ANALYSIS OF PARTITIONED METHODS FOR BIOT SYSTEM

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Abstract. In this work, we present a comprehensive study of several partitioned methods for the coupling of flow and mechanics. We derive energy estimates for each method for the fully discrete problem. We write the obtained stability conditions in terms of a key control parameter defined as a ratio of the coupling strength and the speed of propagation. Depending on the parameters in the problem, give the choice of the partitioned method which allows the largest time step.

1. Introduction. The problem of predicting the response of an elastic and saturated porous medium occurs in several important applications at large scales (earthquake modeling and simulations) and small scales (blood flow and transport through human tissue). The Biot system, presented in detail in Section 3, captures this coupling. It consists of the equations of flow in a porous medium coupled with the elasticity equations describing the skeleton mechanics. The nature of each problem is significantly different. Namely, while the fluid equations describe dissipative, parabolic effects, the structure equations are hyperbolic. As a natural consequence, numerical algorithms that split the fluid dynamics from the structure mechanics are a popular choice. However, the primary concern of partitioned methods is stability. Indeed, asymptotic stability of the continuous problem arises exactly from the subproblems coupling and any useful partitioning will break this coupling.

To date, a few partitioned methods for the Biot system have been proposed and analyzed. Well-known partitioned algorithms are referred to as the methods of *undrained split*, *fixed stress split*, *drained split*, and *fixed strain split*. Using a von Neumann stability analysis (which yields necessary conditions for stability), the work by Kim et al. [9, 10] shows that the latter two methods exhibit stability issues, while the undrained split and the fixed stress split methods satisfy unconditionally the derived necessary conditions. However, their results do not include energy estimates which give sufficient conditions and account for variable coefficients and non-periodic boundary conditions. A proof of unconditional stability (with convergence rates) for the undrained split and the fixed stress split, based on energy estimates, was given in [15] by Mikelić and Wheeler for the discrete time, continuous space and quasi-static Biot system.

The other primary issue is that the wide range of large and small parameter values occurring in applications determine the strength of this coupling and must arise as time step conditions for the partitioned methods. In this work, we derive the stability results based on energy estimates for the fully-discrete Biot problem. We consider the two methods that are known to exhibit stability issues, the drained split and the fixed strain split, and derive stability conditions based on different parameters in the problem. Moreover, we propose and analyze *several other partitioned methods* for the Biot system with complementary stability properties.

Our analysis shows that the coupled Biot system, while containing many physical parameters, possesses one key control parameter. This control parameter we denote by B has the interpretation

$$B = \text{Coupling Strength}/\text{Speed of Propagation}.$$

The identification and definition of B and comparison of the various partitioned methods in terms of B are given in the final, conclusions section. See especially Table 5.3 in Section 5. This important non-dimensional parameter seems to be previously unidentified, e.g. it is not among the many delineated in Bear and Cheng [2] .

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Section 2 presents the (necessarily intricate) Biot system. In Section 2 we show that with the variables

$$\boldsymbol{U} = (\boldsymbol{\eta}, \boldsymbol{u}, p)^T$$

the Biot system can be written in the form:

$$\mathscr{M}\boldsymbol{U}_t + \mathscr{A}\boldsymbol{U} = \Lambda\boldsymbol{U} + F.$$

Here $\mathscr{M}$ is a positive diagonal matrix. Matrix $\mathscr{A}$ includes all the sub-physics terms of each individual component. All the coupling terms are included in the operator Λ. Under idealized (e.g. periodic) boundary conditions $\mathscr{A}$ is symmetric and positive semi-definite, and Λ is skew symmetric in specially constructed inner product $[\cdot,\cdot]$. However, under the physically correct boundary conditions (considered herein) Λ is not skew symmetric, adding to the complexity of the stability analysis. This has a similar form to the fully evolutionary, coupled Stokes-Darcy problem and partitioned methods for the latter can be adapted, analyzed and tested to the former. The extra analytic and algorithmic complexity is due to the natural boundary conditions. Section 3 gives the partitioned methods and their stability analysis. This analysis is via energy methods and includes the influence of boundary conditions and variable coefficients. Since the system and methods are linear, their errors satisfy the same equations as the method driven by their respective truncation errors. Thus, the partitioned methods' errors are also governed by the methods' stability. Section 4 presents numerical experiments confirming the theoretical predictions of Section 3, followed by conclusions in Section 5.

