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Unconditional convergence of high-order extrapolations of the Crank-Nicolson, finite element method for the Navier-Stokes equations

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Abstract Error estimates for the Crank-Nicolson in time, finite element in space (CN-FE) discretization of the Navier-Stokes equations require a discrete version of the Gronwall inequality, which leads to a time-step restriction. We prove herein that *no* restriction on the time-step is necessary for a linear, fully implicit variation of CN-FE obtained by extrapolation of the convecting velocity. Previous convergence analyses of CN-FE with linear extrapolation rely on a similar time-step restriction as the full CN-FE. We show: *CN-FE with linear extrapolation is unconditionally convergent in the energy norm.* We also show optimal convergence of CN-FE with extrapolation in a discrete $L^\infty(H^1)$-norm and convergence of the corresponding discrete time derivative in a discrete $L^2(L^2)$-norm.

1 Introduction

The usual Crank-Nicolson (in time) finite element (in space) (CN-FE) discretization of the Navier-Stokes equations is well-known to be unconditionally and nonlinearly stable.

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The error analysis of the CN-FE method is based on a discrete Gronwall inequality which introduces a time-step restriction (for convergence, not for stability) of the form

$$\Delta t \leq \mathcal{O}(Re^{-5/3}h^{2/3}), \qquad \text{or} \qquad \Delta t \leq \mathcal{O}(Re^{-3}) \tag{1}$$

(implicitly reported for $W^{1,\infty}$-solutions in [13], see Appendix A). Here $h > 0$ is the mesh width, $\Delta t > 0$ is the time-step size, and $Re > 0$ is the Reynolds number. Condition (1)(a) implies *conditional convergence* whereas (1)(b) is a *robustness condition* and both are prohibitively restrictive in practice; for example, (1)(b) suggests

$$Re = 100 \text{ (low-to-moderate value)} \quad \Rightarrow \quad \Delta t \leq \mathcal{O}(10^{-6}).$$

Consequently, an important open question regards whether condition (1) is

– an artifact of imperfect mathematical technique, or

– a special feature of the CN time discretization.

We consider the necessity of a time-step restriction in a linear, fully implicit variant of CN-FE obtained by extrapolation of the convecting velocity u: for example,

$$\mathbf{u} \cdot \nabla \mathbf{u} \approx (\frac{3}{2}\mathbf{u}^{n-1} - \frac{1}{2}\mathbf{u}^{n-2}) \cdot \nabla \frac{\mathbf{u}^n + \mathbf{u}^{n-1}}{2}, \qquad \mathbf{u}^i := \mathbf{u}(x, t_i). \tag{2}$$

This method is often called CNLE and was first studied by Baker [2]. CNLE is linearly implicit, unconditionally and nonlinearly stable, and second order accurate. In this report, we show that *no time-step restriction* is required for the convergence of CNLE (Theorem 1). Additionally, the error in the energy norm satisfies

$$\text{error } \leq \mathcal{O}(h^k + \Delta t^2), \qquad k = \text{ degree of FE-space}$$

(Corollary 1). Our analysis depends on the *extrapolated* convecting velocity in (2), careful majorization of associated bi- and trilinear forms, and application of a particular discrete Gronwall inequality. The key difference between our convergence proof for

CNLE and that of CN-FE is the resulting intermediate estimate: for approximations $\mathbf{U}_h^n$ and constants $\kappa_n > 0$,

$$\text{CN-FE} \Rightarrow ||\mathbf{U}_h^N||^2 + \ldots \leq \sum_{n=0}^{N-1} \kappa_n ||\mathbf{U}_h^{n+1}||^2 + \ldots \tag{3}$$

$$\text{CNLE } \Rightarrow ||\mathbf{U}_h^N||^2 + \ldots \leq \sum_{n=0}^{N-1} \kappa_n ||\mathbf{U}_h^{n}||^2 + \ldots \tag{4}$$

Notice that the term $||\mathbf{U}_h^N||^2$ is included in the right-hand-side of (3), but not of (4). Estimates of the form (3) require a discrete Gronwall inequality (Lemma 1) to proceed, which is the source of a time-step restriction. Conversely, estimates of the form (4) allow application of an alternate discrete Gronwall inequality (Lemma 2), which does not require a time-step restriction.

We also prove convergence estimates in other norms. Under a modest time step restriction

$$\Delta t \leq h^{1/4}, \quad \text{no } Re\text{-dependence} \tag{5}$$

the CNLE velocity approximation converges optimally in a discrete $L^\infty(H^1)$-norm and the corresponding discrete derivative of the velocity approximation converges optimally in a discrete $L^2(L^2)$-norm (Theorem 2 and Corollary 2). The restriction (5) is not a typical artifact of the discrete Gronwall inequality in that it does not depend on *Re* or other problem data. Correspondingly, (5) is much less restrictive than (1). The error estimate is obtained through a bootstrap argument that utilizes the error in the energy norm (Theorem 1 and Corollary 1).

The report is organized as follows: the continuum problem is described in Section 2. The CNLE approximation is introduced in Section 2.1. The main results and their proofs are provided in Section 3.

1.1 Importance of Crank-Nicolson schemes

There are many analyses of Crank-Nicolson time-stepping methods for the Navier-Stokes equations. Heywood and Rannacher [13] provide analysis of CN-FE. The 2nd and 3rd order CNLE methods are introduced and analyzed in [2], [3]. Multilevel methods based on CNLE (building on the work in [20] and [7]) are analyzed in [11], [15]. CNLE approximation of a stochastic Navier-Stokes equation is analyzed in [4]. The authors in [19] analyze a stabilized CNLE method. Each of these analyses requires, explicitly stated or implicitly, a time-step restriction of the form (1) to guarantee convergence. A 1st order CNLE is used in [17] in conjunction with a coupled multigrid and pressure Schur complement schemes for the Navier-Stokes equations. Numerical comparison of various Navier-Stokes time-stepping schemes (excluding CNLE) are provided in [18].

A Crank-Nicolson/Adams-Bashforth (CN-AB) time-stepping, scheme is another linear variant of CN-FE. Unlike CNLE, CN-AB is explicit in the nonlinearity and only *conditionally* stable [10] (i.e. a time-step restriction of form (1)(a) is required for *stability*). CN-AB is a popular method for approximating Navier-Stokes flows because it is fast and easy to implement. For example, it is used to model turbulent flows induced by wind turbine motion [25], turbulent flows transporting particles in [22], and reacting flows in complex geometries (e.g. gas turbine combustors) [1].

The CN method is also applied, for example, to a general class of non-stationary partial differential equations encompassing reaction-diffusion type equations and the Kuramoto-Tsuzuki equation in [**?**], the nonlinear Sobolev equations [23], and the Ginzburg-Landau model [16]. Time-step restrictions of type (1)(b) (where Re has a different meaning) are implicitly required in the convergence analyses of these discrete models.

2 Problem formulation

Let Ω be an open, regular, polygonal domain in $\mathbb{R}^d$ ($d = 2$ or 3). For time $T > 0$, kinematical viscosity $\nu > 0$, and body force $\mathbf{f}$, we consider the Navier-Stokes equation: Find $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, and $p : \Omega \times [0,T] \to \mathbb{R}$ satisfying

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{f} + \nu \Delta \mathbf{u} - \nabla p, \qquad \nabla \cdot \mathbf{u} = 0, \qquad \text{in } \Omega \times (0,T] \tag{6}$$

subject to boundary and initial conditions

$$\begin{aligned} &\mathbf{u}(\mathbf{x},t) = 0, \qquad \forall (\mathbf{x},t) \in \partial\Omega \times (0,T] \\ &\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \, \forall \mathbf{x} \in \Omega \end{aligned} \tag{7}$$

where $\mathbf{u}_0 \in H_0^1(\Omega)^d$ and $\nabla \cdot \mathbf{u}_0 = 0$. Without loss of generality, suppose that $\nu \leq 1$.

The notation used for Sobolev spaces and norms is standard. We denote by $(\cdot,\cdot)$, $||\cdot||$ the $L^2(\Omega)$-inner product and norm. For $k \in \mathbb{R}$ and $1 \leq p \leq \infty$, let $||\cdot||_{k,p} = ||\cdot||_{W^{k,p}(\Omega)}$ be the $W^{k,p}(\Omega)$-Sobolev norm. Identify $H^k(\Omega) = W^{k,2}(\Omega)$ and write $||\cdot||_k = ||\cdot||_{W^{k,2}(\Omega)}$. Let $||\mathbf{u}||_{W^{m,q}(0,T,W^{k,p}(\Omega))}$ denote the $L^q(0,T)$-norm in time of $||\mathbf{u}^{(m)}(t)||_{W^{k,p}(\Omega)}$. Write $W^{m,q}(0,T;W^{k,p}(\Omega)) = W^{m,q}(W^{k,p})$. For example,

$$||\mathbf{u}||_{L^2(W^{k,p})} := \left(\int_0^T ||\mathbf{u}(t)||_{k,p}^2 dt \right)^{1/2}.$$

Write $C^0(W^{k,p}) = C^0([0,T], W^{k,p}(\Omega))$. Lastly, let the context determine whether $W^{k,p}(\Omega)$ denotes a scalar, vector, or tensor function space. For example let $\mathbf{v} : \Omega \to \mathbb{R}^d$. Then, $\mathbf{v} \in H^1(\Omega)$ implies that $\mathbf{v} \in H^1(\Omega)^d$ and $\nabla \mathbf{v} \in H^1(\Omega)$ implies that $\nabla \mathbf{v} \in H^1(\Omega)^{d \times d}$.

Let $H_0^1(\Omega) := \left\{ \mathbf{v} \in H^1(\Omega) : \mathbf{v}|_{\partial\Omega} = 0 \right\}$ and $L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}$. A weak formulation of (6), (7) is: Find $\mathbf{u} : [0,T] \to H_0^1(\Omega)$ and $p : [0,T] \to L_0^2(\Omega)$ for each $t \in (0, T]$ satisfying

$$\int_\Omega \mathbf{u}_t \cdot \mathbf{v} + \int_\Omega \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{v} \\ + \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega) \tag{8}$$

$$\int_\Omega q \nabla \cdot \mathbf{u} = 0, \quad \forall q \in L_0^2(\Omega) \tag{9}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Omega. \tag{10}$$

Let $V := \left\{ \mathbf{v} \in H_0^1(\Omega) : \nabla \cdot \mathbf{v} = 0 \right\}$. Restricting test functions $\mathbf{v} \in V$ reduces (8), (9), (10) to: find $\mathbf{u} : [0, T] \to V$ satisfying (10) and

$$\int_\Omega \mathbf{u}_t \cdot v + \int_\Omega \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{v} + \nu \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in V.$$

2.1 The discrete formulation

Let τ_h be a uniformly regular triangulation (see [8] for a precise definition) of Ω with $E \in \tau_h$ (e.g. triangles for $d = 2$ or tetrahedra for $d = 3$). Set $h = \sup_{E \in \tau_h} \{\text{diameter}(E)\}$. Let $X^h \subset H_0^1(\Omega)^d$ and $Q^h \subset L_0^2(\Omega)$ be a conforming velocity-pressure mixed finite element space. We assume that $X^h \times Q^h$ satisfy the following:

- There exists $C > 0$ such that

$$\inf_{q \in Q^h} \sup_{\mathbf{v} \in X^h} \frac{(q, \nabla \cdot \mathbf{v})}{||\nabla \mathbf{v}|| ||q||} \geq C > 0. \tag{11}$$

- If $\mathbf{u}$, p satisfy Assumption 1 for some fixed $s, k \geq 0$, there exists $C > 0$ such that

$$\begin{aligned} \inf_{\mathbf{v}_h \in X^h} ||\mathbf{u} - \mathbf{v}_h|| + h \inf_{\mathbf{v}_h \in X^h} ||\mathbf{u} - \mathbf{v}_h||_1 &\leq Ch^{k+1} ||\mathbf{u}||_{k+1} \\ \inf_{\mathbf{v}_h \in X^h} ||\partial_t (\mathbf{u} - \mathbf{v}_h)|| &\leq Ch^{k+1} ||\mathbf{u}_t||_{k+1} \\ \inf_{q_h \in Q^h} ||p - q_h|| &\leq Ch^{s+1} ||p||_{s+1}. \end{aligned} \tag{12}$$

Assumption 1 $\mathbf{u} \in L^2(H^{k+1})$, $\mathbf{u}_t \in L^2(H^{k+1})$ and $p \in L^2(H^{s+1})$ for some $k \geq 0$, $s \geq 0$.

