Math 0031 - Final Exam

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For full credit show all work. If you get stuck, write what you think, or what the problem is and you may get partial credit. Each page of the test is worth 10 points.

1. (10 points) Given the points $A = (-3, 7)$ and $B = (-1, -5)$:

(a) Find the midpoint of A and B.

$$\text{midpt}(A,B) = \left(\frac{-3+-1}{2}, \frac{7+-5}{2}\right) = \left(\frac{-4}{2}, \frac{2}{2}\right) = (-2, 1)$$

So the midpt. of the line segment connecting $A(-3,7)$ and $B(-1,-5)$ is $\boxed{(-2,1)}$

(b) Write the equation of the circle passing through the points A and B and centered at their midpoint.

The equation for the circle is $(x-x_0)^2 + (y-y_0)^2 = r^2$, where (x_0, y_0) is the center and r is the radius. Here, $(x_0, y_0) = (-2, 1)$.

Let us find the diameter: $d(A,B) = \sqrt{(-3-(-1))^2 + (7-(-5))^2}$

$= \sqrt{148}$. Then $r = \frac{\sqrt{148}}{2}$

and $r^2 = \frac{148}{4} = 37$.

$$\Rightarrow \boxed{(x+2)^2 + (y-1)^2 = 37}$$

(c) Find the equation of the line passing through the points A and B.

Using point-slope form,

$y - y_0 = m(x - x_0)$,

we have to find the slope $m = \frac{7-(-5)}{-3-(-1)} = \frac{12}{-2} = -6$

For the point, (x_0, y_0), we use $A(-3, 7)$

$$\Rightarrow \boxed{y - 7 = -6(x+3)}$$

2. (10 points) Let $f(x) = \sqrt{x}$, $g(x) = 1 - x$, and $h(x) = x - 2$.

(a) Write

$$k(x) = -2 + \sqrt{1-x},$$

as a composition of $f(x)$, $g(x)$, and $h(x)$.

Consider $(f \circ g)(x) = f(g(x)) = f(1-x) = \sqrt{1-x}$.

Then we need to compose $\sqrt{1-x}$ with $h(x) = x-2$

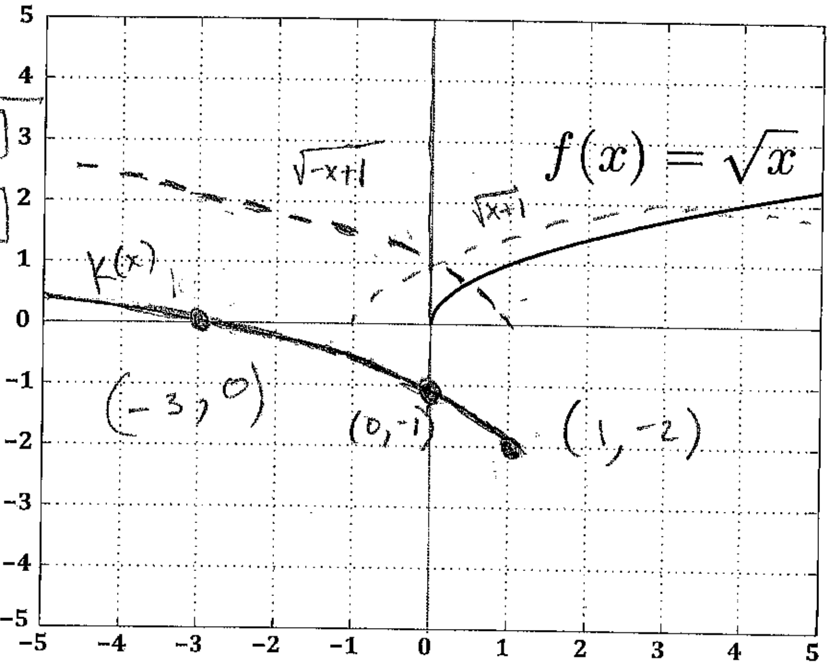
$= -2+x$.

We have $(h \circ f \circ g)(x) = h(f(g(x))) = h(\sqrt{1-x}) = -2+\sqrt{1-x} = k(x)$!!

