Math 0240 - Analytic Geometry & Calculus 3
Final Exam, Fall 2016
Solutions

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1. (10pts) Let C be the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j} + 3e^{t-1}\mathbf{k}$. At the point $P(1,1,3)$, find the curvature of C and parametric or symmetric equations of the line tangent to C.

   Solution: $\mathbf{r}'(t) = 1\mathbf{i} + 2t\mathbf{j} + 3e^{t-1}\mathbf{k}$, $\quad \mathbf{r}''(t) = 0\mathbf{i} + 2\mathbf{j} + 3e^{t-1}\mathbf{k}$.

   At the point $P(1,1,3)$ $t = 1$. Therefore, $\mathbf{r}'(1) = 1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\quad \mathbf{r}''(1) = 0\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

   $\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1,2,3\rangle \times \langle 0,2,3\rangle = \langle 0,-3,2\rangle$.

   The curvature: $\kappa(1) = \dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{0+9+4}}{(\sqrt{1+4+9})^3} = \dfrac{\sqrt{13}}{14\sqrt{14}}$.

   The tangent line is the line through $P(1,1,3)$ and parallel to the vector $\mathbf{r}'(1) = 1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

   Its parametric equation is $x = t+1, \quad y = 2t+1, \quad z = 3t+3$.

   Symmetric equation: $\dfrac{x-1}{1} = \dfrac{y-1}{2} = \dfrac{z-3}{3}$.

2. (10pts) Let $f(x,y,z) = \dfrac{x-y}{z} + 4\sqrt{x+3z}$ and P be the point $P(1,1,1)$. The following three parts are relevant. You do not need to repeat any calculation.

   (a) (4pts) What is the direction in which the maximum rate of change of f occurs at the point P?

   (b) (3pts) Compute the directional derivative of $f(x,y,z)$ at the point P in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

   (c) (3pts) At the point $P(1,1,1)$, the equation $\dfrac{x-y}{z} + 4\sqrt{x+3z} = 8$ holds. Use the Implicit Function Theorem to find $z_x(1,1)$.

   Solution:

   (a) The maximum rate of change of f occurs in the direction of its gradient vector

   $$\nabla f = \langle f_x, f_y, f_z\rangle = \left\langle \frac{1}{z} + 4\cdot\frac{1}{2\sqrt{x+3z}},\ -\frac{1}{z},\ (x-y)\cdot\frac{-1}{z^2} + 4\cdot\frac{1}{2\sqrt{x+3z}}\cdot 3\right\rangle.$$

   $\nabla f(1,1,1) = \langle 2,\ -1,\ 3\rangle$.

(b) The directional derivative is

$$D_u f(1,1,1) = \nabla f(1,1,1) \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \langle 2,\, -1,\, 3\rangle \cdot \frac{\langle 2,\, 3,\, 1\rangle}{\sqrt{14}} = \frac{\langle 2,\, -1,\, 3\rangle \cdot \langle 2,\, 3,\, 1\rangle}{\sqrt{14}} = \frac{4}{\sqrt{14}}.$$

(c) $z_x(1,1) = -\dfrac{f_x(1,1,1)}{f_z(1,1,1)} = -\dfrac{2}{3}$. (Recall $\nabla f(1,1,1) = \langle 2,\, -1,\, 3\rangle$).

3. (10pts) Let S be the ellipsoid given by the equation $x^2 + y^2 - xz + z^2 = 2$. That is, S is a level surface of the function $F(x, y, z) = x^2 + y^2 - xz + z^2$. Find all points on S where the tangent plane is parallel to the plane $x + 2y + z = 10$. (Hint: Use the fact that the coordinates of such a point satisfy the equation of S.)

Solution: $F_x = 2x - z,\ F_y = 2y,\ F_z = -x + 2z.$

The normal vector to the plane $x + 2y + z = 10$ is $\mathbf{n} = \langle 1,\, 2,\, 1\rangle$.

∇F is parallel to $\mathbf{n}$ for some constant λ. Therefore, $\nabla F = \lambda \mathbf{n}$ or

$$\begin{cases} 2x - z = \lambda \\ 2y = 2\lambda \\ x + 2z = \lambda \end{cases}$$

The system gives $x = y = z = \lambda$. Plug in these results into the equation of S to get $\lambda^2 + \lambda^2 - \lambda^2 + \lambda^2 = 2$. Then $\lambda^2 = 1$ or $\lambda = \pm 1$ and the points are $(1, 1, 1)$ and $(-1, -1, -1)$.

4. (10pts) Find all critical points of the function $f(x, y) = 2x^2 + y^2 - x^2 y$. For each critical point determine if it is a local maximum, a local minimum, or a saddle point.