2. Description of the problem. Consider a deformable porous medium $\Omega(t)$ of reference length L, and reference width H, defined as a mixture of an elastic solid material, called skeleton or matrix, and connecting pores filled with fluid. We describe the dynamics of such a medium by the Biot system, whose Eulerian formulation is given by:

$$\rho\frac{\partial^2\boldsymbol{\eta}}{\partial t^2} - \nabla\cdot\boldsymbol{\sigma}^p = \boldsymbol{f} \qquad \text{in } \Omega(t) \text{ for } t\in(0,T), \tag{2.1a}$$

$$\boldsymbol{\kappa}^{-1}\boldsymbol{q} = -\nabla p \qquad \text{in } \Omega(t) \text{ for } t\in(0,T), \tag{2.1b}$$

$$\frac{\partial}{\partial t}(s_0 p + \alpha\nabla\cdot\boldsymbol{\eta}) + \nabla\cdot\boldsymbol{q} = s \qquad \text{in } \Omega(t) \text{ for } t\in(0,T), \tag{2.1c}$$

where $\boldsymbol{\eta}$ is the displacement of the poroelastic medium, p is the fluid pressure, $\boldsymbol{q}$ is the Darcy velocity, and $\boldsymbol{\sigma}^p$ is the total Cauchy stress tensor

$$\boldsymbol{\sigma}^p = \boldsymbol{\sigma}^E - \alpha p\boldsymbol{I}, \tag{2.2}$$

where $\boldsymbol{\sigma}^E$ denotes the elasticity Cauchy stress tensor. Parameters describing the physics of the problem are the density of the saturated porous medium ρ, a symmetric and positive definite hydraulic conductivity tensor $\boldsymbol{\kappa}$, which is the ratio between the permeability and the fluid viscosity, the storage coefficient s_0, and the Biot-Willis constant α. System (2.1) consists of the momentum equation for the balance of total forces (2.1a), Darcy's law (2.1b), and the storage equation (2.1c) for the fluid mass conservation in the pores of the matrix.

Having in mind applications to geomechanics, where $\boldsymbol{\eta}$ represents the displacement of the porous rock, we can assume that the poroelastic medium undergoes infinitesimal displacements. In that case, movement of $\Omega(t)$ can be neglected, and we can assume that the domain is fixed

$$\Omega(t) = \Omega, \quad \forall t\in(0,T). \tag{2.3}$$

Furthermore, assuming the material is isotropic and homogeneous, we describe the relation of the displacement $\boldsymbol{\eta}$ to the stress tensor $\boldsymbol{\sigma}^E$ via the Saint-Venant Kirchhoff elastic model

$$\boldsymbol{\sigma}^E(\boldsymbol{\eta}) = 2\mu\boldsymbol{E}(\boldsymbol{\eta}) + \lambda\,\mathrm{tr}(\boldsymbol{E}(\boldsymbol{\eta}))\boldsymbol{I}, \tag{2.4}$$

where μ and λ denote Lamé parameters, and due to the hypothesis of infinitesimal deformations,

$$\boldsymbol{E}(\boldsymbol{\eta}) = \frac{1}{2}(\nabla\boldsymbol{\eta} + (\nabla\boldsymbol{\eta})^T).$$

Taking into account these assumptions, and eliminating the Darcy velocity $\boldsymbol{q}$, we can write the system (2.1) as a first order system in the following way:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} - \nabla\cdot\boldsymbol{\sigma}^E + \alpha\nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.5a}$$

$$\rho(\boldsymbol{u} - \frac{\partial\boldsymbol{\eta}}{\partial t}) = 0 \quad \text{in } \Omega, \tag{2.5b}$$

$$\frac{\partial}{\partial t}(s_0 p + \alpha\nabla\cdot\boldsymbol{\eta}) - \nabla\cdot(\boldsymbol{\kappa}\nabla p) = s \quad \text{in } \Omega, \tag{2.5c}$$

where $\boldsymbol{u}$ is the velocity of the skeleton.

