN.*

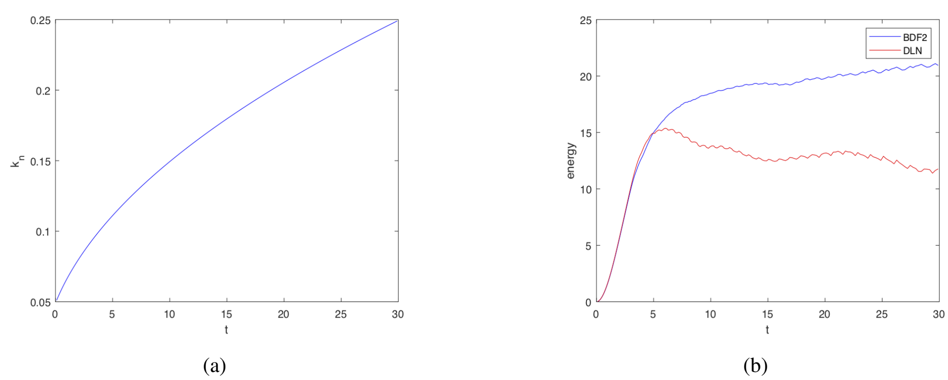


FIG. 6.3. *Energy of DLN and BDF2 with variable timestep.*

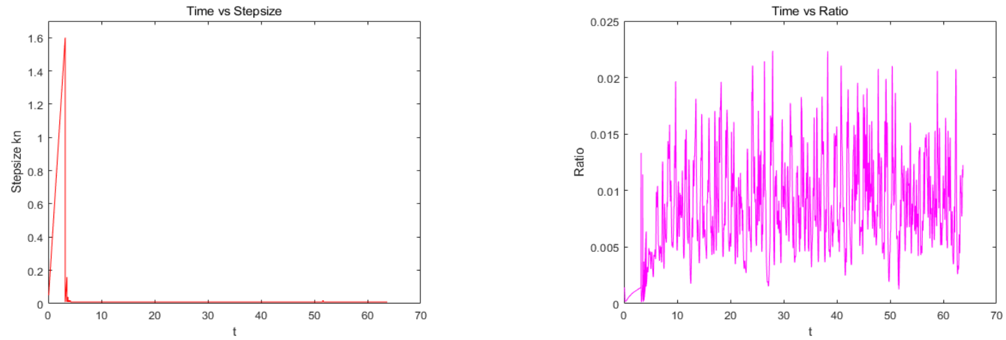
In the test, we set the tolerance for the dissipation ratio δ to be 0.002. The time stepsize is then adapted by halving or doubling according to

$$\Delta t^{n+1} = 2 * \Delta t^n; \quad if\ \chi < \delta,$$
$$\Delta t^n = 0.5 * \Delta t^n; \quad if\ \chi \geq \delta.$$

We adapted the next timestep when the dissipation ratio was out of range. Naturally, other strategies for varying Δt could be tested, such as formula (16) p.2317 of Capuano, Sanderse, De Angelis and Coppola [8]. We select the final time $T = 63.7$ and minimal time stepsize to be 0.01. The adaptive algorithm completed in 6000 steps. Figure 6.4 and Figure 6.5 are line diagrams of time stepsize k_n, energy $\frac{1}{2}\|u\|^2$, numerical dissipation $\sqrt{\varepsilon^{DLN}}$ and ratio χ changing with time T, respectively.

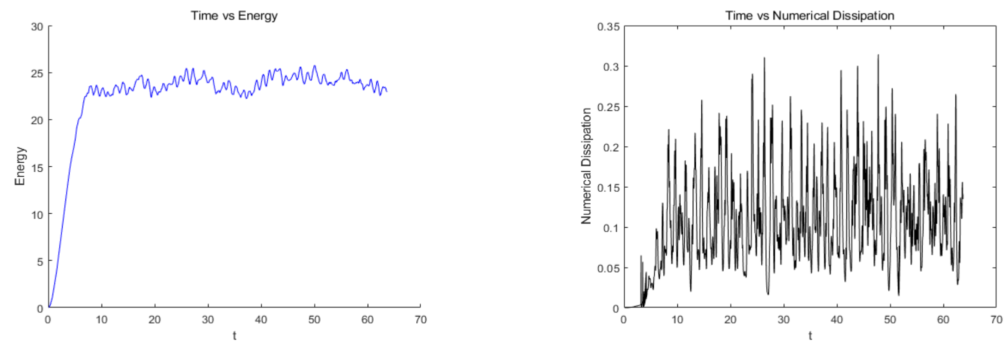
Then we select the same final time $T = 63.7$, the same calculated steps 6000 and use the constant time stepstep $k = T/6000$ to calculate to obtain the line diagram of energy $\frac{1}{2}\|u\|^2$, numerical dissipation $\sqrt{\varepsilon^{DLN}}$ and ratio χ changing with time T, See Figure 6.6 and Figure 6.7.

