Sample Final Exam

Solutions

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1. The points $A = (-1, 2)$ and $B = (7, -4)$ are at the ends of a diameter of a circle.

   (a) Find coordinates of the center of the circle C.

   Solution: C is a midpoint of the diameter, $C = \left(\dfrac{-1+7}{2}, \dfrac{2+(-4)}{2}\right) = (3, -1)$.

   (b) Find the equation of the circle.

   Solution: The radius is the distance between points A and C (or B and C):

   $r = \sqrt{(-1-3)^2 + (2-(-1))^2} = \sqrt{(-4)^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5.$

   The equation of the circle: $(x-3)^2 + (y-(-1))^2 = 5^2, \quad (x-3)^2 + (y+1)^2 = 25.$

   (c) Determine the equation of line passing through points A and B. Write it in the slope-intercept form.

   Solution: The slope of the line is $m = \dfrac{-4-2}{7-(-1)} = \dfrac{-6}{8} = -\frac{3}{4}$.

   The equation of the line: $y-2 = -\frac{3}{4}(x-(-1)), \quad y-2 = -\frac{3}{4}(x+1), \quad y = -\frac{3}{4}x - \frac{3}{4} + 2.$

   The equation of the line in slope-intercept form: $y = -\frac{3}{4}x + 1\frac{1}{4}$.

   (d) Determine the equation of line perpendicular to the diameter AB which passes through the center C. Write it in the slope-intercept form.

   Solution: The slope of the line is $m_\perp = -\frac{1}{m} = \frac{4}{3}$.

   The equation of the line: $y-(-1) = \frac{4}{3}(x-3), \quad y+1 = \frac{4}{3}x - 4.$

   The equation of the line in slope-intercept form: $y = \frac{4}{3}x - 3$.

2. A book, a pen, and a notebook together cost \$100. The book costs \$80 more then the pen and the notebook costs twice as much as the pen. What is the price of each item? To solve the problem use one equation with one unknown.

   Solution: Let x be the price of the pen. Then the price of the book is $x+80$, and the price of the notebook is $2x$.

Then $x+(x+80)+2x=100 \;\Leftrightarrow\; 4x+80=100 \;\Leftrightarrow\; 4x=20 \;\Leftrightarrow\; x=5.$
$x+80=85,\;\; 2x=10.$

Therefore the book costs \$85, the notebook \$10, and the pen \$5.

3. Simplify. Write answers in the form $a+bi$ where a and b are real numbers.

(a) $(5-3i)-(-1+i)$

Solution: $(5-3i)-(-1+i)=5-3i+1-i=6-4i$

(b) $\dfrac{-2+i}{1-i}$

Solution: $\dfrac{-2+i}{1-i}=\dfrac{-2+i}{1-i}\cdot\dfrac{1+i}{1+i}=\dfrac{-2-2i+i-1}{1^2+1^2}=\dfrac{-3-i}{2}=-\tfrac{3}{2}-\tfrac{1}{2}i$

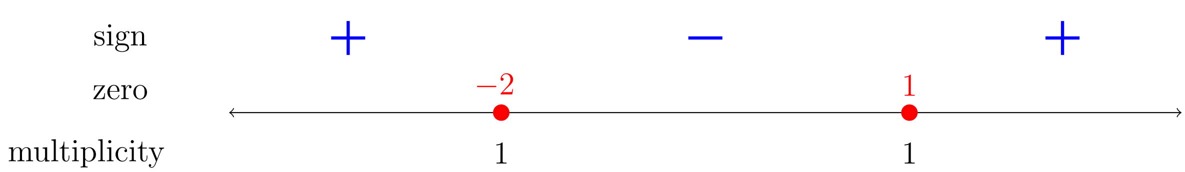
(c) i^{33}

Solution: $i^{33}=i^{32+1}=\left(i^4\right)^8\cdot i^1=1\cdot i=i$

4. Solve inequalities. Write answers in interval notation.

(a) $x^2+x>2.$

Solution: $x^2+x>2 \;\Leftrightarrow\; x^2+x-2>0 \;\Leftrightarrow\; (x+2)(x-1)>0.$



Answer: $(-\infty,-2)\cup(1,\infty).$

(b) $|3x+2|>2.$

Solution: $|3x+2|>2 \;\Leftrightarrow\; 3x+2<-2$ or $3x+2>2$

$\Leftrightarrow\; 3x<-4$ or $3x>0$

$$\Leftrightarrow \quad x < -\tfrac{4}{3} \quad \text{or} \quad x > 0$$

Answer: $\left(-\infty, -\frac{4}{3}\right) \cup (0, \infty)$.