Solution: $f_x = 4x + 0 - 2xy = 2x(2 - y) = 0 \ \Rightarrow\ x = 0$ or $y = 2$.

$f_y = 0 + 2y - x^2 = 0 \ \Rightarrow\ x^2 = 2y$.

Now let's use results $x = 0$ or $y = 2$ obtained from the equation $f_x = 0$ to solve the equation $x^2 = 2y$.

If $x = 0$ then the equality $x^2 = 2y$ gives $y = 0$. If $y = 2$ then $x^2 = 4$ or $x = \pm 2$.

Therefore, critical points are $(0, 0)$, $(-2, 2)$, and $(2, 2)$.

$f_{xx} = 4 - 2y,\ f_{xy} = f_{yx} = -2x,\ f_{yy} = 2.$

$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 8 - 4y - 4x^2 \ \ (= 4(2 - y - x^2)).$

$D(0, 0) = 8 > 0,\ f_{xx}(0, 0) = 4 > 0 \ \Rightarrow$ there is a local minimum at $(0, 0)$.

$D(-2, 2) = -16 < 0 \ \Rightarrow\ (-2, 2)$ is a saddle point.

$D(2, 2) = -16 < 0 \ \Rightarrow\ (2, 2)$ is a saddle point.

5. (10pts) Suppose that the volume of a solid E can be represented by the triple integral

$$\iiint_E dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$

Find the mass of the solid E, if the density function is given by $\rho(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$.

Solution:

$$m = \iiint_E \rho(x,y,z)\, dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dzdydx.$$

In spherical coordinates

$$m = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^1 e^{\rho^3} \rho^2 \sin\phi\, d\rho d\phi d\theta = \frac{\pi}{2} \cdot \int_0^{\pi/4} \sin\phi\, d\phi \cdot \left[\frac{1}{3}e^{\rho^3}\right]_0^1 = \frac{\pi}{2} \cdot \quad (-\cos\phi)\Big|_0^{\pi/4} \cdot \frac{1}{3}(e-1)$$

$$m = \frac{\pi}{6} \cdot \left(1 - \frac{\sqrt{2}}{2}\right) \cdot (e-1).$$

6. (10pts) Given the vector field $\mathbf{F}(x,y) = (2x\ln y - y)\,\mathbf{i} + \left(x^2y^{-1} - x\right)\mathbf{j}$ defined on $\{(x,y) \mid y > 0\}$.
   (a) (6pts) Show that $\mathbf{F}$ is conservative and find a potential function f.
   (b) (4pts) A particle, under the influence of the vector field $\mathbf{F}$, moves along the curve C given by $\mathbf{r}(t) = (3t)\mathbf{i} + (2t^2+1)\mathbf{j}$ from $t = 0$ to $t = 1$. Use the Fundamental Theorem of line integrals to find the work done.

Solution:

(a) $\mathbf{F}(x,y) = P\,\mathbf{i} + Q\,\mathbf{j},\ \ P = 2x\ln y - y,\ \ Q = x^2y^{-1} - x.$

$Q_x - P_y = (2xy^{-1} - 1) - (2xy^{-1} - 1) = 0 \quad\Rightarrow\quad \mathbf{F}$ is conservative.

$f(x,y) = \displaystyle\int P\,dx = \int (2x\ln y - y)\,dx = x^2\ln y - xy + h(y),\ \ f_y = x^2y^{-1} - x + h'(y) = Q = x^2y^{-1} - x$

$\Rightarrow \quad h'(y) = 0 \quad\Rightarrow\quad h(y) = c$, a constant.

$f(x,y) = x^2\ln y - xy + c.$ Take $c = 0$. Then $f(x,y) = x^2\ln y - xy.$

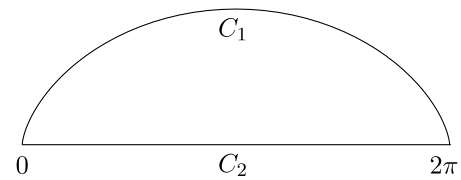
(b) $\mathbf{r}(0) = \langle 0,\, 1\rangle,\ \ \mathbf{r}(1) = \langle 3,\, 3\rangle.$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = f(3,3) - f(0,1) = 9\ln 3 - 9.$$

7. (10pts)

(a) (4pts) Use Green's Theorem to show that a region R enclosed by a simple closed curve C, oriented clockwise, has area $\int_C y\,dx$.