We now compare the constant time stepsize results in Figure 6.6 and Figure 6.7 with the adaptive results in Figure 6.4 and Figure 6.5. We first note that time stepsize under adaptivity reaches maximum value 1.6 in a few



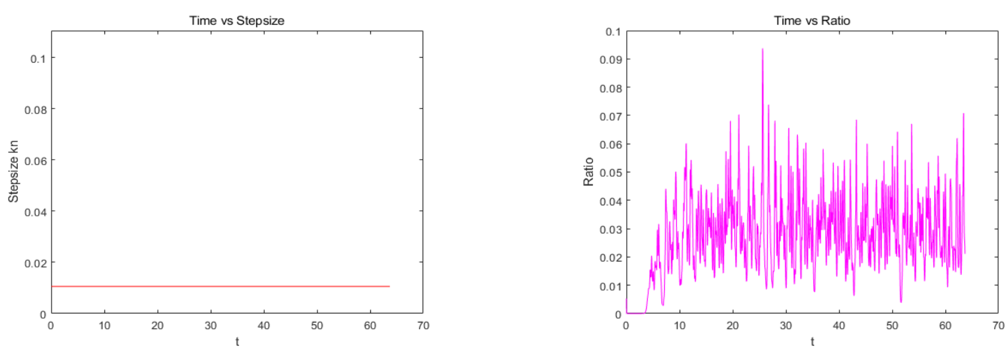
(a) (b)

FIG. 6.4. *The time stepsize k_n and ratio χ changing with adaptive time stepsize.*



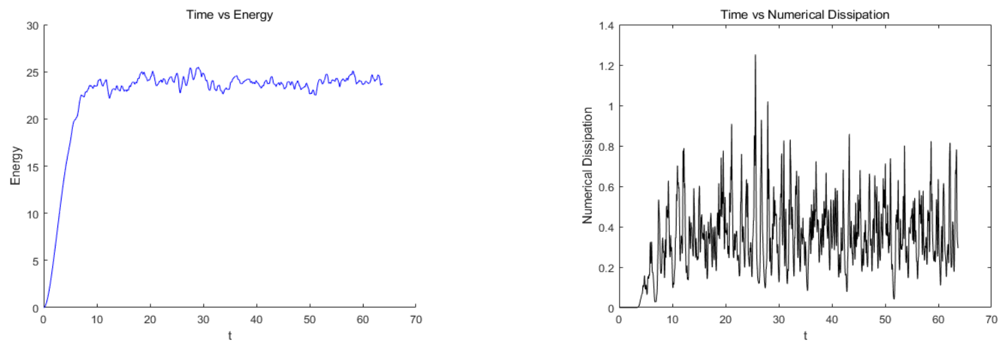
(a) (b)

FIG. 6.5. *The energy $\frac{1}{2}\|u\|^2$ and numerical dissipation $\sqrt{\varepsilon^{DLN}}$ changing with adaptive time stepsize.*



(a) (b)

FIG. 6.6. *The time stepsize k and ratio χ changing with constant time stepsize.*



(a) (b)

FIG. 6.7. *The energy* $\frac{1}{2}\|u\|^2$ *and numerical dissipation* $\sqrt{\varepsilon^{DLN}}$ *changing with constant time stepsize.*

steps then goes down sharply to the minimum stepsize 0.01 thereafter. In the test represented in Figure 6.4(a), the timestep alternates between the minimum stepsize and twice that. This is due to the preset algorithmic choice. DLN under constant stepsize takes 773 timesteps to reach a kinetic energy of approximately 23 which adaptive DLN algorithm reaches that level in 396 timesteps. In comparison of numerical dissipation, Figure 6.5(b) and 6.7(b) show that the numerical dissipation with adaptive time stepsize evolves smoothly with a peak value below 0.35. Similarly the ratio χ has a *order of magnitude smaller for adaptive time stepsize*, Figure 6.4(b), than constant time stepsize, Figure 6.7(b).