(c) $\dfrac{x}{x-1} \le 1.$

Solution: $$\frac{x}{x-1} \le 1 \quad \Leftrightarrow \quad \frac{x}{x-1} - 1 \le 0 \quad \Leftrightarrow \quad \frac{x-(x-1)}{x-1} \le 0$$
$$\Leftrightarrow \quad \frac{1}{x-1} \le 0 \quad \Leftrightarrow \quad x - 1 < 0.$$

Answer: $(-\infty, 1)$.

5. Find all asymptotes of the graph of the function $f(x) = \dfrac{3x+5}{x^2-2x-3}$. Support your answer. If the function doesn't have asymptotes of certain type(s) explain why.

Solution: $f(x) = \dfrac{3x+5}{(x+1)(x-3)}$ is a rational function.

(a) Vertical.

Solution: The denominator $(x+1)(x-3)$ is zero when $x = -1$ and $x = 3$. So the vertical asymptotes are lines $x = -1$ and $x = 3$.

(b) Horizontal.

Solution: The degree of numerator is less than the degree of the denominator. So, the horizontal asymptote is $y = 0$.

(c) Oblique.

Solution: There is no oblique asymptotes.

6. $f(x) = x^2 - 6x + 7$.

(a) Write $f(x)$ in the form $a(x-h)^2 + k$ by completing square. Find its vertex and axis of symmetry.

Solution: $f(x) = x^2 - 6x + 7 = x^2 - 6x + 9 - 2 = (x-3)^2 - 2.$

The vertex is $(3, -2)$. The axis of symmetry is $x = 3$.

(b) Sketch the graph of the function $f(x)$. Mark the vertex and draw the axis of symmetry.

Solution:

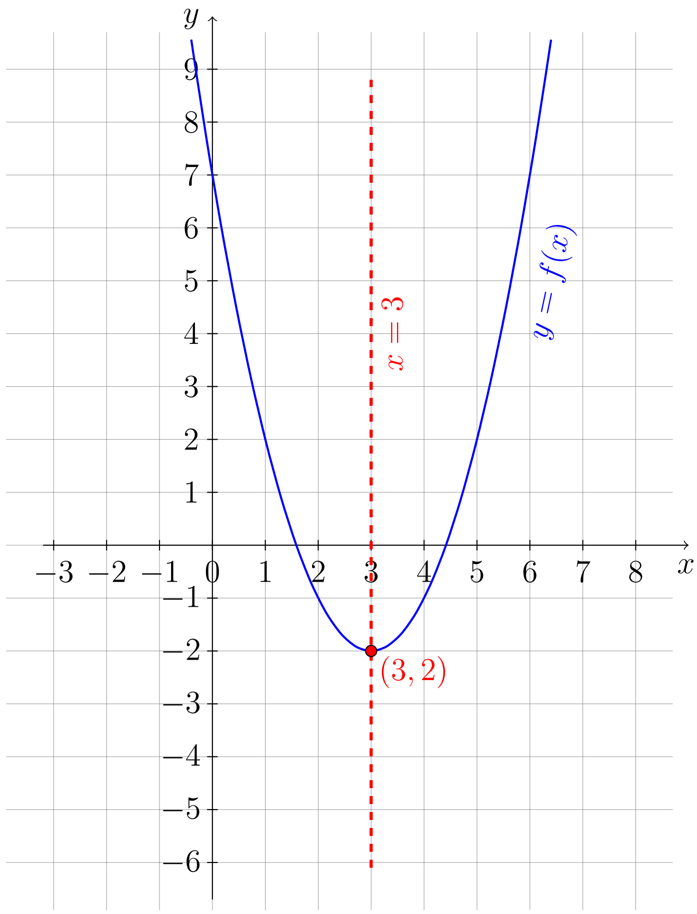


Figure 1: Graph of the function $f(x) = x^2 - 6x + 7$.

7. Consider the polynomial function $h(x) = \frac{1}{4}x^3 + \frac{1}{2}x^2 - x - 2 = \frac{1}{4}(x+2)(x^2-4)$.

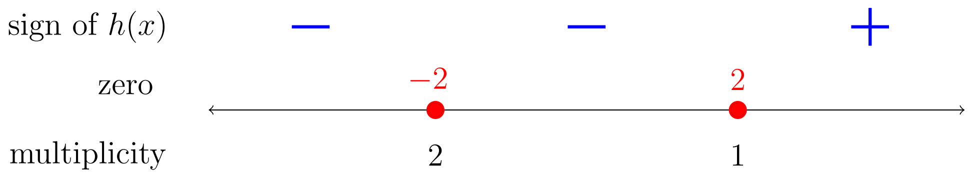
(a) Find zeros of $h(x)$ and their multiplicities.

