Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 53, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MEAN VALUE PROBLEMS OF FLETT TYPE FOR A VOLTERRA OPERATOR

CEZAR LUPU

Dedicated to the memory of Professor Ion Cucurezeanu

Abstract. In this note we give a generalization of a mean value problem which can be viewed as a problem related to Volterra operators. This problem can be seen as a generalization of a result concerning the zeroes of a Volterra operator in the Banach space of continuous functions with null integral on a compact interval.

1. Introduction

Mean value theorems have always been an important tool in mathematical analysis. It is worth mentioning the pioneering contributions of Fermat, Rolle, Lagrange, Cauchy, Darboux and others. A variation of Lagrange's mean value theorem with a Rolle type condition was given by Flett [5] in 1958 and it was later extended in [17] and generalized in [13, 14].

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$, differentiable on (a, b) and $f'(a) = f'(b)$. Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(c) - f(a)}{c - a}.$$

For instance, Trahan [17] extended Theorem 1.1 by replacing the condition $f'(a) = f'(b)$ with $(f'(a) - m)(f'(b) - m) > 0$, where $m = \frac{f(b)-f(a)}{b-a}$. Moreover, Meyers [11] proved in the same condition, $f'(a) = f'(b)$ that there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(c)}{b - c}.$$

Riedel and Sahoo [16] removed the boundary assumption on the derivatives and proved the following

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable on $[a, b]$. Then, there exists a point $c \in (a, b)$ such that*

$$f(c) - f(a) = (c - a)f'(c) - \frac{1}{2} \cdot \frac{f'(b) - f'(a)}{b - a}(c - a)^2.$$

2000 *Mathematics Subject Classification.* 26A24, 26A36, 28A15.
Key words and phrases. Flett's mean value theorem; Volterra operator; integral equation; differentiable function.

Submitted November 10, 2012. Published February 18, 2013.

Moreover, Theorems 1.1 and 1.2 were used in [3] and [7] in proving Hyers-Ulam stability results for Flett's and Sahoo-Riedel's points. Another results of Flett-type appears also in [15],

Theorem 1.3. *If $f : [a, b] \to \mathbb{R}$ is a twice differentiable function such that $f''(a) = f''(b)$, then there exists $c \in (a, b)$ such that*

$$f(c) - f(a) = (c - a)f'(c) - \frac{(c - a)^2}{2} f''(c).$$

Theorems 1.1 and 1.3 have been generalized by Pawlikowska [14]. Moreover, Molnarova [13] gave a new proof for the generalized Flett mean value theorem of Pawlikowska using only Theorem 1.1. Furthermore, Molnarova establishes a Trahan-type condition in the general case. For more details, see [13]. We also point out other contributions in this direction in [1, 2, 3, 7, 10].

In [6] it is proved that a Volterra operator has at least one zero in the space of functions having the integral equal to zero without assuming differentiability of the function. The same problem is also discussed in [12] in the setting of C^1 positive functions with nonnegative derivative. More exactly, in [12] it is showed that for f, g real-valued continuous functions on $[0, 1]$, the problem

$$\int_0^1 f(x)dx \int_0^c \phi(x)g(x)dx = \int_0^1 g(x)dx \int_0^c \phi(x)f(x)dx,$$

has a solution $c \in (0, 1)$ for a class of weight functions. As stated above, the same problem has been studied in [6] for a "smaller" class of weight functions. Curiously, both ideas of proofs from [6] and [12] can be found in [10, page 6.],

In this paper, assuming differentiability, we prove a more general result from the one provided in [6] and [12] by employing an extension of Flett's theorem (Theorem 1.1). The main result and some consequences are given in the section that follows.

2. Main results

The following lemma is an extension of Theorem 1.1 from the previous section.