Let $\partial\Omega = \Gamma_c \cup \Gamma_s$ and $\partial\Omega = \Gamma_d \cup \Gamma_n$. We assume the following boundary conditions:

$$\boldsymbol{\eta} = \boldsymbol{\eta}_D \quad \text{on } \Gamma_c, \tag{2.6a}$$

$$\boldsymbol{\sigma}^p\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma_s, \tag{2.6b}$$

$$p = 0 \quad \text{on } \Gamma_d, \tag{2.6c}$$

$$\boldsymbol{\kappa}\nabla p\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_n. \tag{2.6d}$$

Condition (2.6c) is called the *drained* boundary condition, and if $\boldsymbol{\eta}_D = 0$, condition (2.6a) is called the *clamped* boundary condition. The above system is supplemented with the following initial conditions:

$$\boldsymbol{\eta}(\cdot,0) = \boldsymbol{\eta}_0 \quad \text{in } \Omega, \tag{2.7a}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \quad \text{in } \Omega, \tag{2.7b}$$

$$p(\cdot,0) = p_0 \quad \text{in } \Omega. \tag{2.7c}$$

Define the following functional spaces

$$X^s = \{\boldsymbol{v} \in (H^1(\Omega))^d |\ \boldsymbol{v} = 0 \text{ on } \Gamma_c\}, \tag{2.8}$$

$$X^p = \{\psi \in H^1(\Omega)|\ \psi = 0 \text{ on } \Gamma_d\}. \tag{2.9}$$

Then the weak formulation of the problem (2.5a)-(2.7c) is given by: given $t \in (0,T)$ find $(\boldsymbol{\eta},\boldsymbol{u},p) \in X^s \times X^s \times X^p$, with $\boldsymbol{\eta} = \boldsymbol{\eta}_D$ on Γ_c, such that for all $(\boldsymbol{v},\boldsymbol{w},\psi) \in X^s \times X^s \times X^p$

$$\rho(\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v}) + a_e(\boldsymbol{\eta},\boldsymbol{v}) - b(\boldsymbol{v},p) = (\boldsymbol{f},\boldsymbol{v}) + \int_{\Gamma_s}\boldsymbol{g}\cdot\boldsymbol{v}d\boldsymbol{x}, \tag{2.10a}$$

$$\rho(\boldsymbol{u} - \frac{\partial\boldsymbol{\eta}}{\partial t},\boldsymbol{w}) = 0, \tag{2.10b}$$

$$s_0(\frac{\partial}{\partial t}p,\psi) + b(\frac{\partial\boldsymbol{\eta}}{\partial t},\psi) + a_p(p,\psi) = (s,\psi) \quad , \tag{2.10c}$$

where $(\cdot,\cdot)$ denotes the inner product associated with $L^2(\Omega)$ norm, and the bilinear forms are defined as follows:

$$a_e(\boldsymbol{\eta},\boldsymbol{v}) = 2\mu(E(\boldsymbol{\eta}),E(\boldsymbol{v})) + \lambda(\nabla\cdot\boldsymbol{\eta},\nabla\cdot\boldsymbol{v}), \tag{2.11a}$$

$$b(\boldsymbol{v},\psi) = \alpha(\psi,\nabla\cdot\boldsymbol{v}), \tag{2.11b}$$

$$a_p(p,\psi) = (\boldsymbol{\kappa}\nabla p,\nabla\psi). \tag{2.11c}$$

Define $\mathscr{E}$ to be the sum of the kinetic and elastic energy of the poroelastic medium

$$\mathscr{E} = \frac{\rho}{2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \mu\|E(\boldsymbol{\eta})\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla\cdot\boldsymbol{\eta}\|^2_{L^2(\Omega)} + \frac{s_0}{2}\|p\|^2_{L^2(\Omega)}. \tag{2.12}$$

PROPOSITION 2.1. *A weak solution of the Biot system satisfies the energy equality*

$$\mathscr{E}(t) + 2\int_0^t \|\boldsymbol{\kappa}^{1/2}\nabla p(t')\|^2_{L^2(\Omega)}dt' = \mathscr{E}(0) + 2\int_0^t \left[(\boldsymbol{f}, \frac{\partial\boldsymbol{\eta}}{\partial t}) + (s,p) + \int_{\Gamma_s}\boldsymbol{g}\cdot\frac{\partial\boldsymbol{\eta}}{\partial t}d\boldsymbol{x}\right]dt', \tag{2.13}$$

