Math 0240 - Analytic Geometry & Calculus 3
Final Exam, Fall 2016
Solutions

1. (10pts) Let \( C \) be the curve given by \( r(t) = t \mathbf{i} + t^2 \mathbf{j} + 3e^{t-1} \mathbf{k} \). At the point \( P(1, 1, 3) \), find the curvature of \( C \) and parametric or symmetric equations of the line tangent to \( C \).

Solution: \( r'(t) = 1 \mathbf{i} + 2t \mathbf{j} + 3e^{t-1} \mathbf{k}, \quad r''(t) = 0 \mathbf{i} + 2 \mathbf{j} + 3e^{t-1} \mathbf{k} \).

At the point \( P(1, 1, 3) \) \( t = 1 \). Therefore, \( r'(1) = 1 \mathbf{i} + 2 \mathbf{j} + 3 \mathbf{k}, \quad r''(1) = 0 \mathbf{i} + 2 \mathbf{j} + 3 \mathbf{k} \).

\[ r'(1) \times r''(1) = (1, 2, 3) \times (0, 2, 3) = (0, -3, 2). \]

The curvature: \( \kappa(1) = \frac{|r'(1) \times r''(1)|}{|r'(1)|^3} = \frac{\sqrt{0 + 9 + 4}}{(\sqrt{1 + 4 + 9})^3} = \frac{\sqrt{13}}{14\sqrt{14}}. \)

The tangent line is the line through \( P(1, 1, 3) \) and parallel to the vector \( r'(1) = 1 \mathbf{i} + 2 \mathbf{j} + 3 \mathbf{k} \).

Its parametric equation is \( x = t + 1, \quad y = 2t + 1, \quad z = 3t + 3. \)

Symmetric equation: \( \frac{x - 1}{1} = \frac{y - 1}{2} = \frac{z - 3}{3} \).

2. (10pts) Let \( f(x, y, z) = \frac{x - y}{z} + 4\sqrt{x + 3z} \) and \( P \) be the point \( P(1, 1, 1) \). The following three parts are relevant. You do not need to repeat any calculation.

(a) (4pts) What is the direction in which the maximum rate of change of \( f \) occurs at the point \( P \)?

(b) (3pts) Compute the directional derivative of \( f(x, y, z) \) at the point \( P \) in the direction of the vector \( \mathbf{v} = 2 \mathbf{i} + 3 \mathbf{j} + \mathbf{k} \).

(c) (3pts) At the point \( P(1, 1, 1) \), the equation \( \frac{x - y}{z} + 4\sqrt{x + 3z} = 8 \) holds. Use the Implicit Function Theorem to find \( z_x(1, 1) \).

Solution:

(a) The maximum rate of change of \( f \) occurs in the direction of its gradient vector

\[ \nabla f = (f_x, f_y, f_z) = \left( \frac{1}{z} + 4 \cdot \frac{1}{2\sqrt{x + 3z}}, -\frac{1}{z}, (x - y) \cdot \frac{-1}{z^2} + 4 \cdot \frac{1}{2\sqrt{x + 3z}} \cdot 3 \right). \]

\[ \nabla f(1, 1, 1) = (2, -1, 3). \]
(b) The directional derivative is
\[ D_u f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \frac{v}{|v|} = (2, -1, 3) \cdot \frac{(2, 3, 1)}{\sqrt{14}} = \frac{(2, -1, 3) \cdot (2, 3, 1)}{\sqrt{14}} = \frac{4}{\sqrt{14}}. \]
\[ \left\langle 2, -1, 3 \right\rangle \cdot \left\langle 2, 3, 1 \right\rangle \sqrt{14} = \frac{4}{\sqrt{14}}. \]

(c) \[ z_x(1, 1) = -\frac{f_x(1, 1, 1)}{f_z(1, 1, 1)} = -\frac{2}{3}. \text{ (Recall } \nabla f(1, 1, 1) = (2, -1, 3)). \]

3. (10pts) Let \( S \) be the ellipsoid given by the equation \( x^2 + y^2 - xz + z^2 = 2 \). That is, \( S \) is a level surface of the function \( F(x, y, z) = x^2 + y^2 - xz + z^2 \). Find all points on \( S \) where the tangent plane is parallel to the plane \( x + 2y + z = 10 \). (Hint: Use the fact that the coordinates of such a point satisfy the equation of \( S \).)

Solution: \( F_x = 2x - z, \ F_y = 2y, \ F_z = -x + 2z. \)

The normal vector to the plane \( x + 2y + z = 10 \) is \( \mathbf{n} = (1, 2, 1) \).
\( \nabla F \) is parallel to \( \mathbf{n} \) for some constant \( \lambda \). Therefore, \( \nabla F = \lambda \mathbf{n} \) or
\[ \begin{cases} 2x - z = \lambda \\ 2y = 2\lambda \\ x + 2z = \lambda \end{cases} \]

The system gives \( x = y = z = \lambda \). Plug in these results into the equation of \( S \) to get \( \lambda^2 + \lambda^2 - \lambda^2 + \lambda^2 = 2. \)

Then \( \lambda^2 = 1 \) or \( \lambda = \pm 1 \) and the points are \( (1, 1, 1) \) and \( (-1, -1, -1) \).