Solution: $x^2 - 4 = (x+2)(x-2)$. Therefore $h(x) = \frac{1}{4}(x+2)^2(x-2)$.

Zeros are -2 of multiplicity 2 and 2 of multiplicity 1.

(b) Zeros divide the x-axis into intervals. Determine a sign of $h(x)$ on each such interval.

Solution:



(c) At each zero determine if the graph of $h(x)$ crosses the x-axis or is tangent to it.

Solution: The graph of $h(x)$ crosses the x-axis at 2 and is tangent to it at -2.

(d) Find the y-intercept of the graph of $h(x)$.

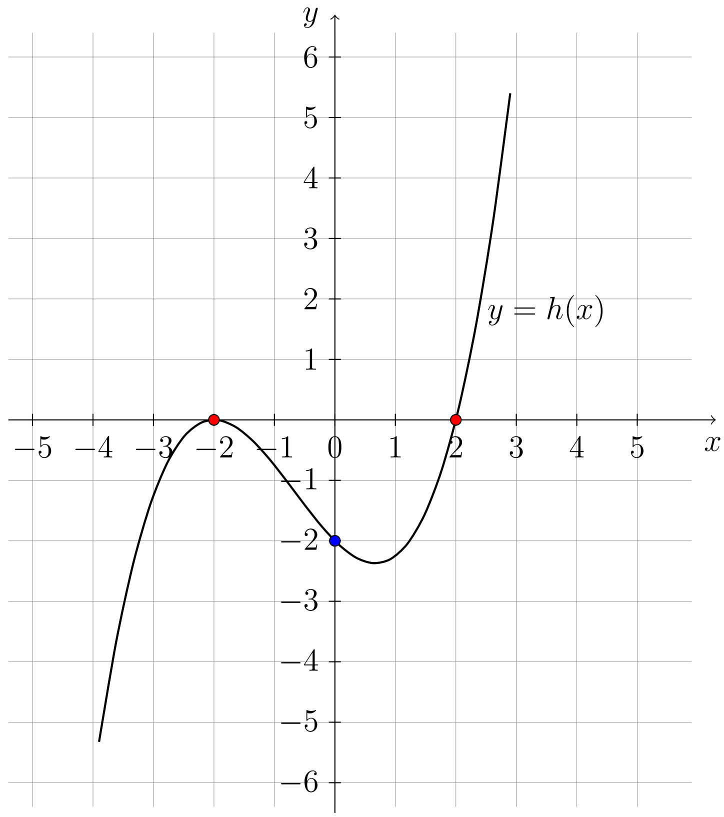
Solution: $h(0) = -2$. Therefore the y-intercept of the graph of $g(x)$ is $(0, 2)$.

(e) Sketch the graph of $h(x)$.