$\Rightarrow \boxed{(h \circ f \circ g)(x) = k(x)}$

(b) Given the graph of $f(x) = \sqrt{x}$, sketch the graph of $k(x)$.

x	$K(x)$
1	$-2+\sqrt{1-1} = \boxed{-2}$
0	$-2+\sqrt{1-0} = \boxed{1}$
-3	$-2+\sqrt{1-(-3)}$ $= -2+\sqrt{4}$ $= -2+2 = \boxed{0}$



First, consider the domain of $K(x) = -2+\sqrt{1-x}$.

Require $1-x \geq 0 \iff 1 \geq x$; i.e. $D: (-\infty, 1]$.

The first composition, $(f \circ g)(x)$, is a shift (in the x-direction) and reflection of $f(x)=\sqrt{x}$. Indeed, $(f \circ g)(x) = \sqrt{1-x} = \sqrt{(-x)+1}$ $\Rightarrow$ shift left 1-unit and reflect across y-axis

Then $h(\sqrt{1-x})$ shifts the graph down 2-units.

3. (5 points) Let $g(x) = x^2 - 3x + 8$. Find and simplify the difference quotient,

$$\frac{g(x+h) - g(x)}{h}.$$

$g(x+h) = (x+h)^2 - 3(x+h) + 8 = x^2 + 2hx + h^2 - 3x - 3h + 8$

$\Rightarrow \dfrac{g(x+h) - g(x)}{h} = \dfrac{1}{h}\left(x^2 + 2hx + h^2 - 3x - 3h + 8 - (x^2 - 3x + 8)\right)$

$= \dfrac{1}{h}\left(\cancel{x^2} + 2hx + h^2 \cancel{- 3x} - 3h \cancel{+ 8} \cancel{- x^2} \cancel{+ 3x} \cancel{- 8}\right) = \dfrac{1}{h}(2hx - 3h + h^2)$

$= \dfrac{h}{h}(2x - 3 + h)$

$= \boxed{2x - 3 + h}$

4. (5 points)

(a) Solve

$$2t^2 - 3t + 4 = 0.$$

Quadratic Formula: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a = 2$, $b = -3$, $c = 4$

$x_{1,2} = \dfrac{3 \pm \sqrt{9 - 4(2)(4)}}{2(2)} = \dfrac{3 \pm \sqrt{9 - 32}}{4} = \dfrac{3 \pm \sqrt{-23}}{4} = \boxed{\dfrac{3}{4} \pm \dfrac{\sqrt{23}}{4} i}$

(b) Simplify

$$\frac{1-i}{6+2i}.$$

$\dfrac{1-i}{6+2i} = \dfrac{(1-i)(6-2i)}{(6+2i)(6-2i)} = \dfrac{1 - 2i - 6i + 2i^2}{36 + 4} = \dfrac{1 - 8i + 2(-1)}{40}$

$= \dfrac{-1 - 8i}{40} = \dfrac{-1}{40} - \dfrac{8}{40} i = \boxed{\dfrac{-1}{40} - \dfrac{1}{5} i}$

(c) Simplify

$$(3+4i)(2-i).$$

$(3+4i)(2-i) = 6 - 3i + 8i - 4i^2$

$= 6 + 5i - 4(-1)$

$= 6 + 10 + 5i$

$= \boxed{16 + 5i}$

5. (10 points) For the graph of

$$f(x) = -x^2 + 2x + 8 = (-x+4)(x+2)$$

(a) Find the coordinates of the vertex.

$(1, f(1)) = \boxed{(1, 9)}$

$0 = f(x) = (-x+4)(x+2)$

$\Rightarrow$ the zeroes : $x = 4, -2$

(b) Find the equation of the axis of symmetry.

The axis of symmetry is the midpt. of the zeroes $\therefore x = \frac{4+(-2)}{2} = 1$

$\boxed{x = 1}$

(c) What is the minimum or maximum value of f?

Since f is a parabola facing down, we know that the y-coordinate of the vertex is a maximum, i.e. $9 = f(1)$ is the max-value of $f(x)$.