– There exists a $C > 0$ such that

$$||\nabla \mathbf{v}_h|| \leq Ch^{-1}||\mathbf{v}_h||, \qquad \forall \mathbf{v}_h \in X^h. \tag{13}$$

Let $V^h = \left\{ \mathbf{v} \in X^h : \int_\Omega q \nabla \cdot \mathbf{v} = 0 \quad \forall q \in Q^h \right\}$. Note that in general $V^h \not\subset V$.

We use the explicitly skew-symmetric convective term:

$$c^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}\left((\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) \right). \tag{14}$$

Replacing $(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})$ with (14) ensures stability, but may have adverse affects on the accuracy of the approximation, see [14]. An equivalent formulation of (14) required in the error analysis of Section 3 is

$$c^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w}) = -(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) + \frac{1}{2}((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w}).$$

Let $0 = t_0 < t_1 < \ldots < t_N = T < \infty$ be a discretization of the time interval $[0, T]$ for a constant time step $\Delta t = t_n - t_{n-1}$. Write $z^n = z(t_n)$ and $z^{n+1/2} = \frac{1}{2}(z(t_{n+1}) + z(t_n))$. Consider the linearization: for some integer $n_0 \geq 0$,

$$c^h(\mathbf{u}^{n+1}, \mathbf{v}, \mathbf{w}) \approx c^h(\xi_n(\mathbf{u}), \mathbf{v}, \mathbf{w}), \qquad \xi_n(\mathbf{u}) := a_0 \mathbf{u}_n + \ldots + a_{n_0} \mathbf{u}_{n-n_0}. \tag{15}$$

For example,

$$\begin{aligned}
\xi_n(\mathbf{u}) &= \mathbf{u}^n && \Rightarrow \xi_n(\mathbf{u}) = \mathbf{u}^{n+1} + \mathcal{O}(\Delta t) \\
\xi_n(\mathbf{u}) &= \tfrac{1}{2}(3\mathbf{u}^n - \mathbf{u}^{n-1}) && \Rightarrow \xi_n(\mathbf{u}) = \mathbf{u}^{n+1} + \mathcal{O}(\Delta t^2) \\
\xi_n(\mathbf{u}) &= 3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2} && \Rightarrow \xi_n(\mathbf{u}) = \mathbf{u}^{n+1} + \mathcal{O}(\Delta t^3).
\end{aligned}$$

In Corollaries 1, 2 we consider extrapolations satisfying $\xi_n(\mathbf{u}) = \mathbf{u}^{n+1} + \mathcal{O}(\Delta t^2)$ to preserve the second order accuracy guaranteed by the full CN method.

Problem 1 (CNLE) Let $\mathbf{u}_h^i \in V^h$ be a good approximation of $\mathbf{u}^i$ for each $i = 0, 1, \ldots, n_0$. For each $n = n_0, n_0+1, \ldots, N-1$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X^h \times Q^h$ satisfying

$$\begin{aligned}
&(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}) + c^h(\xi_n(\mathbf{u}_h), \mathbf{u}_h^{n+1/2}, \mathbf{v}) \\
&\quad + \nu(\nabla \mathbf{u}_h^{n+1/2}, \nabla v) - (p_h^{n+1/2}, \nabla \cdot \mathbf{v}) = (\mathbf{f}^{n+1/2}, \mathbf{v}), \quad \forall v \in X^h \qquad (16)
\end{aligned}$$

$$(\nabla \cdot \mathbf{u}_h^{n+1}, q) = 0, \quad \forall q \in Q^h. \qquad (17)$$

Remark 1 Note that $\xi_n(\mathbf{u}_h) = \mathbf{u}_h^{n+1/2}$ defines the CN-FE method analyzed in [13] and $\xi_n(\mathbf{u}_h) = \frac{1}{2}(3\mathbf{u}^n - \mathbf{u}^{n-1})$ defines the CNLE method of [2], [9], [19].

2.2 Fundamentals of estimation

The discrete Gronwall inequality is essential to the analysis in Section 3.

Lemma 1 (Gronwall - time step restriction) *Let $D \geq 0$ and κ_n, A_n, B_n, $C_n \geq 0$ for any integer $n \geq 0$ and satisfy*

$$A_N + \Delta t \sum_{n=0}^{N} B_n \leq \Delta t \sum_{n=0}^{N} \kappa_n A_n + \Delta t \sum_{n=0}^{N} C_n + D, \quad \forall N \geq 0.$$

Suppose that for all n

$$\Delta t \kappa_n < 1 \qquad (18)$$

and set $g_n = (1 - \Delta t \kappa_n)^{-1}$. Then,

$$A_N + \Delta t \sum_{n=0}^{N} B_n \leq \exp\left(\Delta t \sum_{n=0}^{N} g_n \kappa_n\right) \left[\Delta t \sum_{n=0}^{N} C_n + D\right], \quad \forall N \geq 0.$$

Lemma 2 (Gronwall - no time step restriction) *Let $D \geq 0$ and κ_n, A_n, B_n, $C_n \geq 0$ for any integer $n \geq 0$ and satisfy*

$$A_N + \Delta t \sum_{n=0}^{N} B_n \leq \Delta t \sum_{n=0}^{N-1} \kappa_n A_n + \Delta t \sum_{n=0}^{N} C_n + D, \quad \forall N \geq 0.$$

Then

$$A_N + \Delta t \sum_{n=0}^{N} B_n \leq \exp\left(\Delta t \sum_{n=0}^{N-1} \kappa_n\right) \left[\Delta t \sum_{n=0}^{N} C_n + D\right], \quad \forall N \geq 0.$$

Proof (Lemmas 1, 2) See pp. 369-370 in [13].

Denote by $C > 0$ a generic constant independent of h, Δt, and ν. We collect some estimates from [5] and [8] used in Section 3. For $a, b > 0$ and any $1/p + 1/q = 1$ for $1 \leq p, q \leq \infty$ and any $\varepsilon > 0$

$$ab \leq \frac{1}{p\varepsilon^{p/q}} a^p + \frac{\varepsilon}{q} b^q, \quad \text{Young's inequality.} \tag{19}$$

For $1/p + 1/q + 1/r = 1$, $1 \leq p, q, r \leq \infty$, and $\mathbf{u} \in L^p(\Omega)$, $\nabla \mathbf{v} \in L^q(\Omega)$, $\mathbf{w} \in L^r(\Omega)$

$$(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) \leq ||\mathbf{u}||_{0,p} ||\nabla \mathbf{v}||_{0,q} ||\mathbf{w}||_{0,r}, \quad \text{Hölder's inequality.} \tag{20}$$

For $\mathbf{v} \in H_0^1(\Omega)$,

$$\begin{aligned}
&||\mathbf{v}|| \leq C||\nabla \mathbf{v}||, && \text{Poincaré's inequality} \\
&||\mathbf{v}||_{0,3} \leq C||\mathbf{v}||^{1/2} ||\nabla \mathbf{v}||^{1/2}, && \text{Ladyzhenskaya's inequality, } d = 3 \\
&||\mathbf{v}||_{0,4} \leq C||\mathbf{v}||^{1/2} ||\nabla \mathbf{v}||^{1/2}, && \text{Ladyzhenskaya's inequality, } d = 2 \\
&||\mathbf{v}||_{0,6} \leq C||\nabla \mathbf{v}||, && \text{Ladyzhenskaya's inequality, } d = 2, 3.
\end{aligned} \tag{21}$$

If $\mathbf{u} \in H^2(\Omega)$, then $\mathbf{u} \in L^\infty(\Omega) \cap C^0(0, T)$ and

$$||\mathbf{u}||_{0,\infty} \leq C||\mathbf{u}||_2. \tag{22}$$

The following estimates of the convective term are direct results of the previous inequalities. See [21] for a comprehensive compilation of associated estimates. For $\mathbf{u} \in V$ and $\mathbf{v}, \mathbf{w} \in H^1(\Omega)$,

$$c^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = -(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}), \qquad (\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \tag{23}$$

If, on the other hand, $\mathbf{u} \in H_0^1(\Omega)$,

$$\begin{aligned} c^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) &\leq C\sqrt{||\mathbf{u}||||\nabla \mathbf{u}||}||\nabla \mathbf{v}||||\nabla \mathbf{w}||, \\ c^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) &\leq C||\nabla \mathbf{u}||||\nabla \mathbf{v}||\sqrt{||\mathbf{w}||||\nabla \mathbf{w}||}. \end{aligned} \tag{24}$$

Moreover, if $\mathbf{v} \in H^2(\Omega)$, then

$$c^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C||\mathbf{u}||||\mathbf{v}||_2||\nabla \mathbf{w}||, \quad c^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C||\nabla \mathbf{u}||||\mathbf{v}||_2||\mathbf{w}||. \tag{25}$$

2.3 Fundamentals of approximation

The elliptic projection is used in the error analysis of Section 3. It is given by $P_1 : V \to V^h$ so that $\tilde{\mathbf{v}}_h := P_1(\mathbf{u})$ satisifies

$$\int_\Omega \nabla(\mathbf{u} - \tilde{\mathbf{v}}_h) : \nabla v = 0, \qquad \forall \mathbf{v} \in X^h. \tag{26}$$

The following is a well-known property of P_1.

Lemma 3 $\tilde{\mathbf{v}}_h = P_1(\mathbf{u})$ *defined in (26) is well-defined. Moreover,*

$$||\nabla(\mathbf{u} - \tilde{\mathbf{v}}_h)|| \leq C \inf_{\mathbf{v}_h \in X^h} ||\nabla(\mathbf{u} - \mathbf{v}_h)||. \tag{27}$$

Proof See e.g. [8].

One can show through a duality argument (see e.g. p. 97 in [24]) that

$$||\mathbf{u} - \tilde{\mathbf{v}}_h||_{-1} + h||\mathbf{u} - \tilde{\mathbf{v}}_h|| \leq Ch^2 \inf_{\mathbf{v}_h \in X^h} ||\nabla(\mathbf{u} - \mathbf{v}_h)||^2. \tag{28}$$

The estimates in (29), (30), (31) are used in Corollaries 1, 2: For any $n = 0, 1, \ldots, N-1$,

$$||\frac{z^{n+1} - z(t_n)}{\Delta t}||^2 \leq C\Delta t^{-1} \int_{t_n}^{t_{n+1}} ||z_t(t)||^2 dt \tag{29}$$

$$||z^{n+1/2} - z(t_{n+1/2})||_k^2 \leq C\Delta t^3 \int_{t_n}^{t_{n+1}} ||z_{tt}(t)||_k^2 dt \tag{30}$$

$$||\frac{1}{\Delta t}(z^{n+1} - z^n) - z_t(t_{n+1/2})||^2 \leq C\Delta t^3 \int_{t_n}^{t_{n+1}} ||z_{ttt}(t)||^2 dt. \tag{31}$$

where $z \in H^1(L^2)$, $z \in H^2(H^k)$, and $z \in H^3(L^2)$ is required respectivley. Each estimate (29), (30), (31) is a result of a Taylor expansion with integral remainder. The estimate (29) is used in the analysis of Section 3.