Lemma 2.1. *Let $u, v : [a, b] \to \mathbb{R}$ be two differentiable functions on $[a, b]$ with $v'(x) \neq 0$ for all $x \in [a, b]$ and*

$$\frac{u'(a)}{v'(a)} = \frac{u'(b)}{v'(b)}.$$

Then there exists $c \in (a, b)$ such that

$$\frac{u(c) - u(a)}{v(c) - v(a)} = \frac{u'(c)}{v'(c)}.$$

Proof. Define $w : [a, b] \to \mathbb{R}$ by

$$w(x) = \begin{cases} \frac{u(x)-u(a)}{v(x)-v(a)}, & \text{if } x \neq a \\ \frac{u'(a)}{v'(a)}, & \text{if } x = a \end{cases}$$

Clearly, w is continuous on $[a, b]$, and by Weierstrass theorem w attains its bounds. If w does not attain its bounds simultaneously in a and b, then it follows that there exists $x_0 \in (a, b)$ extremum point. By Fermat's theorem, we have $w'(x_0) = 0$; i.e.,

$$\frac{u(x_0) - u(a)}{v(x_0) - v(a)} = \frac{u'(x_0)}{v'(x_0)}.$$

If w attains its bounds in a and b, then we have the following situations:

$$w(a) \le w(x) \le w(b) \tag{2.1}$$

or

$$w(b) \le w(x) \le w(a) \tag{2.2}$$

for all $x \in [a,b]$. We can assume without loss of generality that (2.1) holds. Moreover, possibly replacing u by $-u$ and v by $-v$, we can also admit that $v'(x) > 0$, $x \in [a,b]$. This enables us to establish the following inequality:

$$u(x) \le u(a) + w(b)(v(x) - v(a)),$$

for all $x \in [a,b]$. Now, for all $x \in [a,b]$ we obtain

$$\frac{u(b)-u(x)}{v(b)-v(x)} \ge \frac{u(b)-u(a)-w(b)(v(x)-v(a))}{v(b)-v(x)} = \frac{u(b)-u(a)}{v(b)-v(a)} = w(b).$$

Passing to the limit as x to b, we obtain

$$w(a) = \frac{u'(a)}{v'(a)} = \frac{u'(b)}{v'(b)} = \lim_{x\to b, x<b} \frac{u(b)-u(x)}{v(b)-v(x)} \ge w(b).$$

This implies in view of (2.1) that w is a constant, and therefore $w' = 0$. $\square$

Remark. Since $v'(x) \neq 0$ for all x, v is a diffeomorphism. Let $u(v^{-1}(x)) = U(x), v(a) = A, v(b) = B$. Then we have $U'(x) = u'(v^{-1}(x))/v'(v^{-1}(x))$ so the constraint reads $U'(A) = U'(B)$. Thus, by Flett's mean value theorem (Theorem 1.1),

$$\frac{U(C)-U(A)}{C-A} = U'(C),$$

for some $C \in (A,B)$. But if $C = v^{-1}(c)$,

$$\frac{U(C)-U(A)}{C-A} = \frac{u(c)-u(a)}{v(c)-v(a)} = U'(C) = \frac{u'(c)}{v'(c)}.$$

This remark yields an alternative proof of Lemma 2.1. The Volterra operator is defined for a function $f(t) \in L^2([0,1])$ (the space of Lebesgue square-integrable function on $[0,1]$), and a value $t \in [0,1]$, as

$$V(f)(x) = \int_0^x f(t)dt.$$

It is a well-known that V is a bounded linear operator between Hilbert spaces, with Hermitian adjoint $V^*(f)(x) = \int_x^1 f(t)dt$.

This operator has been studied intensively in the last decades because it is the simplest operator that exhibits a range of phenomena which can arise when one leaves the normal or finite-dimensional cases. Moreover, the Volterra operator is well-known as a quasinilpotent, but not nilpotent operator with no eigenvalues.