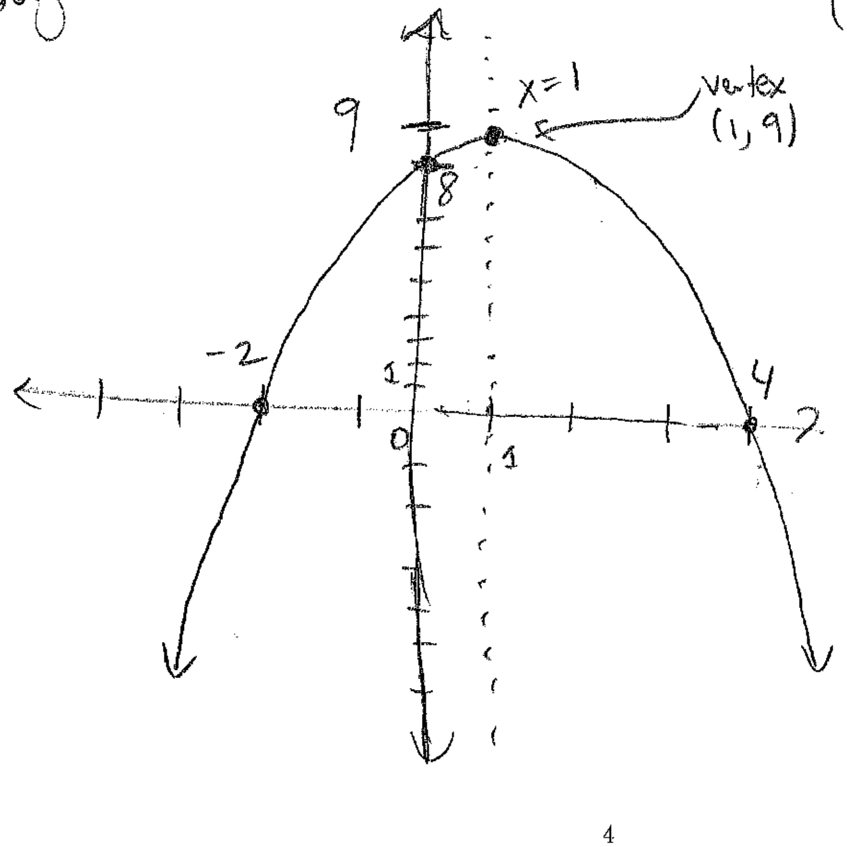
(d) What is the range of f?

(e) Sketch the graph of f.

The range is then

$(-\infty, 9]$.

Rough sketch



y-intercept.

$f(0) = 8$.

6. (10 points) Let $f(x) = x^4 - 2x^3 - 9x^2 + 18x$

(a) Find the leading term and qualitatively describe the end behavior of f.

$\underbrace{LT(f)}_{\text{leading term}} = x^4$, $\underbrace{\deg(f)}_{\text{degree}} = 4$ (even), $\underbrace{LC(f)}_{\text{leading coefficient}} = 1 > 0$ (positive)

qualitative sketch

(b) Use the Intermediate Value Theorem to show f has a zero on the interval $(-1, 1)$. (*no credit will be awarded if the Intermediate Value Theorem is not used*).

$$f(-1) = (-1)^4 - 2(-1)^3 - 9(-1)^2 + 18(-1)$$
$$= 1 + 2 - 9 - 18 = 3 - 27 = -24 < 0$$

$$f(1) = 1^4 - 2\cdot 1^3 - 9\cdot 1^2 + 18\cdot 1 = 1 - 2 - 9 + 18 = 8 > 0$$

Since $f(x)$ is a polynomial and polynomials are continuous, by the Intermediate Value Theorem, $f(-1) < 0$ and $f(1) > 0$ implies there is a point $c \in (-1, 1)$ so that $f(c) = 0$. i.e. there is a zero between in the interval $(-1, 1)$

(c) Verify by substitution that 2 is a zero of $f(x)$ and then find its remaining zeroes.