3 Unconditional convergence of CNLE

We first construct the error equation and then state the main results in Theorems 1, 2 and Corollaries 1, 2. The proofs are contained in the proceeding subsections. The consistency error for the time-discretization is given by, for any $\mathbf{v} \in H_0^1(\Omega)$,

$$\begin{aligned}
\tau_n(\mathbf{v}) := &\int_\Omega \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t_{n+1/2}) \right) \cdot \mathbf{v} - \int_\Omega (p^{n+1/2} - p(t_{n+1/2})) \nabla \cdot \mathbf{v} \\
&+ c^h(\xi_n(\mathbf{u}), \mathbf{u}^{n+1/2}, \mathbf{v}) - c^h(\mathbf{u}(t_{n+1/2}), \mathbf{u}(t_{n+1/2}), \mathbf{v}) \\
&+ \nu \int_\Omega \nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2})) : \nabla \mathbf{v} + \int_\Omega (\mathbf{f}(t_{n+1/2}) - \mathbf{f}^{n+1/2}) \cdot \mathbf{v}.
\end{aligned} \tag{32}$$

Using (32), rewrite (11) in a form conducive to analyzing the error between the continuous and discrete models:

$$\begin{aligned}
&\int_\Omega \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \cdot \mathbf{v} + c^h(\xi_n(\mathbf{u}), \mathbf{u}^{n+1/2}, \mathbf{v}) - \int_\Omega p^{n+1/2} \nabla \cdot \mathbf{v} \\
&\quad + \nu \int_\Omega \nabla \mathbf{u}^{n+1/2} : \nabla \mathbf{v} = \int_\Omega \mathbf{f}^{n+1/2} \cdot \mathbf{v} + \tau_n(\mathbf{v}), \quad \forall \mathbf{v} \in H_0^1.
\end{aligned} \tag{33}$$

Decompose the velocity error

$$\mathbf{E}_u^n = \mathbf{u}_h^n - \mathbf{u}^n = \mathbf{U}_h^n - \boldsymbol{\eta}^n, \qquad \mathbf{U}_h^n = \mathbf{u}_h^n - \tilde{\mathbf{v}}_h^n, \qquad \boldsymbol{\eta}^n = \mathbf{u}^n - \tilde{\mathbf{v}}_h^n.$$

Fix $\tilde{q}_h^n \in Q^h$. Note that $(p_h, \nabla \cdot v) = 0$ for any $\mathbf{v} \in V^h$. Subtract (33) from (16) to get the error equation

$$\begin{aligned}
&\int_\Omega \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} \cdot \mathbf{v} + c^h(\xi_n(\mathbf{u}_h), \mathbf{U}_h^{n+1/2}, \mathbf{v}) + \nu \int_\Omega \nabla \mathbf{U}_h^{n+1/2} : \nabla \mathbf{v} \\
&\quad = \int_\Omega \frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t} \cdot \mathbf{v} - \int_\Omega (p^{n+1/2} - \tilde{q}_h^{n+1/2}) \nabla \cdot \mathbf{v} \\
&\quad + \nu \int_\Omega \nabla \boldsymbol{\eta}^{n+1/2} : \nabla \mathbf{v} - c^h(\xi_n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{v}) + c^h(\xi_n(\boldsymbol{\eta}), \mathbf{u}^{n+1/2}, \mathbf{v}) \\
&\quad + c^h(\xi_n(\mathbf{u}_h), \boldsymbol{\eta}^{n+1/2}, \mathbf{v}) - \tau_n(\mathbf{v}) \quad \forall \mathbf{v} \in V^h. \qquad (34)
\end{aligned}$$

Specifying different $\mathbf{v}$ in (34) results in error estimates in different norms. For instance

$$\begin{aligned}
\mathbf{v} &= \mathbf{U}_h^{n+1/2} && \Rightarrow \text{Theorem 1, Corollary 1} \\
\mathbf{v} &= \tfrac{1}{2}(\mathbf{U}_h^{n+1} - \mathbf{U}_h^n) && \Rightarrow \text{Theorem 2, Corollary 2}
\end{aligned}$$

Successive applications of the estimates given in (13), (19), (23),(24)(a)(b), and (25) lead to an estimate of the form (4). Lemma 2 can then be applied *without imposing a time step restricion* to conclude Theorems 1, 2 and Corollaries 1, 2

First, for Theorem 1 and Corollary 1, write

$$\begin{aligned}
F^*_{h,\Delta t}(u,p,T) &:= E_{*,T} + \int_0^T \left(\inf_{\tilde{q}_h(t) \in Q^h} ||p - \tilde{q}_h||^2 \right) dt \\
&+ \int_0^T \left(h^2 \inf_{\tilde{\mathbf{v}}_h(t) \in X^h} ||\nabla \partial_t (\mathbf{u} - \tilde{\mathbf{v}}_h)||^2 + \inf_{\tilde{\mathbf{v}}_h(t) \in X^h} ||\nabla(\mathbf{u} - \tilde{\mathbf{v}}_h)||^2 \right) dt \qquad (35)
\end{aligned}$$

where $E_{*,T}$ is given by (61). For good choice of $X^h \times Q^h$ and sufficiently smooth u and p, $F^*_{h,\Delta t}(u,p,T) \to 0$ as h, $\Delta t \to 0$. We make this precise in Corollary 1.

Theorem 1 (Unconditional convergence) *Suppose that* $\mathbf{u} \in L^2(H^2) \cap H^1(H^1)$, $p \in C^0(L^2)$, $\mathbf{f} \in C^0(W^{-1,2})$ *and*

$$||\mathbf{u}(t_i) - \mathbf{u}_h^i||^2 \le \mathcal{O}(F^*_{h,\Delta t}(\mathbf{u},p,T)), \quad \textit{for } i = 0, 1, \ldots, n_0. \qquad (36)$$

Then,

$$\sup_{n_0<n\leq N} ||\mathbf{u}^n - \mathbf{u}_h^n||^2 + \nu\Delta t \sum_{n=n_0}^{N-1} ||\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}_h^{n+1/2})||^2 \leq C_*\nu^{-1} F^*_{h,\Delta t}(\mathbf{u}, p, T) \tag{37}$$

where $C_* \geq 0$ is the Gronwall constant given by (65) and $C_* < \infty$ uniformly as h, $\Delta t \to 0$.

Remark 2 Note that $\mathbf{u}_t \in L^2(H^1)$ implies that $\mathbf{u} \in C^0(H^1)$.

Corollary 1 *Under the assumptions of Theorem 1, suppose further Assumption 1, $\mathbf{u}_{tt} \in L^2(H^1)$, $\mathbf{u}_{ttt} \in L^2(W^{-1,2})$, $p_{tt} \in L^2(L^2)$, $\mathbf{f}_{tt} \in L^2(W^{-1,2})$ are satisfied. Then,*

$$\sup_{n} ||\mathbf{u}^n - \mathbf{u}_h^n||^2 + \nu\Delta t \sum_{n=n_0}^{N-1} ||\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}_h^{n+1/2})||^2 \leq C_*\nu^{-1} \left(h^{2k} + h^{2s+2} + \Delta t^4\right) \tag{38}$$

where $C_* \geq 0$ is the Gronwall constant given by (65) and $C_* < \infty$ uniformly as h, $\Delta t \to 0$.

An estimate for $\Delta t \sum_n ||(e^{n+1} - e^n)/\Delta t||$ is needed in the error analysis for pressure and the drag/lift forces by the fluid on imbedded obstacles. Let

$$F^{**}_{h,\Delta t}(\mathbf{u}, p, T) = E_{**,T} + \Delta t \sum_{n=n_0}^{N-1} ||\nabla \mathbf{E}_u^{n+1/2}||^2 + \int_0^T \inf_{\tilde{q}_h(t)\in Q^h} ||p - \tilde{q}_h||_1^2 dt + \int_0^T \left(h \inf_{\tilde{\mathbf{v}}_h(t)\in X^h} ||\nabla \partial_t(\mathbf{u} - \tilde{\mathbf{v}}_h)||^2 + \inf_{\tilde{\mathbf{v}}_h(t)\in X^h} ||\nabla(\mathbf{u} - \tilde{\mathbf{v}}_h)||^2 \right) dt. \tag{39}$$

where $E_{**,T}$ is given by (82). For good choice of $X^h \times Q^h$ and sufficiently smooth $\mathbf{u}$ and p, $F^{**}_{h,\Delta t}(\mathbf{u}, p, T) \to 0$ as h, $\Delta t \to 0$. We make this precise in Corollary 2.

Theorem 2 *Under the assumptions of Theorem 1, suppose further that $\mathbf{u} \in L^\infty(H^2)$, $p \in C^0(H^1)$, $\mathbf{f} \in C^0(L^2)$ and*

$$||\nabla(\mathbf{u}(t_i) - \mathbf{u}_h^i)||^2 \leq \mathcal{O}(F^{**}_{h,\Delta t}(\mathbf{u}, p, T)), \quad i = 0, 1, \ldots, n_0 \tag{40}$$

If

$$h^{-1}\Delta t\sum_{n=n_0}^{N-1}||\nabla\mathbf{E}_u^{n+1/2}||^2\leq C<\infty \tag{41}$$

holds uniformly as h, $\Delta t\to 0$, *then*

$$\begin{aligned}&\nu\sup_n||\nabla(\mathbf{u}^n-\mathbf{u}_h^n)||^2\\&\quad+\Delta t\sum_{n=n_0}^{N-1}||\frac{(\mathbf{u}-\mathbf{u}_h)^{n+1}-(\mathbf{u}-\mathbf{u}_h)^n}{\Delta t}||^2\leq C_{**}F_{h,\Delta t}^{**}(\mathbf{u},p,T)\end{aligned} \tag{42}$$

where $C_{**}>0$ *is the Gronwall constant given by (86) such that* $C_{**}<\infty$ *uniformly as* h, $\Delta t\to 0$ *and where* $\mathbf{E}_u^{n+1/2}$ *is bounded in (37).*

Corollary 2 *Under the conditions of Theorem 2 suppose further that Assumption 1,* $\mathbf{u}_{tt}\in L^2(H^2)$, $\mathbf{u}_{ttt}\in L^2(L^2)$, $p_{tt}\in L^2(H^1)$, $\mathbf{f}_{tt}\in L^2(L^2)$ *are satisfied. Then (41) holds if*

$$\Delta t\leq h^{1/4}. \tag{43}$$

Moreover,

$$\begin{aligned}&\nu\sup_n||\nabla(\mathbf{u}^n-\mathbf{u}_h^n)||^2\\&\quad+\Delta t\sum_{n=n_0}^{N-1}||\frac{(\mathbf{u}-\mathbf{u}_h)^{n+1}-(\mathbf{u}-\mathbf{u}_h)^n}{\Delta t}||^2\leq C_{**}\nu^{-1}\left(h^{2k}+h^{2s+2}+\Delta t^4\right)\end{aligned} \tag{44}$$

where $C_{**}>0$ *is the Gronwall constant given by (86) such that* $C_{**}<\infty$ *uniformly as* h, $\Delta t\to 0$.

3.1 A note on the sharpened estimates

The CNLE method is analyzed in [2] and [19] and the convergence analysis (corresponding to Corollary 1) assumes that $\mathbf{u}\in L^\infty(W^{1,\infty})$ and a time-step restriction. The conclusions of Theorem 1 and Corollary 1 in addition to those of Theorem 2 and Corollary 2 are preserved with the regularity condition $\mathbf{u} \in L^2(H^2)$ replaced by $\mathbf{u} \in L^\infty(W^{1,\infty})$.