4. (10pts) Find all critical points of the function \( f(x, y) = 2x^2 + y^2 - x^2y \). For each critical point determine if it is a local maximum, a local minimum, or a saddle point.

Solution: \( f_x = 4x + 0 - 2xy = 2x(2 - y) = 0 \Rightarrow x = 0 \) or \( y = 2. \)
\( f_y = 0 + 2y - x^2 = 0 \Rightarrow x^2 = 2y. \)

Now let’s use results \( x = 0 \) or \( y = 2 \) obtained from the equation \( f_x = 0 \) to solve the equation \( x^2 = 2y. \)

If \( x = 0 \) then the equality \( x^2 = 2y \) gives \( y = 0. \) If \( y = 2 \) then \( x^2 = 4 \) or \( x = \pm 2. \)

Therefore, critical points are \( (0, 0), (-2, 2), \) and \( (2, 2). \)
\( f_{xx} = 4 - 2y, \ f_{xy} = f_{yx} = -2x, \ f_{yy} = 2. \)
\( D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 8 - 4y - 4x^2 \ (= 4(2 - y - x^2)). \)
\( D(0, 0) = 8 > 0, \ f_{xx}(0, 0) = 4 > 0 \Rightarrow \) there is a local minimum at \( (0, 0). \)
\( D(-2, 2) = -16 < 0 \Rightarrow (-2, 2) \) is a saddle point.
\( D(2, 2) = -16 < 0 \Rightarrow (2, 2) \) is a saddle point.
5. (10pts) Suppose that the volume of a solid $E$ can be represented by the triple integral

$$\iiint_E dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$ 

Find the mass of the solid $E$, if the density function is given by $\rho(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$.

**Solution:**

$$m = \iiint_E \rho(x, y, z) dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} dzdydx.$$ 

In spherical coordinates

$$m = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^1 e^{\rho^2} \rho^2 \sin \phi \, d\rho d\phi d\theta = \frac{\pi}{2} \cdot \int_0^{\pi/4} \sin \phi \, d\phi \cdot \left[ \frac{1}{3} e^{\rho^2} \right]_0^1 = \frac{\pi}{2} \cdot \left. \left( -\cos \phi \right) \right|_0^{\pi/4} \cdot \frac{1}{3} (e - 1)$$

$$m = \frac{\pi}{6} \cdot \left( 1 - \frac{\sqrt{2}}{2} \right) \cdot (e - 1).$$

6. (10pts) Given the vector field $\mathbf{F}(x, y) = (2x \ln y - y) \mathbf{i} + (x^2 y^{-1} - x) \mathbf{j}$ defined on $\{(x, y) \mid y > 0\}$.

(a) (6pts) Show that $\mathbf{F}$ is conservative and find a potential function $f$.

(b) (4pts) A particle, under the influence of the vector field $\mathbf{F}$, moves along the curve $C$ given by $\mathbf{r}(t) = (3t) \mathbf{i} + (2t^2 + 1) \mathbf{j}$ from $t = 0$ to $t = 1$. Use the Fundamental Theorem of line integrals to find the work done.

**Solution:**

(a) $\mathbf{F}(x, y) = P \mathbf{i} + Q \mathbf{j}$, $P = 2x \ln y - y$, $Q = x^2 y^{-1} - x$.

$$Q_x - P_y = (2xy^{-1} - 1) - (2xy^{-1} - 1) = 0 \Rightarrow \mathbf{F} \text{ is conservative.}$$

$$f(x, y) = \int P \, dx = \int (2x \ln y - y) \, dx = x^2 \ln y - xy + h(y), \quad f_y = x^2 y^{-1} - x + h'(y) = Q = x^2 y^{-1} - x$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = c, \text{ a constant.}$$

$$f(x, y) = x^2 \ln y - xy + c. \text{ Take } c = 0. \text{ Then } f(x, y) = x^2 \ln y - xy.$$ 

(b) $\mathbf{r}(0) = \langle 0, 1 \rangle$, $\mathbf{r}(1) = \langle 3, 3 \rangle$.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(3, 3) - f(0, 1) = 9 \ln 3 - 9.$$
7. (10pts)

(a) (4pts) Use Green’s Theorem to show that a region $R$ enclosed by a simple closed curve $C$, oriented clockwise, has area $\int_C y \, dx$.

(b) (6pts) Use part a) to compute the area of the region $D$ enclosed by the arch of the cycloid $C_1 : x = t - \sin t$, $y = 1 - \cos t$ from $(0, 0)$ to $(2\pi, 0)$ and the line segment $C_2 : x = t$, $y = 0$ from $(2\pi, 0)$ to $(0, 0)$. See the sketch below.