2	1	-2	-9	18	0
	↓	2	0	-18	0
	1	0	-9	0	$\boxed{0}$ ← $f(2) = 0$.

$$f(x) = (x-2)(x^3 - 9x)$$
$$= x(x-2)(x^2-9)$$
$$= x(x-2)(x-3)(x+3)$$

Zeroes: $x = 0, 2, \pm 3$

(d) Find the remainder for when $f(x)$ is divided by $(x+1)$.

Part (d)

-1	1	-2	-9	18	0
	↓	-1	3	6	-24
	1	-3	-6	24	$\boxed{-24}$ Remainder

$\Rightarrow f(x) = (x+1)(x^3 - 3x^2 - 6x + 24) - 24 \iff \dfrac{f(x)}{x+1} = x^3 - 3x^2 - 6x + 24 + \dfrac{-24}{x+1}$

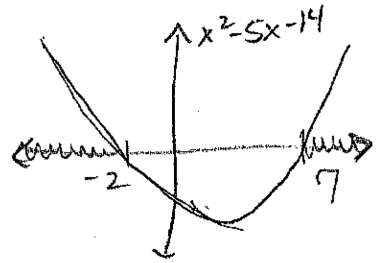
7. (5 points) Solve the inequality: $x^2 + 3x + 1 \geq 8x + 15.$

$-8x-15 \quad -8x-15$

$x^2 - 5x - 14 \geq 0$

$\iff (x-7)(x+2) \geq 0.$ Upward facing parabola w/ roots at $x=-2$, $x=7$.

$(-\infty, -2] \cup [7, \infty)$



8. (5 points) Suppose a 100mg sample of an unknown substance decays radioactively so that after 10 years only 71mg of the sample remain.

(a) What is the growth rate k?

$A(t) = A_0 e^{-kt}$, $\quad 71 = 100e^{-k(10)}$

$A_0 = 100$ mg.

$\Rightarrow .71 = e^{-k10}$

$\Rightarrow \ln(.71) = -k10$

$k = -\frac{1}{10}\ln(.71)$

$\Rightarrow \boxed{-\frac{1}{10}\ln(.71) = k}$

(b) Find an equation which describes the amount $A(t)$ remaining after t years.

$A(t) = 100\, e^{\left[\frac{1}{10}\ln(.71)\right]t}$

(c) How many years will it take the sample to decay to 50mg?

Work.

$50 = 100e^{-kT_H}$

$\Rightarrow \frac{1}{2} = \frac{50}{100} = e^{-kT_H}$

$\Rightarrow \ln(\frac{1}{2}) = -kT_H$

$\Rightarrow -\ln(\frac{1}{2}) = T_H$

Half-life.

$T_H = \frac{\ln(2)}{k} = \boxed{\frac{-\ln(2)}{\frac{1}{10}\ln(.71)}}$

$\iff T_H = \frac{-\ln(\frac{1}{2})}{k} = \frac{\ln(\frac{1}{2})^{-1}}{k} = \boxed{\frac{\ln(2)}{k}}$

9. (10 points) Find the domain and all asymptotes of the rational function $f(x) = \frac{3x^2+5x+1}{x-1}$ then sketch a graph of $f(x)$ below.

domain: $x - 1 = 0 \Rightarrow x = 1 \Rightarrow$ Domain is all real except $x = 1$

$(-\infty, 1) \cup (1, \infty)$.

vertical asymptote:

Since numerator/denominator have no common factors $\Rightarrow$ V.A. @ $x = 1$.