Bounds (72), (80) are crucial in avoiding a sub-optimal convergence estimate $error \leq \mathcal{O}(\Delta t^{-1}(h^{2k} + h^{2s+2} + \Delta t^3))$ in Corollary 2. Correspondingly, the analysis of [19] suggests an associated sub-optimal convergence estimate, in the energy norm, $error \leq \mathcal{O}(h^k + h^{s+1} + h^{-3/2}\Delta t^4 + \Delta t^{3/2})$. Such an estimate requires, for instance, $\Delta t \leq h^{(3+2k)/4}$ for optimal convergence rate as $h \to 0$, but still predicts suboptimal convergence rate with respect to $\Delta t \to 0$.

Lastly, for $k = 2, 3, \ldots$, we have implicitly assumed *sufficient* regularity of $\mathbf{u}_0(x, \cdot)$ and compatibility between $\mathbf{u}_0$ and $\mathbf{f}$ to achieve the estimates (38) and (44). The compatibility condition required (implied by Equation (1.5) in [12]) is infeasible to verify in practice. Consequently the estimates (38), (44) can be formally altered to include a time-dependency factor $t^{(1-k)/2}$ for small time $t \leq 1$ (consequence of Equation (1.6) in [12]).

Moreover, the assumptions in Theorems 1, 2 hold a priori if we assume sufficient smoothness and sufficiently small problem data (see e.g. [6]). Moreover, if $u \in L^\infty(H^1) \cap L^2(H^2)$, then any Navier-Stokes solution u is smooth up to the regularity of the problem data f, u_0, $\partial\Omega$ (independent of a small data restriction). Consequently, the regularity suggested of (u, p) in Theorems 1, 2 implies that the solution is actually smooth corresponding to the smoothness of the problem data. Note, however, that we assume that Ω is polygonal and hence only C^0.

Set $\mathbf{v} = \mathbf{U}_h^{n+1/2}$ in (34) to get

$$\frac{1}{\Delta t}\left(||\mathbf{U}_h^{n+1}||^2 - ||\mathbf{U}_h^n||^2\right) + \nu||\nabla \mathbf{U}_h^{n+1/2}||^2 = \int_\Omega \frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t} \cdot \mathbf{U}_h^{n+1/2}$$
$$- \int_\Omega (p^{n+1/2} - \tilde{q}_h^{n+1/2})\nabla \cdot \mathbf{U}_h^{n+1/2} - c^h(\xi_n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2})$$
$$+ c^h(\xi_n(\mathbf{u}_h), \boldsymbol{\eta}^{n+1/2}, \mathbf{U}_h^{n+1/2}) + c^h(\xi_n(\boldsymbol{\eta}), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) - \tau_n(\mathbf{U}_h^{n+1/2}). \quad (45)$$

We applied (23) and (26).

We bound terms on the right-hand-side of (45) to obtain an a priori estimate for $\mathbf{U}_h$. Although used often, we will not refer explicitly to the estimates (13), (19), (23),(24)(a)(b), or (25). Careful application of these bounds ultimately leads to an estimate of $\mathbf{U}_h$ in the energy norm derived from (45) of the form (4). It is *essential* to absorb all terms including at the *current time-step* $\mathbf{u}^{n+1}$ into the left-hand-side term $||\nabla \mathbf{U}_h^{n+1/2}||^2$ so discrete Gronwall Lemma 2 can be applied to avoid a time-step restriction. Throughout, let $\varepsilon > 0$ be an arbitrary constant.

First, $\mathbf{u}_t \in W^{-1,2}(\Omega)$ and $p \in L^2(\Omega)$ implies

$$(\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}, \mathbf{U}_h^{n+1/2}) \leq C\nu^{-1}||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2 \quad (46)$$
$$(p^{n+1/2} - \tilde{q}_h^{n+1/2}, \nabla \cdot \mathbf{U}_h^{n+1/2}) \leq C\nu^{-1}||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2. \quad (47)$$

We bound the convective terms in the next lemma.

Lemma 4 *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 1. For any* $\varepsilon > 0$ *and for any integer* $n \geq n_0$ *there exists* $C > 0$ *such that*

$$\begin{aligned} & c^h(\xi_n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\ & \quad - c^h(\xi_n(\mathbf{u}_h), \boldsymbol{\eta}^{n+1/2}, \mathbf{U}_h^{n+1/2}) - c^h(\xi_n(\boldsymbol{\eta}), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\ & \leq \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2 + C\nu^{-1}(||\mathbf{u}^{n+1/2}||_2 + h^{-1}||\nabla \boldsymbol{\eta}^{n+1/2}||^2) \sum_{i=0}^{n_0} ||\mathbf{U}_h^{n-i}||^2 \\ & \quad + C\nu^{-1}||\mathbf{u}||^2_{L^\infty(H^1)} \left(\sum_{i=0}^{n_0} ||\nabla \boldsymbol{\eta}^{n-i}||^2 \right) \\ & \quad + C\nu^{-1} \sum_{i=0}^{n_0} (||\mathbf{u}||^2_{L^\infty(H^1)} + ||\nabla \boldsymbol{\eta}^{n-i}||^2)||\nabla \boldsymbol{\eta}^{n+1/2}||^2. \end{aligned} \tag{48}$$

Proof First, $\mathbf{u} \in H^2(\Omega)$ implies

$$c^h(\xi_n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \leq C\nu^{-1}||\xi_n(\mathbf{U}_h)||^2||\mathbf{u}^{n+1/2}||_2^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2 \tag{49}$$

and $\mathbf{u} \in L^\infty(H^1(\Omega))$ implies

$$\begin{aligned} & c^h(\xi_n(\boldsymbol{\eta}), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\ & \quad \leq C\nu^{-1}||\mathbf{u}||^2_{L^\infty(H^1)}||\nabla \xi_n(\boldsymbol{\eta})||^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2. \end{aligned} \tag{50}$$

Next, rewrite the remaining nonlinear term

$$\begin{aligned} c^h(\xi_n(\mathbf{u}_h), \boldsymbol{\eta}^{n+1/2}, \mathbf{U}_h^{n+1/2}) &= c^h(\xi_n(\mathbf{u}), \boldsymbol{\eta}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\ & - c^h(\xi_n(\boldsymbol{\eta}), \boldsymbol{\eta}^{n+1/2}, \mathbf{U}_h^{n+1/2}) + c^h(\xi_n(\mathbf{U}_h), \boldsymbol{\eta}^{n+1/2}, \mathbf{U}_h^{n+1/2}). \end{aligned}$$

Then $\mathbf{u} \in L^\infty(H^1(\Omega))$ implies

$$c^h(\xi_n(\mathbf{u}), \boldsymbol{\eta}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \leq C\nu^{-1}||\mathbf{u}||^2_{L^\infty(H^1)}||\nabla \boldsymbol{\eta}^{n+1/2}||^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2 \tag{51}$$

and similarly for $c^h(\xi_n(\boldsymbol{\eta}), \boldsymbol{\eta}^{n+1/2}, \mathbf{U}_h^{n+1/2})$. Lastly,

$$
\begin{aligned}
&c^h(\xi_n(\mathbf{U}_h), \boldsymbol{\eta}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\
&\quad \le C\sqrt{||\xi_n(\mathbf{U}_h)||||\nabla \xi_n(\mathbf{U}_h)||}||\nabla \boldsymbol{\eta}^{n+1/2}||||\nabla \mathbf{U}_h^{n+1/2}|| \\
&\quad \le C\nu^{-1}h^{-1}||\xi_n(\mathbf{U}_h)||^2||\nabla \boldsymbol{\eta}^{n+1/2}||^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2. \qquad (52)
\end{aligned}
$$

The conclusion follows by noting $\xi_n(\mathbf{u}) = a_0\mathbf{u}_n + \ldots + a_{n_0}\mathbf{u}_{n-n_0}$.

Bounding the time-consistency error remains.

Lemma 5 *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 1. Then, for any* $\varepsilon > 0$ *and any integer* $n \ge n_0$

$$
\tau_n(\mathbf{U}_h^{n+1/2}) \le \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2 + C\nu^{-1}E_{*,\Delta t}^n \qquad (53)
$$

and

$$
\Delta t \sum_{n=n_0}^{N-1} E_{*,\Delta t}^n \le CE_{*,T}. \qquad (54)
$$

where $E_{*,\Delta t}^n \ge 0$ *is given in (60) and* $E_{*,T}$ *in (61).*

Remark 3 We can restrict $\mathbf{u} \in H^1(\Omega)$ for Lemma 5.

Proof First, $\mathbf{u}_t \in W^{-1,2}(\Omega)$ implies

$$
\begin{aligned}
&\int_\Omega \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})\right) \cdot \mathbf{U}_h^{n+1/2} \\
&\le C\nu^{-1}||\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})||_{-1}^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2 \qquad (55)
\end{aligned}
$$

and $\mathbf{u} \in H^1(\Omega)$ implies

$$
\begin{aligned}
&\nu \int_\Omega \nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2})) : \nabla \mathbf{U}_h^{n+1/2} \\
&\qquad \le C\nu||\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2}))||^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2. \qquad (56)
\end{aligned}
$$

Similarly, $p \in L^2(\Omega)$ implies

$$\int_\Omega (p^{n+1/2} - p(t_{n+1/2}))\nabla \cdot \mathbf{U}_h^{n+1/2} \\ \leq C\nu^{-1}||p^{n+1/2} - p(t_{n+1/2})||^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2 \quad (57)$$

and $\mathbf{f} \in W^{-1,2}(\Omega)$ implies

$$\int_\Omega (\mathbf{f}(t_{n+1/2}) - \mathbf{f}^{n+1/2}) \cdot \mathbf{U}_h^{n+1/2} \\ \leq C\nu^{-1}||f(t_{n+1/2}) - f^{n+1/2}||_{-1}^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2. \quad (58)$$

We decompose the nonlinear terms so that

$$c^h(\xi_n(\mathbf{u}), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) - \Big(\mathbf{u}(t_{n+1/2}) \cdot \nabla \mathbf{u}(t_{n+1/2}), \mathbf{U}_h^{n+1/2}\Big) \\ = c^h(\xi_n(\mathbf{u}) - \mathbf{u}(t_{n+1/2}), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\ + c^h(\mathbf{u}(t_{n+1/2}), \mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2}), \mathbf{U}_h^{n+1/2}).$$

Then, $\mathbf{u} \in L^\infty(H^1(\Omega))$ implies

$$c^h(\xi_n(\mathbf{u}) - \mathbf{u}(t_{n+1/2}), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\ \leq C\nu^{-1}||\mathbf{u}||_{L^\infty(H^1)}^2||\nabla(\xi_n(\mathbf{u}) - \mathbf{u}(t_{n+1/2}))||^2 + \frac{\nu}{\varepsilon}||\nabla \mathbf{U}_h^{n+1/2}||^2 \quad (59)$$

and similarly for $c^h(\mathbf{u}(t_{n+1/2}), \mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2}), \mathbf{U}_h^{n+1/2})$. Lastly, set

$$E_{*,\Delta t}^n := ||\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})||_{-1}^2 \\ + ||\mathbf{u}||_{L^\infty(H^1)}^2||\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2}))||^2 \\ + ||\mathbf{u}||_{L^\infty(H^1)}^2||\nabla(\xi_n(\mathbf{u}) - \mathbf{u}(t_{n+1/2}))||^2 + \nu^2||\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2}))||^2 \\ + ||p^{n+1/2} - p(t_{n+1/2})||^2 + ||\mathbf{f}(t_{n+1/2}) - \mathbf{f}^{n+1/2}||_{-1}^2 \quad (60)$$

Then, $\mathbf{u} \in H^1(W^{-1,2}) \cap C^0(H^1)$, $p \in C^0(L^2)$, $\mathbf{f} \in C^0(W^{-1,2})$ imply

$$
\begin{aligned}
E_{*,T} := & \int_0^T \left\| \int_0^1 \left(\mathbf{u}_t(\cdot, t + s\Delta t) - \mathbf{u}_t(\cdot, t + \frac{\Delta t}{2}) \right) ds \right\|_{-1}^2 dt \\
& + \int_0^T ||\nabla (\frac{1}{2}(\mathbf{u}(\cdot, t + \Delta t) + \mathbf{u}(\cdot, t)) - \mathbf{u}(\cdot, t + \frac{\Delta t}{2}))||^2 dt \\
& + \int_0^T ||\nabla (\sum_{i=0}^{n_0} a_i \mathbf{u}(\cdot, t - i\Delta t) - \mathbf{u}(\cdot, t + \frac{\Delta t}{2}))||^2 dt \\
& + \int_0^T ||\frac{1}{2}(p(\cdot, t + \Delta t) + p(\cdot, t)) - p(\cdot, t + \frac{\Delta t}{2})||^2 dt \\
& + \int_0^T ||\frac{1}{2}(\mathbf{f}(\cdot, t + \Delta t) + \mathbf{f}(\cdot, t)) - f(\cdot, t + \frac{\Delta t}{2})||_{-1}^2 dt. \qquad (61)
\end{aligned}
$$

and (54) for some $C > 0$. The conclusion follows by noting $\xi_n(\mathbf{u}) = a_0 u_n + \ldots + a_{n_0} \mathbf{u}_{n-n_0}$.