Proof. Take $(\boldsymbol{v},\boldsymbol{w},\psi) = (\frac{\partial\boldsymbol{\eta}}{\partial t}, \frac{\partial\boldsymbol{u}}{\partial t}, p)$, and add the equations (2.11a)-(2.11c). We then have the monolithic energy satisfying

$$\frac{1}{2}\frac{d}{dt}\mathscr{E} + \|\boldsymbol{\kappa}^{1/2}\nabla p\|^2_{L^2(\Omega)} = (\boldsymbol{f}, \frac{\partial\boldsymbol{\eta}}{\partial t}) + \int_{\Gamma_s}\boldsymbol{g}\cdot\frac{\partial\boldsymbol{\eta}}{\partial t}d\boldsymbol{x} + (s,p), \tag{2.14}$$

from which the result follows. □
This exact energy equality is a stronger result then stability (in the weak sense).

2.1. Structure of the coupled system. Consider the Biot problem (2.5a)-(2.5c). Let the triple of unknowns be denoted by

$$\boldsymbol{U} = (\boldsymbol{\eta}, \boldsymbol{u}, p)^T.$$

Define (via the Riesz representation theorem) operator $A_E : X^s \to (X^s)^*$ and the symmetric, positive definite operator $A_D : X^p \to (X^p)^*$ by

$$A_E\eta := -\nabla\cdot(2\mu\boldsymbol{E}(\boldsymbol{\eta})) + \lambda\nabla(\nabla\cdot\boldsymbol{\eta}), \text{ and } A_D p := -\nabla\cdot(\kappa\nabla p),$$

and define the linear operators on the product space

$$\mathscr{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & s_0 \end{bmatrix}, \mathscr{A} = \begin{bmatrix} 0 & -I & 0 \\ A_E & 0 & 0 \\ 0 & 0 & A_D \end{bmatrix}, \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha\nabla \\ 0 & -\alpha\nabla\cdot & 0 \end{bmatrix}. \tag{2.15}$$

Then the system can be written as

$$\mathscr{M}\boldsymbol{U}_t + \mathscr{A}\boldsymbol{U} = \Lambda\boldsymbol{U} + F.$$

It is easy to show the following property:
PROPOSITION 2.2. *Assume $\boldsymbol{\eta}_D = 0$ and $\boldsymbol{\sigma}^E\boldsymbol{n} = 0$ on Γ_s. Then, operator A_E is symmetric and positive definite.*

We define a special inner product on the product space (adapted to the Biot equations) as follows
DEFINITION 2.3. *For all $\boldsymbol{U} = (\boldsymbol{\eta}, \boldsymbol{u}, p)^T$, $\boldsymbol{V} = (\boldsymbol{\gamma}, \boldsymbol{v}, q)^T \in X^s \times X^s \times X^p$*

$$\begin{aligned}[\boldsymbol{U},\boldsymbol{V}] \;:&= (A_E\boldsymbol{\eta},\boldsymbol{\gamma}) + (\boldsymbol{u},\boldsymbol{v}) + (p,q) = \\ &= \int_\Omega (2\mu E(\boldsymbol{\eta}) : E(\boldsymbol{\gamma}) + \lambda(\nabla\cdot\boldsymbol{\eta})(\nabla\cdot\boldsymbol{\gamma}) + \boldsymbol{u}\cdot\boldsymbol{v} + pq)\,d\boldsymbol{x}.\end{aligned}$$