For a continuous real-valued function Ψ and $\phi : [0,1] \to \mathbb{R}$ differentiable with $\phi'(x) \neq 0$ for all $x \in (0,1)$. Let $V\Psi$ be the function mapping given by $V\Psi(t) = \int_0^t \Psi(x)dx$ and similarly define

$$V_\phi \Psi(t) = \int_0^t \phi(x)\Psi(x)dx.$$

Let

$$\tilde{C}^1([a,b]) := \{\phi : [a,b] \to \mathbb{R} : \phi \in C^1([a,b]); \phi'(x) \neq 0, x \in [a,b], \phi(a) = 0\}.$$

Denote $C_{\text{null}}([a,b])$ the space of continuous functions having null integral on the interval $[a,b]$.

Now, we are ready to prove the main results of this note.

Theorem 2.2. *Let $f \in C_{\text{null}}([a,b])$ and $g \in C^1([a,b])$, with $g'(x) \neq 0$ for all $x \in [a,b]$. Then there exists $c \in (a,b)$ such that*

$$V_g f(c) = g(a) \cdot V f(c).$$

Proof. The conclusion asks to prove the existence of $c \in (a,b)$ such that

$$\int_a^c f(x)g(x)dx = g(a) \int_a^c f(x)dx.$$

Let us consider the functions $u, v : [a,b] \to \mathbb{R}$ given by

$$u(t) = \int_a^t f(x)g(x)dx - g(t) \int_a^t f(x)dx,$$
$$v(t) = g(t), \quad \text{for all } t \in [a,b].$$

Now, it is easy to see that $u'(t) = -g'(t) \int_a^t f(x)dx$. By the Lemma 2.1, there exists $c \in (a,b)$ such that

$$\frac{u(c) - u(a)}{v(c) - v(a)} = \frac{u'(c)}{v'(c)},$$

which is equivalent to

$$\frac{\int_a^c f(x)g(x)dx - g(c) \int_a^c f(x)dx}{g(c) - g(a)} = \frac{-g'(c) \int_a^c f(x)dx}{g'(c)}$$

which finally reduces to

$$\int_a^c f(x)g(x)dx - g(c) \int_a^c f(x)dx = -g(c) \int_a^c f(x)dx + g(a) \int_a^c f(x)dx,$$

and the conclusion follows. □

Remark. In the same setting as Theorem 2.2 if we apply Meyers [11] mean value theorem we obtain the existence of $c \in (a,b)$ such that

$$(b-c)u'(c) = u(b) - u(c)$$

which is equivalent to

$$-(b-c)g'(c) \int_a^c f(x)dx = \int_a^c f(x)g(x)dx - g(c) \int_a^c f(x)dx,$$
$$(g(c) - (b-c)g'(c)) \int_a^c f(x)dx = \int_a^c f(x)g(x)dx,$$

and this can be rewritten as

$$(g(c) - g'(c)(b-c)) V f(c) = V_g f(c).$$

Corollary 2.3. *If $f \in C_{\text{null}}([a,b])$ and $g \in \tilde{C}^1([a,b])$, then there is $c \in (a,b)$ such that*

$$\int_a^c f(x)g(x)dx = 0.$$

The above corollary is evident from Theorem 2.2.

Theorem 2.4. *If f, g are continuous real-valued functions on $[0,1]$, then there exists $x_0 \in (0,1)$ such that*

$$\begin{aligned} &V_\phi f(x_0)\int_0^1 g(x)dx - V_\phi g(x_0)\int_0^1 f(x)dx \\ &= \phi(0)\cdot\Big(Vf(x_0)\int_0^1 g(x)dx - Vg(x_0)\int_0^1 f(x)dx\Big) \end{aligned}$$

Proof. Let us consider the functions $u, v : [0,1] \to \mathbb{R}$,

$$u(t) = (\phi(t)Vf(t) - V_\phi f(t))\int_0^1 g(x)dx - (\phi(t)Vg(t) - V_\phi g(t))\int_0^1 f(x)dx,$$
$$v(t) = \phi(t).$$