Apply estimates from (46), (47), (48) and (53) to (45). Set $\varepsilon = 8$ and absorb all terms including $||\nabla \mathbf{U}_h^{n+1/2}||$ from the right into left-hand-side of (45). Sum the resulting inequality on both sides from $n = n_0$ to $n = N - 1$. Apply the estimate (54), (61). The result is

$$
\begin{aligned}
||\mathbf{U}_h^N||^2 + \nu \Delta t \sum_{n=n_0}^{N-1} ||\nabla \mathbf{U}_h^{n+1/2}||^2 \leq & ||\mathbf{U}_h^{n_0}||^2 + C\nu^{-1} \Delta t \sum_{n=n_0}^{N-1} E_{*,\Delta t}^n \\
& + C\nu^{-1} \Delta t \sum_{n=n_0}^{N-1} ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2 + C\nu^{-1} \Delta t \sum_{n=n_0}^{N-1} ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 \\
& + C\nu^{-1} \Delta t \sum_{n=n_0}^{N} \sum_{i=0}^{n_0} \Big(||\nabla \boldsymbol{\eta}^{n-i}|| + (1 + ||\nabla \boldsymbol{\eta}^{n-i}||^2) ||\nabla \boldsymbol{\eta}^{n+1/2}||^2 \Big) \\
& + C\nu^{-1} \Delta t \sum_{n=n_0}^{N} \sum_{i=0}^{n_0} (||\mathbf{u}^{n+1/2}||_2^2 + \frac{1}{h} ||\nabla \boldsymbol{\eta}^{n+1/2}||^2) ||\mathbf{U}_h^{n-i}||^2. \qquad (62)
\end{aligned}
$$

The approximation (12) and $\mathbf{u} \in C^0(H^1) \cap L^2(H^2)$ imply

$$
\sup_n ||\nabla \boldsymbol{\eta}^n|| \leq C||\mathbf{u}||_{L^\infty(H^1)} < \infty, \qquad \frac{\Delta t}{h} \sum_{n=n_0}^{N-1} ||\nabla \boldsymbol{\eta}^n||^2 \leq Ch||\mathbf{u}||_{L^2(H^2)}^2 < \infty.
$$

Applying these results, (62) becomes

$$||\mathbf{U}_h^N||^2 + \nu\Delta t \sum_{n=n_0}^{N-1} ||\nabla \mathbf{U}_h^{n+1/2}||^2 \le ||\mathbf{U}_h^{n_0}||^2 + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} E_{*,\Delta t}^n$$
$$+ C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2 + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2$$
$$+ C\nu^{-1}\Delta t \sum_{n=0}^{N} ||\nabla \boldsymbol{\eta}^n||^2 + C\nu^{-1}(1+h)||\mathbf{u}^n||_2^2 \Delta t \sum_{n=0}^{N} ||\mathbf{U}_h^n||^2. \tag{63}$$

In order to apply discrete Gronwall Lemma 2, we need, for $i = 0, 1, \ldots, n_0$,

$$||\mathbf{U}_h^i||^2 \le C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} E_{*,\Delta t}^n + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2$$
$$+ C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 + C\nu^{-1}\Delta t \sum_{n=0}^{N} ||\nabla \boldsymbol{\eta}^n||^2$$

which is implied by (36). Thus, (63) becomes

$$||\mathbf{U}_h^N||^2 + \nu\Delta t \sum_{n=n_0}^{N-1} ||\nabla \mathbf{U}_h^{n+1/2}||^2 \le C_*\nu^{-1} \left(E_{*,T} + \Delta t \sum_{n=n_0}^{N-1} ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2 \right)$$
$$+ C_*\nu^{-1}\Delta t \sum_{n=0}^{N-1} \left(||\nabla \boldsymbol{\eta}^n||^2 + ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 \right) \tag{64}$$

where

$$C_* = C \exp\left(\nu^{-1} \int_0^T ||\mathbf{u}(t)||_2^2 dt \right) \tag{65}$$

Lastly, the triangle inequality $||\mathbf{E}_u|| \le ||\mathbf{U}|| + ||\boldsymbol{\eta}||$ applied to (64) implies

$$||\mathbf{u}^N - \mathbf{u}_h^N||^2 + \nu\Delta t \sum_{n=n_0}^{N-1} ||\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}_h^{n+1/2})||^2$$
$$\le C_*\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} \mathbf{E}_{*,\Delta t}^n + C_*\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2$$
$$+ C_*\nu^{-1}\Delta t \sum_{n=0}^{N} \left(||\nabla \boldsymbol{\eta}^n||^2 + ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 \right). \tag{66}$$

Apply estimates (29) and (27), (28) to (66). Then, after simplification, using $\mathbf{u}_t \in L^2(H^1)$ to bound the discrete derivative on the right-hand-side, (66) results in (37) which proves Theorem 1. Lastly, to prove Corollary 1, apply estimates (29), (30), (31) and (12) to the preliminary estimate (66).

3.3 Proof of Theorem 2 and Corollary 2

Set $\mathbf{v} = \Delta t^{-1}(\mathbf{U}_h^{n+1} - \mathbf{U}_h^n)$ in (34). This gives,

$$\begin{aligned}
&||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2 + \frac{\nu}{\Delta t}\left(||\nabla \mathbf{U}_h^{n+1}||^2 - ||\nabla \mathbf{U}_h^n||^2\right) = -\tau_n\left(\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}\right) \\
&+ \int_\Omega \frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t} \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} + \int_\Omega (p^{n+1/2} - \tilde{q}_h^{n+1/2})\nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} \\
&\quad - c^h(\xi_n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) - c^h(\xi_n(\mathbf{u}_h), \mathbf{U}_h^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \\
&\quad + c^h(\xi_n(\boldsymbol{\eta}), \mathbf{u}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) + c^h(\xi_n(\mathbf{u}_h), \boldsymbol{\eta}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}). \qquad (67)
\end{aligned}$$

We applied the properties of the elliptic projection given in (26).

Our strategy is similar to the proof of Theorem 1 and Corollary 1: bound the terms on the right-hand-side of (67) to obtain an a priori estimate for $\mathbf{U}_h$ for the discrete $L^\infty(L^2)$-norm of $\mathbf{U}_h$ and the discrete $L^2(L^2)$-norm of the discrete time derivative. Although used often, we will not refer explicitly to the estimates (13), (19), (23),(24)(a)(b), or (25). Careful application of these bounds leads to an estimate of a form similar to (4):

$$||\nabla \mathbf{u}^N||^2 + \ldots \leq \sum_{n=0}^{N-1} \kappa_n ||\nabla \mathbf{u}^n||^2 + \ldots$$

It is *essential* to absorb all terms including at the *current time-step* $\mathbf{u}^{n+1}$ into the left-hand-side term $||\Delta t^{-1}(\mathbf{U}_h^{n+1} - \mathbf{U}_h^n)||^2$ so that we can apply discrete Gronwall Lemma 2 to avoid a time-step restriction. Throughout, let $\varepsilon > 0$ be an arbitrary constant.

First, $\mathbf{u} \in L^2(\Omega)$ and $p \in H^1(\Omega)$ implies

$$(\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \leq C||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2 \qquad (68)$$

$$\begin{aligned}
&(p^{n+1/2} - \tilde{q}_h^{n+1/2}, \nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \\
&\quad \leq C||p^{n+1/2} - \tilde{q}_h^{n+1/2}||_1^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2. \qquad (69)
\end{aligned}$$

We bound the convective terms in the next lemma.

Lemma 6 *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 2. For any* $\varepsilon > 0$ *and integer* $n \geq n_0$*, there exists* $C > 0$ *such that*

$$\begin{aligned}
&c^h(\xi_n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) + c^h(\xi_n(\mathbf{u}_h), \mathbf{U}_h^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \\
&\quad - c^h(\xi_n(\boldsymbol{\eta}), \mathbf{u}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) - c^h(\xi_n(\mathbf{u}_h), \boldsymbol{\eta}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \\
&\leq \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2 + C\sum_{i=0}^{n_0}(h^{-1}||\nabla \mathbf{E}_u^{n+1/2}||^2 + ||\mathbf{u}||^2_{L^\infty(H^2)})||\nabla \mathbf{U}_h^{n-i}||^2 \\
&\quad + Ch^{-1}\sum_{i=0}^{n_0}||\nabla \boldsymbol{\eta}^{n-i}||^2||\nabla \mathbf{E}_u^{n+1/2}||^2 \\
&\quad + C||\mathbf{u}||^2_{L^\infty(H^2)}\sum_{i=0}^{n_0}\left(||\nabla \mathbf{E}_u^{n+1/2}||^2 + ||\nabla \boldsymbol{\eta}^{n-i}||^2\right). \qquad (70)
\end{aligned}$$

Proof First, $\mathbf{u} \in L^\infty(H^2(\Omega))$ implies

$$\begin{aligned}
&c^h(\xi_n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \\
&\quad \leq C||\mathbf{u}||^2_{L^\infty(H^2)}||\nabla \xi_n(\mathbf{U}_h)||^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2. \qquad (71)
\end{aligned}$$

Similarly, for $c^h(\xi_n(\boldsymbol{\eta}), \mathbf{u}^{n+1/2}, \frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^n}{\Delta t})$. Rewrite the remaining terms to get

$$\begin{aligned}
&c^h(\xi_n(\mathbf{u}_h), \mathbf{U}_h^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) - c^h(\xi_n(\mathbf{u}_h), \boldsymbol{\eta}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \\
&= c^h(\xi_n(\mathbf{u}), \mathbf{E}_u^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) - c^h(\xi_n(\boldsymbol{\eta}), \mathbf{E}_u^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \\
&\quad + c^h(\xi_n(\mathbf{U}_h), \mathbf{E}_u^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}).
\end{aligned}$$

Next, $\nabla \cdot \mathbf{u} = 0$, $\mathbf{u} \in L^\infty(H^2(\Omega))$ and estimates (20), (21), (22) imply

$$\begin{aligned}
&c^h(\xi_n(\mathbf{u}), \mathbf{E}_u^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) = \int_\Omega \xi_n(\mathbf{u}) \cdot \nabla \mathbf{E}_u^{n+1/2} \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} \\
&\quad \leq C||\xi_n(\mathbf{u})||_{L^\infty}||\nabla \mathbf{E}_u^{n+1/2}||||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}|| \\
&\quad \leq C||\mathbf{u}||^2_{L^\infty(H^2)}||\nabla \mathbf{E}_u^{n+1/2}||^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2. \qquad (72)
\end{aligned}$$

Lastly,

$$
\begin{aligned}
&c^h(\xi_n(\mathbf{U}_h), \mathbf{E}_u^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \\
&\quad \le C||\nabla \xi_n(\mathbf{U}_h)||||\nabla \mathbf{E}_u^{n+1/2}||||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^{1/2}||\nabla \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^{1/2} \\
&\quad \le Ch^{-1}||\nabla \mathbf{E}_u^{n+1/2}||^2||\nabla \xi_n(\mathbf{U}_h)||^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2 \qquad (73)
\end{aligned}
$$

and similarly for $c^h(\xi_n(\boldsymbol{\eta}), \mathbf{E}_u^{n+1/2}, \frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^n}{\Delta t})$. The conclusion follows by noting $\xi_n(\mathbf{u}) = a_0\mathbf{u}_n + \ldots + a_{n_0}\mathbf{u}_{n-n_0}$.

