

Math 0240 - Analytic Geometry & Calculus 3
Final Exam, Fall 2016

Solutions

1. (10pts) Let C be the curve given by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3e^{t-1}\mathbf{k}$. At the point $P(1, 1, 3)$, find the curvature of C and parametric or symmetric equations of the line tangent to C .

Solution: $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3e^{t-1}\mathbf{k}$, $\mathbf{r}''(t) = 0\mathbf{i} + 2\mathbf{j} + 3e^{t-1}\mathbf{k}$.

At the point $P(1, 1, 3)$ $t = 1$. Therefore, $\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{r}''(1) = 0\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 3 \rangle \times \langle 0, 2, 3 \rangle = \langle 0, -3, 2 \rangle$.

The curvature: $\kappa(1) = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{\sqrt{0+9+4}}{(\sqrt{1+4+9})^3} = \frac{\sqrt{13}}{14\sqrt{14}}$.

The tangent line is the line through $P(1, 1, 3)$ and parallel to the vector $\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Its parametric equation is $x = t + 1$, $y = 2t + 1$, $z = 3t + 3$.

Symmetric equation: $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-3}{3}$.

2. (10pts) Let $f(x, y, z) = \frac{x-y}{z} + 4\sqrt{x+3z}$ and P be the point $P(1, 1, 1)$. The following three parts are relevant. You do not need to repeat any calculation.

(a) (4pts) What is the direction in which the maximum rate of change of f occurs at the point P ?

(b) (3pts) Compute the directional derivative of $f(x, y, z)$ at the point P in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

(c) (3pts) At the point $P(1, 1, 1)$, the equation $\frac{x-y}{z} + 4\sqrt{x+3z} = 8$ holds. Use the Implicit Function Theorem to find $z_x(1, 1)$.

Solution:

(a) The maximum rate of change of f occurs in the direction of its gradient vector

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{1}{z} + 4 \cdot \frac{1}{2\sqrt{x+3z}}, -\frac{1}{z}, (x-y) \cdot \frac{-1}{z^2} + 4 \cdot \frac{1}{2\sqrt{x+3z}} \cdot 3 \right\rangle.$$

$$\nabla f(1, 1, 1) = \langle 2, -1, 3 \rangle.$$

(b) The directional derivative is

$$D_{\mathbf{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \langle 2, -1, 3 \rangle \cdot \frac{\langle 2, 3, 1 \rangle}{\sqrt{14}} = \frac{\langle 2, -1, 3 \rangle \cdot \langle 2, 3, 1 \rangle}{\sqrt{14}} = \frac{4}{\sqrt{14}}.$$

(c) $z_x(1, 1) = -\frac{f_x(1, 1, 1)}{f_z(1, 1, 1)} = -\frac{2}{3}$. (Recall $\nabla f(1, 1, 1) = \langle 2, -1, 3 \rangle$).

3. (10pts) Let S be the ellipsoid given by the equation $x^2 + y^2 - xz + z^2 = 2$. That is, S is a level surface of the function $F(x, y, z) = x^2 + y^2 - xz + z^2$. Find all points on S where the tangent plane is parallel to the plane $x + 2y + z = 10$. (Hint: Use the fact that the coordinates of such a point satisfy the equation of S .)

Solution: $F_x = 2x - z$, $F_y = 2y$, $F_z = -x + 2z$.

The normal vector to the plane $x + 2y + z = 10$ is $\mathbf{n} = \langle 1, 2, 1 \rangle$.

∇F is parallel to \mathbf{n} for some constant λ . Therefore, $\nabla F = \lambda \mathbf{n}$ or

$$\{ * 2x - z = \lambda, 2y = 2\lambda, x + 2z = \lambda$$

The system gives $x = y = z = \lambda$. Plug in these results into the equation of S to get $\lambda^2 + \lambda^2 - \lambda^2 + \lambda^2 = 2$. Then $\lambda^2 = 1$ or $\lambda = \pm 1$ and the points are $(1, 1, 1)$ and $(-1, -1, -1)$.

4. (10pts) Find all critical points of the function $f(x, y) = 2x^2 + y^2 - x^2y$. For each critical point determine if it is a local maximum, a local minimum, or a saddle point.

Solution: $f_x = 4x + 0 - 2xy = 2x(2 - y) = 0 \Rightarrow x = 0$ or $y = 2$.

$$f_y = 0 + 2y - x^2 = 0 \Rightarrow x^2 = 2y.$$

Now let's use results $x = 0$ or $y = 2$ obtained from the equation $f_x = 0$ to solve the equation $x^2 = 2y$.