Clearly, these two functions satisfy the conditions from the Lemma 2.1, and thus, there exists $x_0 \in (0,1)$ such that

$$\frac{u(x_0) - u(0)}{v(x_0) - v(0)} = \frac{u'(x_0)}{v'(x_0)},$$

which is equivalent to

$$\begin{aligned} &\int_0^{x_0} \phi(x)f(x)dx\int_0^1 g(x)dx - \int_0^{x_0}\phi(x)g(x)dx\int_0^1 f(x)dx \\ &= \phi(0)\Big(\int_0^{x_0} f(x)dx\int_0^1 g(x)dx - \int_0^{x_0} g(x)\int_0^1 f(x)dx\Big), \end{aligned}$$

which is rewritten as

$$\begin{aligned} &V_\phi f(x_0)\int_0^1 g(x)dx - V_\phi g(x_0)\int_0^1 f(x)dx \\ &= \phi(0)\cdot\Big(Vf(x_0)\int_0^1 g(x)dx - Vg(x_0)\int_0^1 f(x)dx\Big). \end{aligned}$$

□

Corollary 2.5 ([6]). *If $\phi(0) = 0$, then there exists $x_0 \in (0,1)$ such that*

$$\int_0^1 f(x)dx\cdot V_\phi g(x_0) = \int_0^1 g(x)dx\cdot V_\phi f(x_0),$$

The above corollary follows immediately from Theorem 2.4.

Corollary 2.6 ([9]). *If $f, g : [0,1] \to \mathbb{R}$ are two continuous functions, then there exists $x_1 \in (0,1)$ such that*

$$\int_0^1 f(x)dx\int_0^{x_1} xg(x)dx = \int_0^1 g(x)dx\int_0^{x_1} xf(x)dx.$$

The proof of the above corollary follows by applying Corollary 2.5 wiht $\phi(x) = x$.

3. Discussion and some examples

In the proof of Theorem 2.2 we considered the auxiliary function $\varphi : [0,1] \to \mathbb{R}$,

$$\varphi(t) = \int_0^t \phi(x) f(x) dx - \phi(t) \int_0^t f(x) dx.$$

If we apply Theorem 1.1 for $f \in C_{\mathrm{null}}([0,1])$, then there exists $c \in (0,1)$ such that $\varphi(c) - \varphi(0) = c\varphi'(c)$ which is equivalent to

$$\int_0^c \phi(x) f(x) dx = (\phi(c) - c\phi'(c)) \int_0^c f(x) dx.$$

One can remark that this result is completely different from Theorem 2.2. On the other hand, if we consider $f \in C([0,1])$ such that

$$\int_0^1 \phi(x) f(x) dx = \phi(1) \int_0^1 f(x) dx,$$

then by Rolle's theorem there exists $\tilde{c} \in (0,1)$ such that $\varphi'(\tilde{c}) = 0$,

$$\int_0^{\tilde{c}} f(x) dx = 0.$$

Now, we present some examples that follow as consequences from Theorem 2.2 on interval $[0,1]$.

Example 3.1. If we replace functions f, g by their squares in Theorem 2.4, we have the equality

$$\begin{aligned} &\int_0^{x_0} \phi(x) f^2(x) dx \int_0^1 g^2(x) dx - \int_0^{x_0} \phi(x) g^2(x) dx \int_0^1 g^2(x) dx \\ &= \phi(0) \Big(\int_0^{x_0} f^2(x) dx \int_0^1 g^2(x) dx - \int_0^{x_0} g^2(x) dx \int_0^1 f^2(x) dx \Big). \end{aligned}$$

This equality actually says that given any two continuous functions f, g we have

$$\begin{aligned} &\|f\|_{L^2_\phi(0,x_0)} \|g\|_{L^2(0,1)} - \|g\|_{L^2_\phi(0,x_0)} \|f\|_{L^2(0,1)} \\ &= \phi(0) (\|f\|_{L^2(0,x_0)} \|g\|_{L^2(0,1)} - \|g\|_{L^2(0,x_0)} \|f\|_{L^2(0,1)}), \end{aligned}$$

where the quantities are the norms in their respective spaces of (weighted) square integrable functions. Moreover, if $\phi(0) = 0$ we recover [12, Example 3].

