Talk 3: The τ-fixed point property for left reversible semigroups

In 1965 W. Kirk proved that every Banach space with weak normal structure satisfies the w-FPP for nonexpansive mappings. In a similar way it can be proved that weak* normal structure implies the weak*-FPP in dual Banach spaces.

In the seventies Kirk's result was generalized by T-C- Lim, R. Holmes and A. Lau in the setting of nonexpansive actions of left reversible semigroups, that is, weak normal structure implies the w-FPP for left reversible semigroups. In case of dual Banach spaces, such general statement is still unknown for the weak* normal structure and the weak*-fixed point property for left reversible semigroups.

Particular examples of dual Banach spaces are known to satisfy the weak*-FPP for left reversible semigroups. In 1980 T.C. Lim proved that the sequence space ℓ_1 satisfies the weak*-FPP for left reversible semigroups. In 2010, A.T-M. Lau and P.F. Mah generalized Lim's result by proving that the Fourier-Stieltjes algebra $B(G)$ of a separable compact group verifies the weak*-FPP for left reversible semigroups. In 2010, N. Randrianantoanina proved that the Hardy Banach space and $\mathcal{T}(H)$, the trace class operators on a Hilbert space, also satisfy the weak*-FPP for left reversible semigroups. However, the techniques used in these articles cannot be extended to more general dual Banach spaces, since they are mainly based on the following fact: in the above-mentioned Banach spaces, the asymptotic centers of a weak* compact set with respect to a decreasing net of bounded subsets, are proved to be either norm compact or weakly compact. The w^*-FPP for left reversible semigroups is then derived from this fact and the weak normal structure.

The asymptotic centers of a weak*-compact set are not always norm compact or weakly compact, as we will check with some examples. In this case, new techniques are needed to prove that w^*-FPP for left reversible semigroups.

In this talk we develop new arguments to deduce whether a dual Banach space satisfies the weak*-FPP for left reversible semigroups. More generally, we will consider τ as any translation invariant topology on a separable Banach space X and we give sufficient conditions to assure the τ-FPP for left reversible semigroups. The strict Opial condition and the generalized Gossez-Lami Dozo property will be our main tools. Most of the previous known results will be deduced from ours but we will also achieve new examples of Banach spaces which satisfy the τ-FPP for left reversible semigroups. Here we will consider different types of topologies. Firstly we will regard the weak* topology in Musielak-Orlicz sequence spaces, in some renormings of ℓ_1 and in some other dual Banach spaces non-isomorphic to ℓ_1. We will also consider the topology of the convergence locally in measure in some function spaces, the abstract measure topology in L-embedded Banach spaces and the topology of ρ-almost everywhere convergence in modular function spaces.

Moreover we will extend some known results for nonexpansive mappings to the setting of the fixed point property for left reversible semigroups.

Some open problems:

- Does c_0 have the w-FPP for left reversible semigroups?
- Does there exist any property implying the w-FPP for left reversible semigroups in absence of weak normal structure?