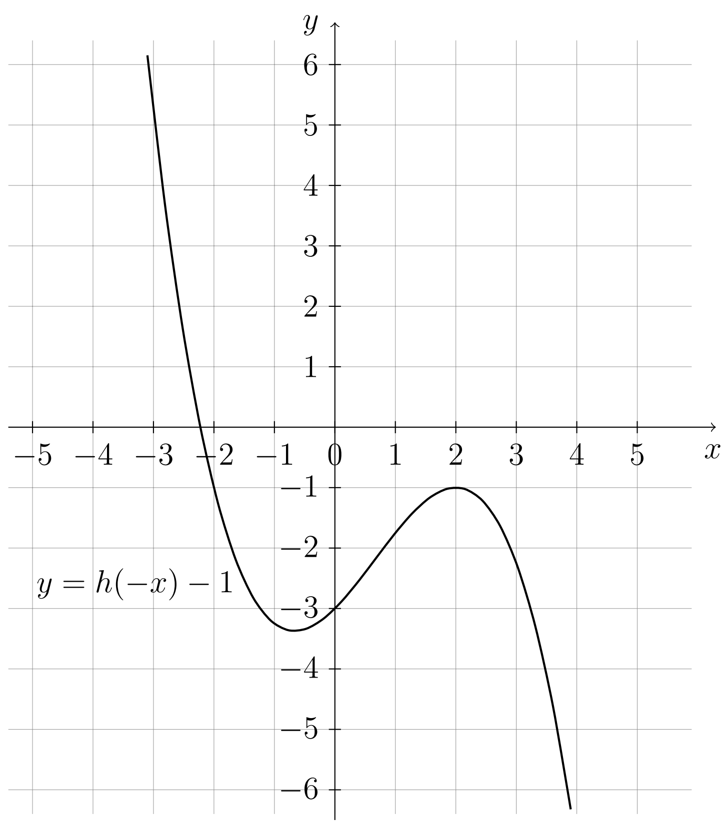
Solution: See next page

(f) Using the graph of $h(x)$ sketch the graph of the function $f(x) = h(-x) - 1$.

Solution: See next page



y
x
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−5 −4 −3 −2 −1 0 1 2 3 4 5
y = h(x)



y
x
6
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4
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2
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0
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−2
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−4
−5
−6
−5 −4 −3 −2 −1 0 1 2 3 4 5
y = h(−x) − 1

8. $g(x) = x^3 - 5x^2 + x - 5$.

(a) By using the rational zeros theorem find all possible rational zeros of $g(x)$.

Solution: possible values for p are $-1,\ 1,\ -5,\ 5$.

possible values for q are $-1,\ 1$.

possible rational zeros are $-1,\ 1,\ -5,\ 5$.

(b) $g(x)$ has only one rational zero. By using synthetic division find it.

Solution:

$$\begin{array}{r|rrr|r} 1 & 1 & -5 & 1 & -5 \\ & & 1 & -4 & -3 \\ \hline & 1 & -4 & -3 & -8 \end{array} \qquad \begin{array}{r|rrr|r} -1 & 1 & -5 & 1 & -5 \\ & & -1 & 6 & -7 \\ \hline & 1 & -6 & 7 & -12 \end{array}$$

$$\begin{array}{r|rrr|r} -5 & 1 & -5 & 1 & -5 \\ & & -5 & 50 & -255 \\ \hline & 1 & -10 & 51 & -260 \end{array} \qquad \begin{array}{r|rrr|r} 5 & 1 & -5 & 1 & -5 \\ & & 5 & 0 & 5 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

Therefore the rational zero is 5.

(c) Find all other zeros and factor $g(x)$.

Solution: $g(x) = (x-5)(x^2+1)$.

The other two zeros can be found from the equation $x^2+1=0$. They are i and $-i$.

Therefore $g(x) = (x-5)(x+i)(x-i)$.

9. Find a polynomial function $f(x)$ of least degree with rational coefficients that has 2 and $\sqrt{5}$ as zeros.

Solution: If $\sqrt{5}$ is a zero then $-\sqrt{5}$ is also a zero.

Therefore $f(x) = (x-2)(x-\sqrt{5})(x+\sqrt{5}) = (x-2)(x^2-5)$.

$f(x) = x^3 - 2x^2 - 5x + 10$.

10. (a) The function $f(x) = 5^{4x-12}$ is one-to-one. Find a formula for the inverse $f^{-1}(x)$.

Solution: 1. We replace $f(x)$ with y: $y = 5^{4x-12}$

2. Interchange x and y: $x = 5^{4y-12}$

3. Solve for y: $\log_5 x = 4y - 12 \quad \Leftrightarrow \quad \log_5 x + 12 = 4y \quad \Leftrightarrow \quad y = \frac{1}{4}\log_5 x + 3$

4. Replace y with $f^{-1}(x)$: $f^{-1}(x) = \frac{1}{4}\log_5 x + 3.$

Answer: $f^{-1}(x) = \frac{1}{4}\log_5 x + 3.$

(b) Find the value of x for which $f^{-1}(x) = 3.5.$

Solution: $\frac{1}{4}\log_5 x + 3 = 3.5 \quad \Leftrightarrow \quad \frac{1}{4}\log_5 x = 0.5 \quad \Leftrightarrow \quad \log_5 x = 2 \quad \Leftrightarrow \quad x = 25.$

Answer: $x = 25.$

Alternative *solution:* $f(3.5) = 5^{4 \cdot 3.5 - 12} = 5^{14-12} = 5^2 = 25.$

11. Simplify

(a) $\dfrac{\ln 9}{\ln 3}$

Solution: Using the Change-of-Base formula: $\dfrac{\ln 9}{\ln 3} = \log_3 9 = \log_3 3^2 = 2.$