Example 3.2. For $i \neq j$ consider $f(x) = P_i(x) P_j(x)$, where P_i, P_j are orthogonal functions on $[0,1]$; i.e.,

$$\int_0^1 P_i(x) P_j(x) dx = 0.$$

Now, by Theorem 2.2, there is a point $c_{ij} \in (0,1)$ such that

$$\int_0^{c_{ij}} P_i(x) P_j(x) \phi(x) dx = \phi(0) \int_0^{c_{ij}} P_i(x) P_j(x) dx.$$

If $\phi(0) = 0$ we obtain $\int_0^{c_{ij}} P_i(x) P_j(x) \phi(x) dx = 0$ which is precisely [12, Example 4].

Acknoledgments. The author is indebted to the department of mathematics of the Politehnica University of Bucharest for hospitality during May-June, 2012, when this work was done.

References

[1] U. Abel, M. Ivan, T. Riedel; The mean value of Flett and divided differences, *J. Math. Anal. Appl.* **295**(2004), 1–9.

[2] D. Cakmak, A. Tiryaki; Mean value theorems for holomorphic functions, *Electronic J. Diff. Eqn.* **2012**(2012), No. 34, 1–6.

[3] M. Das, T. Riedel, P. K. Sahoo; Hyers-Ulam stability of Flett's points, *Appl. Math. Lett.* **16**(2003), 269–271.

[4] J. B. Diaz, R. Vyborny; On some mean value theorems for differential calculus, *Bull. Aus. Math.Soc.* **5**(1971), 227–238.

[5] T. M. Flett; A mean value problem, *Math. Gaz.* **42**(1958), 38–39.

[6] R. Gologan, C. Lupu; An olympiad problem: Zeroes of functions in the image of a Volterra operator, *Gaz. Mat. (A-series)*, **27**(2009), 209-213.

[7] W. Lee, S. Xu, F. Ye; Hyers-Ulam stability of Sahoo-Riedel's point, *Appl. Math. Lett.* **22**(2009), 1649–1652.

[8] I. Jedrzejewska, B. Szkopinska; On generalizations of Flett's Theorem, *Real Anal. Exchange*, **30**(2004), 75–86.

[9] C. Lupu, T. Lupu, Problem 11290, *Amer. Math. Mon.*, **114**(2007), 359.

[10] C. Lupu, T. Lupu; Mean value theorems for some linear integral operators, *Electron. J. Diff. Eqn.*, **2009**(2009), no. 117, 1-15.

[11] R. E. Meyers; Some elementary results related to the mean value theorem, *The two-year college Mathematics Journal* **8**(1977), 51–53.

[12] A. B. Mingarelli, J. M. Pacheco, A. Plaza; A mean value property for pair of integrals, *Acta Math. Univ. Comenianae*, **87**(2009), 65-70.

[13] J. Molnarova; On generalized Flett's mean value theorem, *preprint* at http://arxiv.org/PS_cache/arxiv/pdf/1107/1107.3264v1.pdf

[14] I. Pawlikowska; An extension of a theorem of Flett, *Demonstratio Math.*, **32** (1999), 281-286.

[15] T.L. Rădulescu, V.D. Rădulescu, T. Andreescu; *Problems in Real Analysis: advanced calculus on the real axis*, Springer Verlag, 2009.

[16] P.K. Sahoo, T. Riedel; *Mean Value Theorems and Functional Equations*, World Scientific, River Edge, NJ, 1998.

[17] D. H . Trahan; A new type of mean value theorem, *Math. Mag.*, **39**(1966), 264–268.

[18] R. Witula, E. Hetmaniok, D. Slota; A stronger version of the second mean value theorem for integrals, *Computer Math. Appl.*, in press at www.doi:10.1016/j.camwa.2012.01.008

Cezar Lupu

University of Pittsburgh, Department of Mathematics, Pittsburgh, PA 15260, USA.

University of Craiova, Department of Mathematics, Str. Alexandru Ioan Cuza 13, RO – 200585, Craiova, Romania

E-mail address: `cel47@pitt.edu, lupucezar@gmail.com`