A bound for the time-consistency error remains.

Lemma 7 *Let u satisfy the regularity assumptions of Theorem 2. For any $\varepsilon > 0$ and integer $n \ge n_0$, there exists $C > 0$ such that*

$$
\tau_n\left(\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}\right) \le \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2 + CE_{**,\Delta t}^n \qquad (74)
$$

and

$$
\Delta t \sum_{n=n_0}^{N-1} E_{**,\Delta t}^n \le CE_{**,T}. \qquad (75)
$$

where $E_{**,\Delta t}^n \ge 0$ is given in (81) and $E_{**,T}$ in (82).

Proof First, $\mathbf{u}_t \in L^2(\Omega)$ implies

$$
\begin{aligned}
&\int_\Omega \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})\right) \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} \\
&\quad \le C||\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})||^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2. \qquad (76)
\end{aligned}
$$

Next, $\mathbf{u} \in H^2(\Omega)$ imlies

$$
\begin{aligned}
&\nu \int_\Omega \nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2})) : \nabla(\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \\
&\quad \le C\nu^2||\mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2})||_2^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2. \qquad (77)
\end{aligned}
$$

Similarly, $p \in H^1(\Omega)$ implies

$$\int_\Omega (p^{n+1/2} - p(t_{n+1/2}))\nabla \cdot (\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t})$$
$$\leq C||p^{n+1/2} - p(t_{n+1/2})||_1^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2 \tag{78}$$

and $f \in L^2(\Omega)$ implies

$$\int_\Omega (\mathbf{f}(t_{n+1/2}) - \mathbf{f}^{n+1/2}) \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}$$
$$\leq C||\mathbf{f}(t_{n+1/2}) - \mathbf{f}^{n+1/2}||^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2. \tag{79}$$

Rewrite the convective terms

$$c^h(\xi_n(\mathbf{u}), \mathbf{u}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) - \left(\mathbf{u}(t_{n+1/2}) \cdot \nabla \mathbf{u}(t_{n+1/2}), \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}\right)$$
$$= c^h(\xi_n(\mathbf{u}) - \mathbf{u}(t_{n+1/2}), \mathbf{u}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t})$$
$$+ c^h(\mathbf{u}(t_{n+1/2}), \mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2}), \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t})$$

Next, $\nabla \cdot \mathbf{u} = 0$, $\mathbf{u} \in L^\infty(H^2)$ and estimates (20), (21), (22) imply

$$c^h(\xi_n(\mathbf{u}) - \mathbf{u}(t_{n+1/2}), \mathbf{u}^{n+1/2}, \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t})$$
$$= \int_\Omega (\xi_n(\mathbf{u}) - \mathbf{u}(t_{n+1/2})) \cdot \nabla \mathbf{u}^{n+1/2} \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}$$
$$\leq C||\mathbf{u}||_{L^\infty(H^2)}^2 ||\nabla(\xi_n(\mathbf{u}) - \mathbf{u}(t_{n+1/2}))||^2 + \frac{1}{\varepsilon}||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2 \tag{80}$$

and similarly for $c^h(\mathbf{u}(t_{n+1/2}), \mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2}), \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t})$. Let

$$E_{**,\Delta t}^n := ||\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})||^2 + \nu^2||\mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2})||_2^2$$
$$+ ||\mathbf{u}||_{L^\infty(H^2)}^2 ||\nabla(\xi_n(\mathbf{u}) - \mathbf{u}(t_{n+1/2}))||^2$$
$$+ ||\mathbf{u}||_{L^\infty(H^2)}^2 ||\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(t_{n+1/2}))||^2$$
$$+ ||p^{n+1/2} - p(t_{n+1/2})||_1^2 + ||\mathbf{f}(t_{n+1/2}) - \mathbf{f}^{n+1/2}||^2 \tag{81}$$

and $\mathbf{u} \in H^1(L^2) \cap C^0(H^1)$, $p \in C^0(H^1)$, and $\mathbf{f} \in C^0(L^2)$ imply

$$
\begin{aligned}
E_{**,T} &:= \int_0^T ||\frac{\mathbf{u}(\cdot, t+\Delta t) - \mathbf{u}(t)}{\Delta t} - \mathbf{u}_t(\cdot, t + \frac{\Delta t}{2})||^2 dt \\
&+ \int_0^T ||\nabla(\frac{1}{2}(\mathbf{u}(\cdot, t+\Delta t) + \mathbf{u}(\cdot, t)) - \mathbf{u}(\cdot, t + \frac{\Delta t}{2}))||^2 dt \\
&+ \int_0^T ||\nabla(\sum_{i=0}^{n_0} a_i \mathbf{u}(\cdot, t - i\Delta t) - \mathbf{u}(\cdot, t + \frac{\Delta t}{2}))||^2 dt \\
&+ \int_0^T ||\frac{1}{2}(p(\cdot, t+\Delta t) + p(\cdot, t)) - p(\cdot, t + \frac{\Delta t}{2})||_1^2 dt \\
&+ \int_0^T ||\frac{1}{2}(\mathbf{f}(\cdot, t+\Delta t) + \mathbf{f}(\cdot, t)) - f(\cdot, t + \frac{\Delta t}{2})||^2 dt. \qquad (82)
\end{aligned}
$$

Then $\mathbf{u} \in L^\infty(H^2)$ implies (75). The conclusion follows by noting $\xi_n(\mathbf{u}) = a_0 \mathbf{u}_n + \ldots + a_{n_0} \mathbf{u}_{n-n_0}$.

Apply estimates from (68), (69), (70) and (74) to (67). Set $\varepsilon = 8$ and absorb all terms including $||\Delta t^{-1}(\mathbf{U}_h^{n+1} - \mathbf{U}_h^n)||$ from the right into left-hand-side of (67).

Sum the resulting inequality on both sides from $n = n_0$ to $n = N-1$. Apply the estimate (75), (82). Assuming that $\mathbf{u} \in L^\infty(H^2)$, the result is

$$
\begin{aligned}
&\nu||\nabla \mathbf{U}_h^N||^2 + \Delta t \sum_{n=n_0}^{N-1} ||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2 \le \nu||\nabla \mathbf{U}_h^{n_0}||^2 + C\Delta t \sum_{n=n_0}^{N-1} E_{**,\Delta t}^n \\
&\quad + C\Delta t \sum_{n=n_0}^{N-1} \left(||p^{n+1/2} - \tilde{q}_h^{n+1/2}||_1^2 + ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||^2 \right) + C\Delta t \sum_{n=0}^{N-1} ||\nabla \boldsymbol{\eta}^n||^2 \\
&\quad + C\Delta t \sum_{n=n_0}^{N-1} \sum_{i=0}^{n_0} \left(||\mathbf{u}||_{L^\infty(H^2)}^2 + h^{-1}||\nabla \boldsymbol{\eta}^{n-i}||^2 \right) ||\nabla \mathbf{E}_u^{n+1/2}||^2 \\
&\quad + C\Delta t \sum_{n=n_0}^{N-1} \sum_{i=0}^{n_0} (||\mathbf{u}||_{L^\infty(H^2)}^2 + h^{-1}||\nabla \mathbf{E}_u^{n+1/2}||^2)||\nabla \mathbf{U}_h^{n-i}||^2. \qquad (83)
\end{aligned}
$$

The approximation (12) and $\mathbf{u} \in L^\infty(H^2)$ imply

$$
h^{-1} \sup_n ||\nabla \boldsymbol{\eta}^n||^2 \le Ch||\mathbf{u}||_{L^\infty(H^2)}^2 < \infty.
$$

Applying this result, (83) becomes

$$\nu||\nabla \mathbf{U}_h^N||^2 + \Delta t\sum_{n=n_0}^{N-1}||\frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^n}{\Delta t}||^2 \le \nu||\nabla \mathbf{U}_h^{n_0}||^2 + C\Delta t\sum_{n=n_0}^{N-1}E_{**,\Delta t}^n$$

$$+ C\Delta t\sum_{n=n_0}^{N-1}\left(||p^{n+1/2}-\tilde{q}_h^{n+1/2}||_1^2 + ||\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n}{\Delta t}||^2\right) + C\Delta t\sum_{n=0}^{N}||\nabla \boldsymbol{\eta}^n||^2$$

$$+ C\Delta t\left(\sum_{n=n_0}^{N-1}||\nabla \mathbf{E}_u^{n+1/2}||^2 + \sum_{n=0}^{N-1}(||\mathbf{u}||_2^2 + \frac{1}{h}||\nabla \mathbf{E}_u^{n+1/2}||^2)||\nabla \mathbf{U}_h^n||^2\right). \qquad (84)$$

In order to apply discrete Gronwall Lemma 2, we need

$$||\nabla \mathbf{U}_h^i||^2 \le C\Delta t\sum_{n=n_0}^{N-1}E_{**,\Delta t}^n + C\Delta t\sum_{n=0}^{N-1}||\nabla \boldsymbol{\eta}^n||^2 + C\Delta t\sum_{n=n_0}^{N-1}||\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n}{\Delta t}||^2$$

$$+ C\Delta t\sum_{n=n_0}^{N-1}\left(||\nabla \mathbf{E}_u^{n+1/2}||^2 + ||p^{n+1/2}-\tilde{q}_h^{n+1/2}||_1^2\right), \quad \text{for } i = 0, 1, \ldots, n_0$$

which is satisfied under (40). Thus, (84) becomes

$$\nu||\nabla \mathbf{U}_h^N||^2 + \Delta t\sum_{n=n_0}^{N-1}||\frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^n}{\Delta t}||^2$$

$$\le C_{**}\Delta t\sum_{n=n_0}^{N-1}E_{**,\Delta t}^n + C_{**}\Delta t\sum_{n=n_0}^{N-1}||\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n}{\Delta t}||^2$$

$$+ C_{**}\Delta t\sum_{n=n_0}^{N-1}\left(||p^{n+1/2}-\tilde{q}_h^{n+1/2}||_1^2 + ||\nabla \boldsymbol{\eta}^n||^2 + ||\nabla \mathbf{E}_u^{n+1/2}||^2\right) \qquad (85)$$

where

$$C_{**} = C\exp\left(\nu^{-1}(h^{-1} + ||\mathbf{u}||_{L^\infty(H^2)}^2)\Delta t\sum_{n=n_0}^{N-1}||\nabla \mathbf{E}_u^{n+1/2}||^2\right). \qquad (86)$$

Note that C_{**} is uniformly bounded with h, $\Delta t \to 0$ as long as (41) is satisfied. Apply the triangle inequality $||\mathbf{E}_u|| \le ||\mathbf{U}|| + ||\boldsymbol{\eta}||$ to (85). Simplifying, (85) becomes

$$\nu||\nabla(\mathbf{u}^N - \mathbf{u}_h^N)||^2 + \Delta t\sum_{n=n_0}^{N-1}||\frac{(\mathbf{u}-\mathbf{u}_h)^{n+1}-(\mathbf{u}-u_h)^n}{\Delta t}||^2$$

$$\le C_{**}\Delta t\sum_{n=n_0}^{N-1}\left(E_{**,\Delta t}^n + ||p^{n+1/2}-\tilde{q}_h^{n+1/2}||_1^2 + ||\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n}{\Delta t}||^2\right)$$

$$+ C_{**}\Delta t\sum_{n=0}^{N}||\nabla \boldsymbol{\eta}^n||^2 + C_{**}\Delta t\sum_{n=n_0}^{N-1}||\nabla \mathbf{E}_u^{n+1/2}||^2. \qquad (87)$$

Apply estimates (29) and (27), (28) to (66). After simplification, (87) results in (42) which proves Theorem 2.