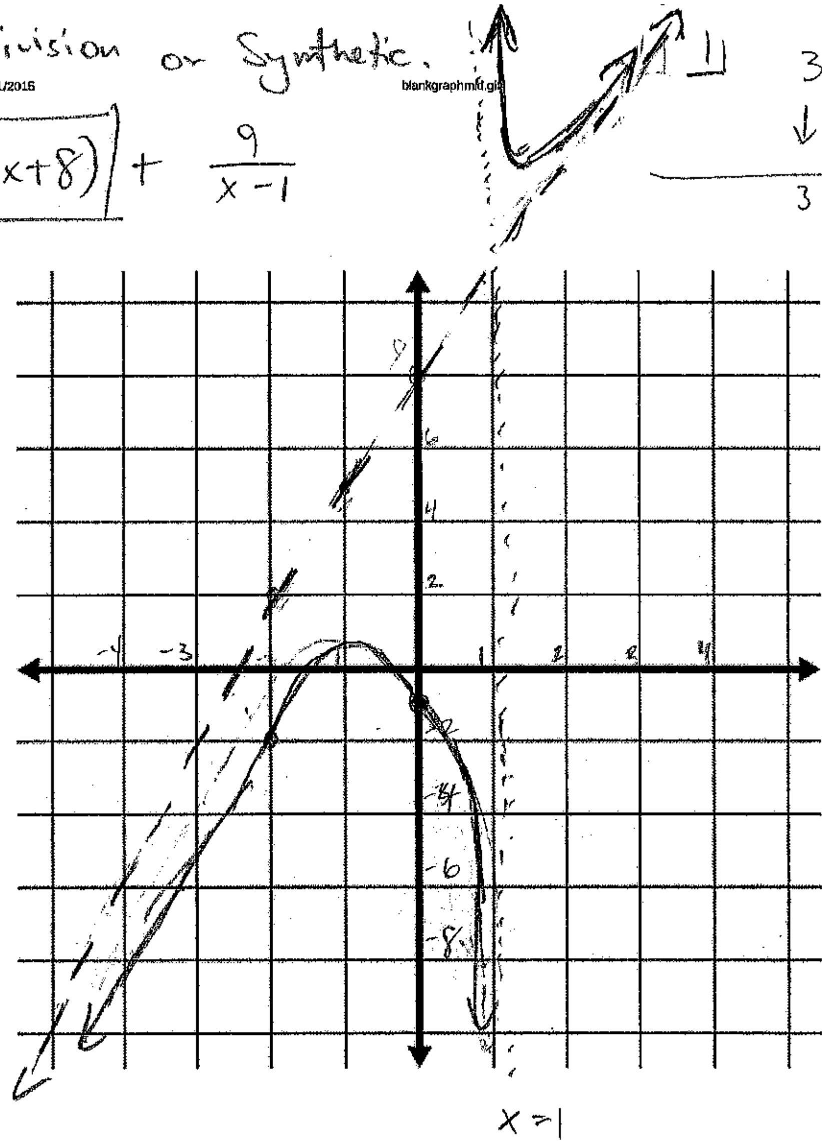
oblique asymptote:

Long-Division or Synthetic.

1	3	5	1
	↓	3	8
	3	8	9

$f(x) = \boxed{(3x+8)} + \frac{9}{x-1}$

oblique Asymptote



$f(2) = (6+8) + 9 = 23$

$f(0) = -1 < 0$

$f(-1) = 5 - \frac{9}{2} = \frac{1}{2} > 0$

$f(-2) = 2 - 3 = -1$

10. (5 points) Let $f(x) = \frac{x+4}{x-3}$.

(a) Show that $f(x)$ is **one-to-one**.

$\boxed{\text{WTS if } f(a) = f(b), \text{ then } a = b.}$

$$\frac{a+4}{a-3} = \frac{b+4}{b-3} \iff (b-3)(a+4) = (b+4)(a-3)$$

$$\iff \cancel{ab} + 4b - 3a - \cancel{12} = \cancel{ab} - 3b + 4a - \cancel{12}$$

$$\iff 7b = 7a \iff \boxed{a = b}.$$

(b) Find the inverse function $f^{-1}(x)$.

$$y = \frac{x+4}{x-3} \qquad x \longleftrightarrow y$$

Solve for y:

$$x = \frac{y+4}{y-3} \iff x(y-3) = y+4$$

$$\iff xy - 3x = y + 4$$

$$\iff xy - y = 3x + 4$$

$$\iff (x-1)y = 3x + 4$$

$$\iff y = \frac{3x+4}{x-1}$$

$$\boxed{f^{-1}(x) = \frac{3x+4}{x-1}}$$

(c) Find the range of f^{-1}.