(b) (6pts) Use part a) to compute the area of the region D enclosed by the arch of the cycloid $C_1 :\ x = t - \sin t,\ y = 1 - \cos t$ from $(0, 0)$ to $(2\pi, 0)$ and the line segment $C_2 :\ x = t,\ y = 0$ from $(2\pi, 0)$ to $(0, 0)$. See the sketch below.



Solution:

(a) Let $\mathbf{F} = \langle P, Q\rangle = \langle y, 0\rangle$. Then by Green's theorem

$$\int_C P dx + Q dy = \int_C y\,dx = -\iint_R (Q_x - P_y)\,dA = -\iint_R (0-1)\,dA = \iint_R dA = A(R), \quad \text{area of R}$$

(b) $t = 0$ at $(0, 0)$ and $t = 2\pi$ at $(2\pi, 0)$.

$$A(D) = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt + \int_{C_2} 0\,dx = \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt$$

$$= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\right)\,dt = \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right)\,dt = \frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\,\Big|_0^{2\pi}$$

$A(D) = 3\pi - 0 + 0 = 3\pi.$

8. (10pts) Find the area of the surface S that is the part of the cylinder $x^2 + y^2 = 1$, below the plane $z = 3 - x - y$ and above the plane $z = 0$.

Solution: S: $x = \cos\theta,\ y = \sin\theta,\ z = z,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 3 - x - y = 3 - \cos\theta - \sin\theta.$

$\mathbf{r}(\theta, z) = \langle \cos\theta, \sin\theta, z\rangle,\ \mathbf{r}_\theta = \langle -\sin\theta, \cos\theta, z\rangle,\ \mathbf{r}_z = \langle 0, 0, 1\rangle.$

$$\mathbf{n} = \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0\rangle, \qquad |\mathbf{n}| = 1$$

$$A(S) = \int_0^{2\pi} \int_0^{3-\cos\theta-\sin\theta} 1 \cdot 1\,dz d\theta = \int_0^{2\pi} (3 - \cos\theta - \sin\theta)\,d\theta = 6\pi.$$

[An alternative solution: $A(S) = \int_C f(x, y)\,ds$

where C: $x = \cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi$ and $f(x, y) = z = 3 - x - y = 3 - \cos\theta - \sin\theta.$

Then $A(S) = \int_0^{2\pi} (3 - \cos\theta - \sin\theta)\,d\theta = 6\pi.$]

9. (10pts) Use Stoke's Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = x^2y\,\mathbf{i} + \dfrac{1}{3}x^3\;\mathbf{j} + xy\;\mathbf{k}$, and where C is the curve of intersection of the cylinder $x^2+y^2=1$ and the hyperbolic paraboloid $z = y^2 - x^2$, oriented counterclockwise when viewed from above.

Solution: S: $x = x,\ y = y,\ z = y^2 - x^2$. D: $x^2 + y^2 = 1$.

$\mathbf{n} = \langle -z_x, -z_y, 1\rangle = \langle 2x, -2y, 1\rangle$.

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^y & 0 & xe^y + 3y^2 \end{vmatrix} = \langle x, -y, 0\rangle.$$

By Stoke's Theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot dS = \iint_D \langle x,-y,0\rangle\cdot\langle 2x,-2y,1\rangle\, dA = \iint_D \left(2x^2+2y^2\right)\, dA$$

$$= 2\int_0^{2\pi}\int_0^1 r^2\cdot r\,dr d\theta = 2\cdot 2\pi\cdot \left.\frac{r^4}{4}\right|_0^1 = \pi.$$

10. (10pts) Let S be the boundary surface of the solid E enclosed by the paraboloids $z = 1 + x^2 + y^2$ and $z = 2(x^2+y^2)$, with the normal pointing outward. Compute the flux integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = \left(e^{\sin z} - x^2\right)\mathbf{i} + 2xy\mathbf{j} + (z^2 - \cos y)\mathbf{k}$.

Solution: $\operatorname{div}\mathbf{F} = -2x + 2x + 2z = 2z$. By the Divergence Theorem

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 2z\,dV = \int_0^{\pi/2}\int_0^1\int_{2r^2}^{1+r^2} 2zr\ dzdrd\theta = 2\pi\cdot\int_0^1 \left[\, z^2\,\right]_{2r^2}^{1+r^2}\cdot r\,dr$$

$$= 2\pi\cdot\int_0^1 \left((1+r^2)^2 - (2r^2)^2\right)\cdot r\,dr = 2\pi\cdot\int_0^1 \left(r + 2r^3 - 3r^5\right)\,dr = 2\pi\cdot\left[\frac{r^2}{2} + \frac{r^4}{2} - \frac{r^6}{2}\right]_0^1$$

$$= 2\pi\cdot\left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\right) = \pi.$$