Lastly, to prove Corollary 2, apply estimates (29), (30), (31) and (12) to the preliminary estimate (87). Note that the time-step restriction (43) implies (41). Indeed, combining (41) and (38), $k \geq 1/2$ and (43) imply

$$\Delta t \sum_{n=n_0}^{N-1} h^{-1} ||\nabla \mathbf{E}_u^{n+1/2}||^2 \leq C_* \nu^{-1} \left(h^{2k-1} + h^{2s+1} + \Delta t^4 h^{-1} \right) < \infty. \tag{88}$$

4 Conclusions

The analysis in this report was performed for a fully implicit, linearly extrapolated version of the Crank-Nicolson finite element method (CNLE) for approximating Navier-Stokes flows. Our analysis includes the general case of arbitrary (high) order extrapolations of the form

$$\mathbf{u} \cdot \nabla \mathbf{u} \approx \xi_n(\mathbf{u}) \cdot \nabla \frac{\mathbf{u}^n + \mathbf{u}^{n-1}}{2}, \quad \xi_n(\mathbf{u}) = a_0 \mathbf{u}^{n-1} + a_1 \mathbf{u}^{n-2} + \ldots + a_{n_0} \mathbf{u}^{n-n_0}.$$

We proved that CNLE is converges *without any time-step restriction* in the energy norm. We also proved that the approximating velocity converges optimally to the true Navier Stokes velocity in the discrete $L^\infty(H^1)$-norm and that the discrete time derivative converges in the discrete $L^2(L^2)$-norm under the mild time step restriction $\Delta t \leq \mathcal{O}(h^{1/4})$. Convergence in these norms is required to derive convergence rates for pressure and drag/lift forces the fluid exerts on imbedded obstacles.

The full CN method is believed to be more accurate than CNLE. However, the accuracy of CNLE is easily improvable by increasing the order of extrapolation. Moreover, CNLE methods are linearly implicit (simple to implement and fast to solve). The additional guarantee that CNLE approximations converge *unconditionally* is another important property not shared by full CN methods. Consequently, CNLE methods are of great interest in practical computations in which speed, robustness, ease of implementation, and accuracy are required. A comparative study of CNLE against full CN methods and other CN-variants (like Adams-Bashforth linearizations) should be investigated to determine the robustness and accuracy of CNLE methods in practice.

A Derivation of Condition (1)

In this section, we provide details on how each (1)(a) and (b) are derived for the CN-FE approximation of the Navier-Stokes equation. In fact, we present a best time-step restriction based on Kolmogorov's energy-cascade/micro-scale theory of turbulent (high-Re) flows suggests the relationship $h = \mathcal{O}(\nu^{3/4})$ is required. We note that $||u||_{1,p} \leq \mathcal{O}(\nu^{-1})$ for $p = 2, \infty$. Note that $Re = \mathcal{O}(\nu^{-1})$. Also note that the time-step restrictions (89) and (91) can be replaced with $\Delta t \leq \mathcal{O}(\nu^{13/3} h^{2/3})$ and condition (1) (either (a) or (b)) respectively (see Remark 4).

Theorem 3 *Under the assumptions of Corollary 1, suppose further that*

$$\Delta t \leq \mathcal{O}(\nu h^{3/2}). \tag{89}$$

Then

$$\sup_n ||\mathbf{u}^n - \mathbf{w}_h^n||^2 + \nu \Delta t \sum_{n=n_0}^{N-1} ||\nabla(\mathbf{u}^{n+1/2} - \mathbf{w}_h^{n+1/2})||^2 \leq C_{CNFE} \nu^{-1} \left(h^{2k} + h^{2s+2} + \Delta t^4 \right) \tag{90}$$

Moreover, if $u \in L^2(H^2(\Omega))$ is replaced with $u \in L^2(W^{1,\infty}(\Omega))$, then condition (89) can be replaced with

$$\Delta t \leq \mathcal{O}(\nu h). \tag{91}$$

so that (90) still holds.

Proof First, CN-FE is obtained by letting $\xi_n(\mathbf{u}) = u^{n+1/2}$ in (16): Let $\mathbf{w}_h^i \in V^h$ be a good approximation of $\mathbf{u}^i$ for each $i = 0, 1, \ldots, n_0$. For each $n = n_0, n_0 + 1, \ldots, N - 1$, find $\mathbf{w}_h^{n+1} \in$

V^h satisfying

$$
\begin{aligned}
&(\frac{\mathbf{w}_h^{n+1} - \mathbf{w}_h^n}{\Delta t}, \mathbf{v}) + c^h(\mathbf{w}_h^{n+1/2}, \mathbf{w}_h^{n+1/2}, \mathbf{v}) \\
&\quad + \nu(\nabla \mathbf{w}_h^{n+1/2}, \nabla v) = (\mathbf{f}^{n+1/2}, \mathbf{v}), \quad \forall v \in X^h
\end{aligned} \tag{92}
$$

We first construct the error equation for CN-FE. Decompose the velocity error

$$
\mathbf{E}_w^n = \mathbf{w}_h^n - \mathbf{u}^n = \mathbf{W}_h^n - \boldsymbol{\eta}^n, \qquad \mathbf{W}_h^n = \mathbf{w}_h^n - \tilde{\mathbf{v}}_h^n, \qquad \boldsymbol{\eta}^n = \mathbf{u}^n - \tilde{\mathbf{v}}_h^n.
$$

Fix $\tilde{q}_h^n \in Q^h$. Note that $(p_h, \nabla \cdot v) = 0$ for any $\mathbf{v} \in V^h$. Subtract (33) from (92) to get the error equation

$$
\begin{aligned}
&\int_\Omega \frac{\mathbf{W}_h^{n+1} - \mathbf{W}_h^n}{\Delta t} \cdot \mathbf{v} + c^h(\mathbf{w}_h^{n+1/2}, \mathbf{W}_h^{n+1/2}, \mathbf{v}) + \nu \int_\Omega \nabla \mathbf{W}_h^{n+1/2} : \nabla \mathbf{v} \\
&\quad = \int_\Omega \frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t} \cdot \mathbf{v} - \int_\Omega (p^{n+1/2} - \tilde{q}_h^{n+1/2}) \nabla \cdot \mathbf{v} \\
&\quad + \nu \int_\Omega \nabla \boldsymbol{\eta}^{n+1/2} : \nabla \mathbf{v} - c^h(\mathbf{W}_h^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{v}) + c^h(\boldsymbol{\eta}^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{v}) \\
&\quad + c^h(\mathbf{w}_h^{n+1/2}, \boldsymbol{\eta}^{n+1/2}, \mathbf{v}) - \tau_n(\mathbf{v}) \quad \forall \mathbf{v} \in V^h.
\end{aligned} \tag{93}
$$

Set $v = W_h^{n+1/2}$ in (93) to get

$$
\begin{aligned}
&\frac{1}{\Delta t}\left(||\mathbf{W}_h^{n+1}||^2 - ||\mathbf{W}_h^n||^2\right) + \nu ||\nabla \mathbf{W}_h^{n+1/2}||^2 = \int_\Omega \frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t} \cdot \mathbf{W}_h^{n+1/2} \\
&\quad - \int_\Omega (p^{n+1/2} - \tilde{q}_h^{n+1/2}) \nabla \cdot \mathbf{W}_h^{n+1/2} - c^h(\mathbf{W}_h^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{W}_h^{n+1/2}) \\
&\quad + c^h(\mathbf{u}_h^{n+1/2}, \boldsymbol{\eta}^{n+1/2}, \mathbf{W}_h^{n+1/2}) + c^h(\boldsymbol{\eta}^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{W}_h^{n+1/2}) - \tau_n(\mathbf{W}_h^{n+1/2}).
\end{aligned} \tag{94}
$$

We proceed as in Section 3.2: bound the right-hand-side, absorb like-terms into the left-hand-side, apply the discrete Gronwall lemma 1 or 2 and simplify. The estimates are generally the same, except for the convective term $c^h(\cdots)$ since $\xi^n(u) = u^{n+1/2}$ here. Lemma 4 is replaced with the following estimate:

Lemma 8 *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 1. For any* $\varepsilon > 0$ *and for any integer* $n \geq n_0$ *there exists* $C > 0$ *such that*

$$\begin{aligned} & c^h(\mathbf{W}_h^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{W}_h^{n+1/2}) \\ & \quad - c^h(\mathbf{u}_h^{n+1/2}, \boldsymbol{\eta}^{n+1/2}, \mathbf{W}_h^{n+1/2}) - c^h(\boldsymbol{\eta}^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{W}_h^{n+1/2}) \\ & \leq \frac{\nu}{\varepsilon}||\nabla \mathbf{W}_h^{n+1/2}||^2 + C\Phi_n(h,\nu)||\mathbf{W}_h^{n+1/2}||^2 \\ & \quad + C\nu^{-1}||\mathbf{u}||^2_{L^\infty(H^1)}||\nabla \boldsymbol{\eta}^{n+1/2}||^2 \\ & \quad + C\nu^{-1}(||\mathbf{u}||^2_{L^\infty(H^1)} + ||\nabla \boldsymbol{\eta}^{n+1/2}||^2)||\nabla \boldsymbol{\eta}^{n+1/2}||^2. \end{aligned} \tag{95}$$

where, for any $0 \leq s \leq 3/2$ *and* $0 \leq r \leq 1$,

$$\Phi_n(h,\nu) = \begin{cases} \nu^{-(3-2s)/(1+2s)} h^{-4s/(1+2s)} ||\nabla \mathbf{u}^{n+1/2}||^{4/(1+2s)}, \ \textit{if} \quad u \in L^{4/(1+2s)}(H^1) \\ \nu^{-(1-r)/(1+r)} h^{-2r/(1+r)} ||\mathbf{u}^{n+1/2}||^{2/(1+r)}_{1,\infty}, \quad \textit{if} \quad u \in L^{2/(1+r)}(W^{1,\infty}) \end{cases}$$