![Cycloid Arch](image)

**Solution:**

(a) Let $F = \langle P, Q \rangle = \langle y, 0 \rangle$. Then by Green’s theorem

$$ \int_C P \, dx + Q \, dy = \int_C y \, dx = -\iint_R (Q_x - P_y) \, dA = -\iint_R (0 - 1) \, dA = \iint_R dA = A(R), \quad \text{area of } R $$

(b) $t = 0$ at $(0, 0)$ and $t = 2\pi$ at $(2\pi, 0)$.

$$ A(D) = \int_{C_1} y \, dx + \int_{C_2} y \, dx = \int_0^{2\pi} (1 - \cos t)(1 - \cos t) \, dt + \int_0^{2\pi} 0 \, dx = \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) \, dt $$

$$ = \int_0^{2\pi} \left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2} \cos 2t\right) \, dt = \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2} \cos 2t\right) \, dt = \frac{3}{2} t - 2\sin t + \frac{1}{4} \sin 2t \bigg|_0^{2\pi} $$

$$ A(D) = 3\pi - 0 + 0 = 3\pi. $$

8. (10pts) Find the area of the surface $S$ that is the part of the cylinder $x^2 + y^2 = 1$, below the plane $z = 3 - x - y$ and above the plane $z = 0$.

**Solution:** $S : x = \cos \theta, y = \sin \theta, z = z, \ 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq 3 - x - y = 3 - \cos \theta - \sin \theta$.

$r(\theta, z) = (\cos \theta, \sin \theta, z), \ r_\theta = (-\sin \theta, \cos \theta, z), \ r_z = (0, 0, 1)$.

$n = r_\theta \times r_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos \theta, \sin \theta, 0 \rangle, \ |n| = 1$

$$ A(S) = \int_0^{2\pi} \int_0^{3-\cos \theta - \sin \theta} 1 \cdot 1 \, dz \, d\theta = \int_0^{2\pi} (3 - \cos \theta - \sin \theta) \, d\theta = 6\pi. $$

[An alternative solution: $A(S) = \int_C f(x, y) \, ds$

where $C : x = \cos \theta, y = \sin \theta, \ 0 \leq \theta \leq 2\pi$ and $f(x, y) = z = 3 - x - y = 3 - \cos \theta - \sin \theta$.

Then $A(S) = \int_0^{2\pi} (3 - \cos \theta - \sin \theta) \, d\theta = 6\pi.$]
9. (10pts) Use Stoke’s Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F}(x,y,z) = x^2y \mathbf{i} + \frac{1}{3}x^3 \mathbf{j} + xy \mathbf{k} \), and where 
\( C \) is the curve of intersection of the cylinder \( x^2 + y^2 = 1 \) and the hyperbolic paraboloid \( z = y^2 - x^2 \), oriented counterclockwise when viewed from above.

Solution:
\[ S: \quad x = x, \quad y = y, \quad z = y^2 - x^2. \]
\[ D: \quad x^2 + y^2 = 1. \]
\[ \mathbf{n} = (-z_x, -z_y, 1) = (2x, -2y, 1). \]

\[ \text{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^y & 0 & xe^y + 3y^2 \end{vmatrix} = (x, -y, 0). \]

By Stoke’s Theorem
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot dS = \iint_D (x, -y, 0) \cdot (2x, -2y, 1) \, dA = \iint_D (2x^2 + 2y^2) \, dA \]
\[ = 2 \int_0^{2\pi} \int_0^1 r^2 \cdot r \, dr \, d\theta = 2 \cdot 2\pi \cdot \frac{r^4}{4} \bigg|_0^1 = \pi. \]

10. (10pts) Let \( S \) be the boundary surface of the solid \( E \) enclosed by the paraboloids \( z = 1 + x^2 + y^2 \) and \( z = 2(x^2 + y^2) \), with the normal pointing outward. Compute the flux integral \( \iint_S \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F}(x,y,z) = (e^{\sin z} - x^2) \mathbf{i} + 2xy \mathbf{j} + (z^2 - \cos y) \mathbf{k} \).

Solution:
\[ \text{div} \mathbf{F} = -2x + 2x + 2z = 2z. \] By the Divergence Theorem
\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div} \mathbf{F} \, dV = \iiint_E 2z \, dV = \int_0^{\pi/2} \int_0^1 \int_{2\pi}^{1+r^2} 2zr \, dz \, dr \, d\theta = 2\pi \cdot \int_0^1 [z^2]_{2\pi}^{1+r^2} \cdot r \, dr \]
\[ = 2\pi \cdot \int_0^1 ((1 + r^2)^2 - (2r^2)^2) \cdot r \, dr = 2\pi \cdot \int_0^1 (r + 2r^3 - 3r^5) \, dr = 2\pi \cdot \left[ \frac{r^2}{2} + \frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 \]
\[ = 2\pi \cdot \left( \frac{1}{2} + \frac{1}{4} - \frac{1}{2} \right) = \pi. \]