If $x = 0$ then the equality $x^2 = 2y$ gives $y = 0$. If $y = 2$ then $x^2 = 4$ or $x = \pm 2$.

Therefore, critical points are $(0, 0)$, $(-2, 2)$, and $(2, 2)$.

$$f_{xx} = 4 - 2y, f_{xy} = f_{yx} = -2x, f_{yy} = 2.$$

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 8 - 4y - 4x^2 \quad (= 4(2 - y - x^2)).$$

$$D(0, 0) = 8 > 0, f_{xx}(0, 0) = 4 > 0 \Rightarrow \text{there is a local minimum at } (0, 0).$$

$$D(-2, 2) = -16 < 0 \Rightarrow (-2, 2) \text{ is a saddle point.}$$

$$D(2, 2) = -16 < 0 \Rightarrow (2, 2) \text{ is a saddle point.}$$

5. (10pts) Suppose that the volume of a solid E can be represented by the triple integral

$$\iiint_E dV = \int_0^{\frac{\sqrt{2}}{2}} \int_0^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz dy dx.$$

Find the mass of the solid E , if the density function is given by $\rho(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$.

Solution:

$$m = \iiint_E \rho(x, y, z) dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} dz dy dx.$$

In spherical coordinates

$$m = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^1 e^{\rho^3} \rho^2 \sin \phi \, d\rho d\phi d\theta = \frac{\pi}{2} \cdot \int_0^{\pi/4} \sin \phi \, d\phi \cdot \left[\frac{1}{3} e^{\rho^3} \right]_0^1 = \frac{\pi}{2} \cdot (-\cos \phi) \Big|_0^{\pi/4} \cdot \frac{1}{3} (e - 1)$$

$$m = \frac{\pi}{6} \cdot \left(1 - \frac{\sqrt{2}}{2} \right) \cdot (e - 1).$$

6. (10pts) Given the vector field $\mathbf{F}(x, y) = (2x \ln y - y) \mathbf{i} + (x^2 y^{-1} - x) \mathbf{j}$ defined on $\{(x, y) \mid y > 0\}$.

(a) (6pts) Show that \mathbf{F} is conservative and find a potential function f .

(b) (4pts) A particle, under the influence of the vector field \mathbf{F} , moves along the curve C given by $\mathbf{r}(t) = (3t)\mathbf{i} + (2t^2 + 1)\mathbf{j}$ from $t = 0$ to $t = 1$. Use the Fundamental Theorem of line integrals to find the work done.

Solution:

(a) $\mathbf{F}(x, y) = P \mathbf{i} + Q \mathbf{j}$, $P = 2x \ln y - y$, $Q = x^2 y^{-1} - x$.

$$Q_x - P_y = (2xy^{-1} - 1) - (2xy^{-1} - 1) = 0 \Rightarrow \mathbf{F} \text{ is conservative.}$$

$$f(x, y) = \int P dx = \int (2x \ln y - y) dx = x^2 \ln y - xy + h(y), \quad f_y = x^2 y^{-1} - x + h'(y) = Q = x^2 y^{-1} - x$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = c, \text{ a constant.}$$

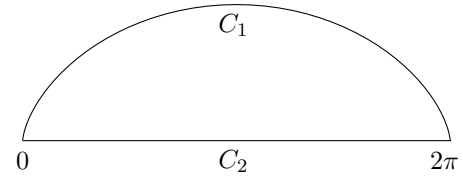
$$f(x, y) = x^2 \ln y - xy + c. \quad \text{Take } c = 0. \quad \text{Then } f(x, y) = x^2 \ln y - xy.$$

(b) $\mathbf{r}(0) = \langle 0, 1 \rangle$, $\mathbf{r}(1) = \langle 3, 3 \rangle$.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(3, 3) - f(0, 1) = 9 \ln 3 - 9.$$

7. (10pts)

- (a) (4pts) Use Green's Theorem to show that a region R enclosed by a simple closed curve C , oriented clockwise, has area $\int_C y \, dx$.
- (b) (6pts) Use part a) to compute the area of the region D enclosed by the arch of the cycloid $C_1 : x = t - \sin t, y = 1 - \cos t$ from $(0, 0)$ to $(2\pi, 0)$ and the line segment $C_2 : x = t, y = 0$ from $(2\pi, 0)$ to $(0, 0)$. See the sketch below.