Remark 4 The time-step restriction from the discrete Gronwall Lemma 1, for some $C_0 > 0$, is exactly

$$C_0 \Delta t \Phi_n(h,\nu) < 1, \qquad \forall n \geq 0 \tag{96}$$

Noting that $||u||_{1,p} \leq C\nu^{-1}$ for $p = 2, \infty$ (when well-defined) and assuming that the discretization parameter scales via Kolmogorov's theory $h = \mathcal{O}(\nu^{3/4})$, then Φ_n scales in such a way that the time-step restriction is of the form:

$$\Delta t \leq \begin{cases} \mathcal{O}(\nu^{(7-2s)/(1+2s)} h^{4s/(1+2s)}) \approx \mathcal{O}(\nu^{(7+s)/(1+2s)}) \text{ if } \quad u \in L^{4/(1+2s)}(H^1) \\ \mathcal{O}(\nu^{(3-r)/(1+r)} h^{-2r/(1+r)}) \quad \approx \mathcal{O}(\nu^{(6+r)/(2+2r)}) \text{ if } \quad u \in L^{2/(1+r)}(W^{1,\infty}) \end{cases}$$

So, $s = 0$, $r = 0$ implies

$$\Phi_n(h,\nu) = \begin{cases} \nu^{-3}||\nabla \mathbf{u}^{n+1/2}||^4 \ \leq C\nu^{-7}, \text{ if } \quad u \in L^4(H^1) \\ \nu^{-1}||\mathbf{u}^{n+1/2}||^2_{1,\infty} \leq C\nu^{-3}, \text{ if } \quad L^2(W^{1,\infty}) \end{cases} \tag{97}$$

and $s = 3/2$, and $r = 1$ respectively implies

$$\Phi_n(h,\nu) = \begin{cases} h^{-3/2}||\nabla \mathbf{u}^{n+1/2}|| \leq C\nu^{-1}h^{-3/2} \leq C\nu^{-17/8} \text{ if } \quad u \in L^1(H^1) \\ h^{-1}||\mathbf{u}^{n+1/2}||_{1,\infty} \leq C\nu^{-1}h^{-1} \quad \leq C\nu^{-7/4} \text{ if } \quad u \in L^1(W^{1,\infty}) \end{cases} \tag{98}$$

and $s = 1/4$, $s = 1/2$, and $r = 1/2$ respectively implies

$$\Phi_n(h,\nu) = \begin{cases} \nu^{-5/3}h^{-2/3}||\nabla \mathbf{u}^{n+1/2}||^{8/3}, \leq C\nu^{-13/3}h^{-2/3} \leq C\nu^{-29/6} \text{ if } & u \in L^{8/3}(H^1) \\ \nu^{-1}h^{-1}||\nabla \mathbf{u}^{n+1/2}||^2, \quad \leq C\nu^{-3}h^{-1} \quad \leq C\nu^{-15/4} \text{ if } & u \in L^2(H^1) \\ \nu^{-1/3}h^{-2/3}||\mathbf{u}||_{1,\infty}^{4/3}, \quad \leq C\nu^{-5/3}h^{-2/3} \leq C\nu^{-13/6} \text{ if } & u \in L^{4/3}(W^{1,\infty}) \end{cases} \tag{99}$$

We interpret *best* Δt-condition by the least-restrictive ν-dependence (based on Kolmogorov's theory), which is attained for the $s = 3/2$, $r = 1$ cases. Consequently, we report a less restrictive condition on Δt in (89) and (91), which is implied by estimate (98)(a) and (b) respectively. Condition (1)(a), (suggested in [13] without ν-dependence) is implied by (99)(c) when $u \in L^{4/3}(W^{1,\infty})$ and should be replaced with $\Delta t \leq \mathcal{O}(\nu^{13/3}h^{2/3})$ when $u \in L^{8/3}(H^1)$.

Proof Estimates (50) and (51) remain relatively unchanged: simply change U_h, u_h with W_h and w_h respectively along with setting $\xi_n(z) = z^{n+1/2}$. We consider the following options for the estimate corresponding to (49). Fix $0 \leq s \leq 3/2$. For the first cases, consider $u \in L^{4/(1+2s)}(H^1)$

$$c^h(\mathbf{W}_h^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{W}_h^{n+1/2}) \leq C||\nabla \mathbf{u}^{n+1/2}||||\mathbf{W}_h^{n+1/2}||^{1/2}||\nabla \mathbf{W}_h^{n+1/2}||^{3/2}$$

Then, $p = 4/(3-2s)$ and $1/p + 1/q = 1$ implies that $q = 4/(1+2s)$ so that (13) implies $||\nabla \mathbf{W}_h^{n+1/2}||^{3/2} \leq Ch^{-s}||\mathbf{W}_h^{n+1/2}||^s||\nabla \mathbf{W}_h^{n+1/2}||^{3/2-s}$ and together with (19) gives

$$\begin{aligned} c^h(\mathbf{W}_h^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{W}_h^{n+1/2}) &\leq \frac{\nu}{\varepsilon}||\nabla \mathbf{W}_h^{n+1/2}||^2 \\ &+ C\nu^{-(3-2s)/(1+2s)}h^{-4s/(1+2s)}||\nabla \mathbf{u}^{n+1/2}||^{4/(1+2s)}||\mathbf{W}_h^{n+1/2}||^2 \end{aligned} \tag{100}$$

Alternative, fix $0 \leq r \leq 1$ and consider $u \in L^{2/(1+r)}(W^{1,\infty})$. Then

$$c^h(\mathbf{W}_h^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{W}_h^{n+1/2}) \leq C||\mathbf{u}^{n+1/2}||_{1,\infty}||\mathbf{W}_h^{n+1/2}||||\nabla \mathbf{W}_h^{n+1/2}||$$

Then, $p = 2/(1-r)$ and $1/p + 1/q = 1$ implies that $q = 2/(1+r)$ so that (13) implies $||\nabla \mathbf{W}_h^{n+1/2}|| \leq Ch^{-r}||\mathbf{W}_h^{n+1/2}||^R||\nabla \mathbf{W}_h^{n+1/2}||^{1-r}$ and together with (19) gives

$$\begin{aligned} c^h(\mathbf{W}_h^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{W}_h^{n+1/2}) &\leq \frac{\nu}{\varepsilon}||\nabla \mathbf{W}_h^{n+1/2}||^2 \\ &+ C\nu^{-(1-r)/(1+r)}h^{-2r/(1+r)}||\mathbf{u}^{n+1/2}||_{1,\infty}^{2/(1+r)}||\mathbf{W}_h^{n+1/2}||^2 \end{aligned} \tag{101}$$

Last rewrite the remaining nonlinear term as in the proof of Lemma 4 to get

$$\begin{aligned} c^h(\mathbf{w}_h^{n+1/2}, \boldsymbol{\eta}^{n+1/2}, \mathbf{W}_h^{n+1/2}) &= c^h(\mathbf{u}^{n+1/2}, \boldsymbol{\eta}^{n+1/2}, \mathbf{W}_h^{n+1/2}) \\ &- c^h(\boldsymbol{\eta}^{n+1/2}, \boldsymbol{\eta}^{n+1/2}, \mathbf{W}_h^{n+1/2}) + c^h(\mathbf{W}_h^{n+1/2}, \boldsymbol{\eta}^{n+1/2}, \mathbf{W}_h^{n+1/2}). \end{aligned}$$

Only the estimate corresponding to (52) is different. Suppose that $||\eta||_{1,p} \le C||u||_{1,p}$. Then we bound $c^h(\mathbf{W}_h^{n+1/2}, \boldsymbol{\eta}^{n+1/2}, \mathbf{W}_h^{n+1/2})$ is a similar way to (100), (101).

The time consistency estimates in Lemma 5 follow similarly here. Thus, apply the discussed estimates, including (95), to (94). Set ε appropriately and absorb all terms including $||\nabla \mathbf{W}_h^{n+1/2}||$ from the right into left-hand-side of (94). Sum the resulting inequality on both sides from $n = n_0$ to $n = N-1$. The result is

$$
\begin{aligned}
&||\mathbf{W}_h^N||^2 + \nu \Delta t \sum_{n=n_0}^{N-1} ||\nabla \mathbf{W}_h^{n+1/2}||^2 \le ||\mathbf{W}_h^{n_0}||^2 + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} E_{*,\Delta t}^n \\
&\quad + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2 + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 \\
&\quad + C\nu^{-1}\Delta t \sum_{n=n_0}^{N} \left(||\nabla \boldsymbol{\eta}^n|| + (1 + ||\nabla \boldsymbol{\eta}^n||^2)||\nabla \boldsymbol{\eta}^n||^2\right) \\
&\quad + C_0\nu^{-1}\Delta t \sum_{n=n_0}^{N} \Phi_n(h,\nu)||\mathbf{U}_h^n||^2. \qquad (102)
\end{aligned}
$$

The approximation (12) and $\mathbf{u} \in C^0(H^1)$ implies $\sup_n ||\nabla \boldsymbol{\eta}^n|| \le C||\mathbf{u}||_{L^\infty(H^1)} < \infty$. Thus, (102) becomes

$$
\begin{aligned}
&||\mathbf{W}_h^N||^2 + \nu \Delta t \sum_{n=n_0}^{N-1} ||\nabla \mathbf{W}_h^{n+1/2}||^2 \le ||\mathbf{W}_h^{n_0}||^2 + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} E_{*,\Delta t}^n \\
&\quad + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2 + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 \\
&\quad + C\nu^{-1}\Delta t \sum_{n=0}^{N} ||\nabla \boldsymbol{\eta}^n||^2 + C_0 \Delta t \sum_{n=0}^{N} \Phi_n(h,\nu)||\mathbf{W}_h^n||^2. \qquad (103)
\end{aligned}
$$

In order to apply discrete Gronwall Lemma 1, we need, for $i = 0, 1, \dots, n_0$,

$$
\begin{aligned}
||\mathbf{W}_h^i||^2 \le\ & C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} E_{*,\Delta t}^n + C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2 \\
&+ C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 + C\nu^{-1}\Delta t \sum_{n=0}^{N} ||\nabla \boldsymbol{\eta}^n||^2
\end{aligned}
$$

which is implied by (36). Moreover, we need Δt to be small enough; in particular, (96) must be satisfied, which relates to conditions (89), (91). Thus, (103) becomes

$$
\begin{aligned}
&||\mathbf{W}_h^N||^2 + \nu \Delta t \sum_{n=n_0}^{N-1} ||\nabla \mathbf{W}_h^{n+1/2}||^2 \le C_{CNFE}\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} \left(E_{*,\Delta t}^n + ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||_{-1}^2\right) \\
&\quad + C_{CNFE}\nu^{-1}\Delta t \sum_{n=0}^{N-1} \left(||\nabla \boldsymbol{\eta}^n||^2 + ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2\right) \qquad (104)
\end{aligned}
$$

where C_{CNFE} is the Gronwall factor from Lemma (1) (depending on Δt, T, u, ν via the bound on Φ_n in Lemma 8 and Remark 4). Lastly, the triangle inequality $||\mathbf{E}_w|| \leq ||\mathbf{W}|| + ||\boldsymbol{\eta}||$ applied to (104) implies

$$\begin{aligned} &||\mathbf{u}^N - \mathbf{w}_h^N||^2 + \nu \Delta t \sum_{n=n_0}^{N-1} ||\nabla(\mathbf{u}^{n+1/2} - \mathbf{w}_h^{n+1/2})||^2 \\ &\leq C_{CNFE} \nu^{-1} \Delta t \sum_{n=n_0}^{N-1} \left(E^n_{*,\Delta t} + ||\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}||^2_{-1} \right) \\ &+ C_{CNFE} \nu^{-1} \Delta t \sum_{n=0}^{N-1} \left(||\nabla \boldsymbol{\eta}^n||^2 + ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 \right). \end{aligned} \tag{105}$$

Apply estimates (29) and (27), (28) to (105). Then, after simplification, use fact that $\mathbf{u}_t \in L^2(H^1)$ to bound the discrete derivative on the right-hand-side, and apply estimates (29), (30), (31) and (12) to estimate (105) to prove (90).

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