$$\text{Range}(f^{-1}) = \text{Domain}(f) = \{x \in \mathbb{R} \mid x \neq 3\}$$

(d) The graph of $f^{-1}(x)$ is a reflection of the graph of $f(x)$ across what line?

$$\boxed{y = x}$$

11. (5 points) Solve

(a)

$$4^{3x-5} = 16$$

$$4^{3x-5} = 16 = 4^2 \iff 4^{3x-5} = 4^2$$

$$\iff 3x - 5 = 2$$

$$\iff 3x = 7 \iff \boxed{x = \frac{7}{3}}$$

(b)

$$2\ln x - \ln 5 = \ln(x+10)$$

$$2\ln(x) - \ln(5) = \ln(x)^2 - \ln(5) = \ln\left(\frac{x^2}{5}\right).$$

$$\implies \ln\left(\frac{x^2}{5}\right) = \ln(x+10) \implies \frac{x^2}{5} = x + 10$$

$$\implies \frac{x^2}{5} - x - 10 = 0$$

$$\implies x^2 - 5x - 50 = 0$$

$$(x+5)(x-10) = 0$$

can't plug in negative number to ln(x).

$x = \cancel{-5}, 10$

12. (5 points) Solve

$$\begin{array}{lrrrl} (1) & x & +2y & -z & = 4 \\ (2) & x & & +z & = 3 \\ (3) & & y & -z & = \frac{1}{2}. \end{array}$$

$$X + 2y - z = 4$$
$$-(x \quad + z = 3)$$

$$2y - 2z = 1$$

$\Longrightarrow \ y - z = 1/2$ which is the same as the eqn (3). So we have a line of solutions.

$$\boxed{x = 3 - z \ , \ y = z + \tfrac{1}{2} \ , \ z \in \mathbb{R}.}$$

13. (5 points) Compute the determinant

$$\begin{vmatrix} 2 & -1 & 4 \\ -3 & 1 & -2 \\ 5 & 3 & -1 \end{vmatrix} \ A \qquad \boxed{|A| = -33}$$

Cofactor Expansion Across the 1st row.

$$2A_{11} + (-1)A_{12} + 4A_{13}.$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = -1 - (-2)(3) = \boxed{5}$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} -3 & -2 \\ 5 & -1 \end{vmatrix} = (-1)(3 + 10) = \boxed{-13}$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 1 \\ 5 & 3 \end{vmatrix} = -9 - 5 = \boxed{-14}$$

9

$$\Longrightarrow \det(A) = |A| = 2(5) - 1(-13) + 4(-14) = 10 + 13 - 56 \ \boxed{-33}$$

14. (10 points) For the matrices

2×3 $\quad$ 2×2

$$A = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 1 \\ -2 & 4 \end{bmatrix},$$

either compute the following, or explain why it is not possible.

(a) AB

$(2 \times 3)(2 \times 2)$ $\times$

inner dimensions don't match, i.e. the number of columns of A does <u>not</u> equal the number of rows of B. $AB \Rightarrow$ <u>not possible</u>.

(b) BA

$(\boxed{2} \times \underline{2}) \cdot (\underline{2} \times \boxed{3})$

2×3

$$BA = \begin{pmatrix} -5 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ -2 & 5 & 2 \end{pmatrix} = \begin{pmatrix} -7 & 10 & -13 \\ -10 & 22 & 2 \end{pmatrix}$$

(c) $A + B$

$\times$ not possible. $A^{(2\times 3)}$ and $B^{(2\times 2)}$ must have the same dimensions.

(d) $B + 2I$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$B+2I$

$$= \begin{pmatrix} -5 & 1 \\ 2 & 4 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -5+2 & 1+0 \\ 2+0 & 4+2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -3 & 1 \\ 2 & 6 \end{pmatrix}}$$

(e) B^{-1}

$$\det(B) = -20 - (-2)$$
$$= -20 + 2 = \boxed{-18}.$$

$$\boxed{B^{-1} = \frac{-1}{18}\begin{pmatrix} 4 & -1 \\ 2 & -5 \end{pmatrix}},$$