(b) $\log_2 \dfrac{\sqrt[4]{x}}{y^2} - \log_2 \dfrac{x^{5/4}}{y^3}$

Solution:

$$\log_2 \frac{\sqrt[4]{x}}{y^2} - \log_2 \frac{x^{5/4}}{y^3} = \log_2\left(\frac{x^{1/4}}{y^2} \div \frac{x^{5/4}}{y^3}\right) = \log_2\left(\frac{x^{1/4}}{y^2} \cdot \frac{y^3}{x^{5/4}}\right)$$

$$= \log_2\left(\frac{y^{3-2}}{x^{5/4-1/4}}\right) = \log_2 \frac{y}{x} = \log_2 y - \log_2 x.$$

12. Solve equations

(a) $3^{2x-5} = 27$

Solution:

$$3^{2x-5} = 27 \quad \Leftrightarrow \quad 3^{2x-5} = 3^3 \quad \Leftrightarrow \quad 2x - 5 = 3$$

$$\Leftrightarrow \quad 2x = 8 \quad \Leftrightarrow \quad x = 4.$$

(b) $\log_2(x-1) + \log_2 x = 1$

Solution: $\log_2(x-1) + \log_2 x = 1 \quad \Rightarrow \quad \log_2(x-1)x = \log_2 2 \quad \Leftrightarrow \quad x^2 - x = 2$

$x^2 - x - 2 = 0 \quad \Leftrightarrow \quad (x+1)(x-2) = 0 \quad \Leftrightarrow \quad x = -1$ and $x = 2$ are possible solutions.

Domain: $x - 1 > 0$ and $x > 0 \quad \Leftrightarrow \quad x \in (1, \infty)$.

Answer: $x = 2$.

13. Magnesium-27 has a half-life of 10 minutes.

(a) What is the exponential decay rate k for Magnesium-27 if time is measured in minutes?

Solution: The exponential decay is given by the formula $P(t) = P_0 e^{-kt}$, where $k > 0$ and t is measured in minutes.

We have $P(10) = \frac{1}{2}P_0$. Then $P_0 e^{-k \cdot 10} = \frac{1}{2}P_0$.

$e^{-10k} = \frac{1}{2}, \quad -10k = \ln\left(\frac{1}{2}\right) = -\ln 2, \quad k = \dfrac{-\ln 2}{-10}, \quad k = \dfrac{\ln 2}{10}.$

(b) A sample contains 100 grams of Magnesium-27. How much of Magnesium-27 is left in the sample after 20 minutes?

Solution: $P(t) = 100e^{-kt} = 100e^{-\ln 2 \cdot t/10}$,

$P(20) = 100e^{-\ln 2 \cdot 20/10} = 100e^{-2\ln 2} = 100e^{\ln(1/4)} = 100 \cdot \frac{1}{4} = 25.$

Answer: 25 grams of Magnesium-27 is left after 20 minutes.

14. One day Mike bought two cups of coffee and two donuts for the cost of \$9. The next day he bought one cup of coffee and six donuts for a total of \$12. What is the cost of one cup of coffee and the cost of one donut?

(a) Write a system of two equations with two unknowns that describes the problem.

Solution: Let x be the cost of a cup of coffee and y be the cost of a donut. Then

$$\begin{array}{rcrcr} 2x & + & 2y & = & 9 \\ x & + & 6y & = & 12 \end{array}$$

(b) Apply Cramer's rule to solve the system. For that calculate determinants of three 2×2 matrices.

Solution: $D = \begin{vmatrix} 2 & 2 \\ 1 & 6 \end{vmatrix} = 2 \cdot 6 - 1 \cdot 2 = 12 - 2 = 10$

$$D_x = \begin{vmatrix} 9 & 2 \\ 12 & 6 \end{vmatrix} = 9 \cdot 6 - 12 \cdot 2 = 54 - 24 = 30$$

$$D_y = \begin{vmatrix} 2 & 9 \\ 1 & 12 \end{vmatrix} = 2 \cdot 12 - 1 \cdot 9 = 24 - 9 = 15$$

$$x = \frac{D_x}{D} = \frac{30}{10} = 3, \quad y = \frac{D_y}{D} = \frac{15}{10} = 1.5$$

Answer: The cost of a cup of coffee is \$3 and the cost of a donut is \$1.5.

15. Graph $3x + 2y > 6$.

Solution: First we graph the related equation $3x + 2y = 6$ using a dashed line.

The test point is $(0, 0)$: $0 > 6$ is false. So we shade the half-plane that does not contain the point $(0, 0)$.