*Solution:*

- (a) Let $\mathbf{F} = \langle P, Q \rangle = \langle y, 0 \rangle$. Then by Green's theorem

$$\int_C P dx + Q dy = \int_C y \, dx = - \iint_R (Q_x - P_y) \, dA = - \iint_R (0 - 1) \, dA = \iint_R dA = A(R), \quad \text{area of } R$$

- (b) $t = 0$ at $(0, 0)$ and $t = 2\pi$ at $(2\pi, 0)$.

$$\begin{aligned} A(D) &= \int_{C_1} y \, dx + \int_{C_2} y \, dx = \int_0^{2\pi} (1 - \cos t)(1 - \cos t) \, dt + \int_{C_2} 0 \, dx = \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t) \, dt \\ &= \int_0^{2\pi} \left(1 - 2 \cos t + \frac{1}{2} + \frac{1}{2} \cos 2t \right) \, dt = \int_0^{2\pi} \left(\frac{3}{2} - 2 \cos t + \frac{1}{2} \cos 2t \right) \, dt = \left. \frac{3}{2}t - 2 \sin t + \frac{1}{4} \sin 2t \right|_0^{2\pi} \\ A(D) &= 3\pi - 0 + 0 = 3\pi. \end{aligned}$$

8. (10pts) Find the area of the surface S that is the part of the cylinder $x^2 + y^2 = 1$, below the plane $z = 3 - x - y$ and above the plane $z = 0$.

Solution: $S: x = \cos \theta, y = \sin \theta, z = z, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 3 - x - y = 3 - \cos \theta - \sin \theta$.

$$\mathbf{r}(\theta, z) = \langle \cos \theta, \sin \theta, z \rangle, \quad \mathbf{r}_\theta = \langle -\sin \theta, \cos \theta, z \rangle, \quad \mathbf{r}_z = \langle 0, 0, 1 \rangle.$$

$$\mathbf{n} = \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos \theta, \sin \theta, 0 \rangle, \quad |\mathbf{n}| = 1$$

$$A(S) = \int_0^{2\pi} \int_0^{3 - \cos \theta - \sin \theta} 1 \cdot 1 \, dz d\theta = \int_0^{2\pi} (3 - \cos \theta - \sin \theta) \, d\theta = 6\pi.$$

[An alternative solution: $A(S) = \int_C f(x, y) \, ds$

where $C: x = \cos \theta, y = \sin \theta, 0 \leq \theta \leq 2\pi$ and $f(x, y) = z = 3 - x - y = 3 - \cos \theta - \sin \theta$.

Then $A(S) = \int_0^{2\pi} (3 - \cos \theta - \sin \theta) \, d\theta = 6\pi.$]

9. (10pts) Use Stoke's Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2y \mathbf{i} + \frac{1}{3}x^3 \mathbf{j} + xy \mathbf{k}$, and where C is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the hyperbolic paraboloid $z = y^2 - x^2$, oriented counterclockwise when viewed from above.

Solution: $S: x = x, y = y, z = y^2 - x^2$. $D: x^2 + y^2 = 1$.

$$\mathbf{n} = \langle -z_x, -z_y, 1 \rangle = \langle 2x, -2y, 1 \rangle.$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^y & 0 & xe^y + 3y^2 \end{vmatrix} = \langle x, -y, 0 \rangle.$$

By Stoke's Theorem

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle x, -y, 0 \rangle \cdot \langle 2x, -2y, 1 \rangle dA = \iint_D (2x^2 + 2y^2) dA \\ &= 2 \int_0^{2\pi} \int_0^1 r^2 \cdot r dr d\theta = 2 \cdot 2\pi \cdot \frac{r^4}{4} \Big|_0^1 = \pi. \end{aligned}$$

10. (10pts) Let S be the boundary surface of the solid E enclosed by the paraboloids $z = 1 + x^2 + y^2$ and $z = 2(x^2 + y^2)$, with the normal pointing outward. Compute the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (e^{\sin z} - x^2) \mathbf{i} + 2xy \mathbf{j} + (z^2 - \cos y) \mathbf{k}$.

Solution: $\text{div } \mathbf{F} = -2x + 2x + 2z = 2z$. By the Divergence Theorem

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \text{div } \mathbf{F} dV = \iiint_E 2z dV = \int_0^{\pi/2} \int_0^1 \int_{2r^2}^{1+r^2} 2zr dz dr d\theta = 2\pi \cdot \int_0^1 \left[z^2 \right]_{2r^2}^{1+r^2} \cdot r dr \\ &= 2\pi \cdot \int_0^1 ((1+r^2)^2 - (2r^2)^2) \cdot r dr = 2\pi \cdot \int_0^1 (r + 2r^3 - 3r^5) dr = 2\pi \cdot \left[\frac{r^2}{2} + \frac{r^4}{2} - \frac{r^6}{2} \right]_0^1 \\ &= 2\pi \cdot \left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2} \right) = \pi. \end{aligned}$$