7. Conclusions. Based on the theory and the simple numerical tests that for time discretization of flow problems the 2-step DLN method is to be preferred over the common BDF2 method. It is second order, unconditionally, long time, nonlinearly stable. For increasing step-sizes, BDF2 injects nonphysical kinetic energy in the discrete solution (disrupting long time behavior and statistical equilibrium) while DLN does not. Important open questions include how to perform error estimation in a memory and computationally efficient (and effective) way. In particular, finding a memory efficient estimator, as was done in Gresho, Sani and Engelman [19] for the trapezoid rule, is a necessary step. It would be useful if the DLN method could be embedded in a family of different orders with good properties or if it could be induced from simpler methods by added time filters. Both are open problems.

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Appendix A. Proof of Lemma 5.1.

Proof. For first part, if M is even integer

$$\sum_{n=1}^{M-1}(k_n+k_{n-1})\|v_{n+1}\|_k^p \le \sum_{\ell=1}^{M/2}(k_{2\ell-2}+k_{2\ell-1})\|v_{2\ell}\|_k^p$$

$$+\left(k_0\|v_1\|_k^p+\sum_{\ell=1}^{M/2-1}(k_{2\ell-1}+k_{2\ell})\|v_{2\ell+1}\|_k^p+k_{M-1}\|v_M\|_k^p\right)$$

$$=\left(\||v|\|_{p,k}^{P_1,R}\right)^p+\left(\||v|\|_{p,k}^{P_2,R}\right)^p.$$

And

$$\sum_{n=1}^{M-1}(k_n+k_{n-1})\|v_{n-1}\|_k^p \le \sum_{\ell=0}^{M/2-1}(k_{2\ell}+k_{2\ell+1})\|v_{2\ell}\|_k^p$$
$$+\left(k_0\|v_0\|_k^p+\sum_{\ell=0}^{M/2-2}(k_{2\ell+1}+k_{2\ell+2})\|v_{2\ell+1}\|_k^p+k_{M-1}\|v_{M-1}\|_k^p\right)$$
$$=\left(|||v|||_{p,k}^{P_1,L}\right)^p+\left(|||v|||_{p,k}^{P_2,L}\right)^p .$$

If M is odd integer

$$\sum_{n=1}^{M-1}(k_n+k_{n-1})\|v_{n+1}\|_k^p \le \left(\sum_{\ell=1}^{(M-1)/2}(k_{2\ell-2}+k_{2\ell-1})\|v_{2\ell}\|_k^p+k_{M-1}\|v_M\|_k^p\right)$$
$$+\left(k_0\|v_1\|_k^p+\sum_{\ell=1}^{(M-1)/2}(k_{2\ell-1}+k_{2\ell})\|v_{2\ell+1}\|_k^p\right)=\left(\||v|\|_{p,k}^{P_1,R}\right)^p+\left(\||v|\|_{p,k}^{P_2,R}\right)^p .$$

And

$$\sum_{n=1}^{M-1}(k_n+k_{n-1})\|v_{n-1}\|_k^p \le \left(\sum_{\ell=1}^{(M-1)/2}(k_{2\ell-2}+k_{2\ell-1})\|v_{2\ell-2}\|_k^p+k_{M-1}\|v_{M-1}\|_k^p\right)$$
$$+\left(k_0\|v_0\|_k^p+\sum_{\ell=1}^{(M-1)/2}(k_{2\ell-1}+k_{2\ell})\|v_{2\ell-1}\|_k^p\right)=\left(\||v|\|_{p,k}^{P_1,L}\right)^p+\left(\||v|\|_{p,k}^{P_2,L}\right)^p .$$

It's easy to check

$$\sum_{n=1}^{M-1}(k_n+k_{n-1})\|v_n\|_k^p=\left(\||v|\|_{p,k}^{P_0,L}\right)^p+\left(\||v|\|_{p,k}^{P_0,R}\right)^p .$$

Thus we have proved the first part. For second part, we can check

$$\sum_{n=1}^{M-1}(k_n+k_{n-1})\|v(t_{n,*})\|_k^p=\left(|||v_*|||_{p,k}^{\widetilde{P}_1}\right)^p+\left(|||v_*|||_{p,k}^{\widetilde{P}_2}\right)^p .$$

whenever M is even integer or odd integer. □