The following properties hold.
PROPOSITION 2.4 (Coercivity and skew-symmetry). *For all $\boldsymbol{U} = (\boldsymbol{\eta}, \boldsymbol{u}, p)^T, \boldsymbol{V} = (\boldsymbol{\gamma}, \boldsymbol{v}, q)^T \in X^s \times X^s \times X^p$, we have*

$$[\mathscr{A}\boldsymbol{U},\boldsymbol{U}] = -\int_{\Gamma_s}\boldsymbol{\sigma}^E(\boldsymbol{u})\boldsymbol{n}\cdot\boldsymbol{\eta}dx + \int_{\Gamma_s}\boldsymbol{\sigma}^E(\boldsymbol{\eta})\boldsymbol{n}\cdot\boldsymbol{u}dx + \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p\|^2,$$

$$[\Lambda \boldsymbol{U}, \boldsymbol{V}] = -[\boldsymbol{U}, \Lambda \boldsymbol{V}] - \alpha \int_{\Gamma_n} p \boldsymbol{v} \cdot \boldsymbol{n} dx - \alpha \int_{\Gamma_n} q \boldsymbol{u} \cdot \boldsymbol{n} dx.$$

If $\boldsymbol{\sigma}^E \boldsymbol{n} = 0$ on Γ_s, then $\mathscr{A}$ is coercive. If $p = 0$ on Γ_n or $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_n, then Λ is skew symmetric.

Proof. We calculate

$$\begin{aligned}
&[\mathscr{A}\boldsymbol{U}, \boldsymbol{U}] = (A_E(-\boldsymbol{u}), \boldsymbol{\eta}) + (A_E \boldsymbol{\eta}, \boldsymbol{u}) + (\kappa \nabla p, \nabla p) \\
&= \int_\Omega \boldsymbol{\sigma}^E(\boldsymbol{u}) : \nabla \boldsymbol{\eta} d\boldsymbol{x} - \int_{\Gamma_s} \boldsymbol{\sigma}^E(\boldsymbol{u}) \boldsymbol{n} \cdot \boldsymbol{\eta} dx + (A_E \boldsymbol{\eta}, \boldsymbol{u}) + (\kappa \nabla p, \nabla p) \\
&= -(A_E \boldsymbol{\eta}, \boldsymbol{u}) - \int_{\Gamma_s} \boldsymbol{\sigma}^E(\boldsymbol{u}) \boldsymbol{n} \cdot \boldsymbol{\eta} dx + \int_{\Gamma_s} \boldsymbol{\sigma}^E(\boldsymbol{\eta}) \boldsymbol{n} \cdot \boldsymbol{u} dx + (A_E \boldsymbol{\eta}, \boldsymbol{u}) + (\kappa \nabla p, \nabla p).
\end{aligned}$$

If $\boldsymbol{\sigma}^E \boldsymbol{n} = 0$ on Γ_s, then we have

$$[\mathscr{A}\boldsymbol{U}, \boldsymbol{U}] = \|\kappa^{\frac{1}{2}} \nabla p\|^2 \geq 0.$$

For the second claim we have, using integration by parts,

$$\begin{aligned}
[\Lambda \boldsymbol{U}, \boldsymbol{V}] &= 0 + (-\alpha \nabla p, \boldsymbol{v}) + (-\alpha \nabla \cdot \boldsymbol{u}, q) \\
&= (p, \alpha \nabla \cdot \boldsymbol{v}) + (\boldsymbol{u}, \alpha \nabla q) - \alpha \int_{\Gamma_n} p \boldsymbol{v} \cdot \boldsymbol{n} dx - \alpha \int_{\Gamma_n} q \boldsymbol{u} \cdot \boldsymbol{n} dx \\
&= -[\boldsymbol{U}, \Lambda \boldsymbol{V}] - \alpha \int_{\Gamma_n} p \boldsymbol{v} \cdot \boldsymbol{n} dx - \alpha \int_{\Gamma_n} q \boldsymbol{u} \cdot \boldsymbol{n} dx.
\end{aligned}$$

If $p = 0$ or $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_n, then we have

$$[\Lambda \boldsymbol{U}, \boldsymbol{V}] = -[\boldsymbol{U}, \Lambda \boldsymbol{V}].$$

□

REMARK 1. *Partitioned methods for the Biot system are based on this formulation. Partitioning is achieved by treating implicitly the sub-physics terms, collected in the operator $\mathscr{A}$, and treating explicitly the coupling terms in Λ (either in parallel as in IMEX methods or sequentially implicitly as in splitting methods). Often, after a partitioned method is formulated, the intermediate variable $\boldsymbol{u} = \partial_t \boldsymbol{\eta}$ can be eliminated and the method stated in an equivalent form in the variables $\boldsymbol{\eta}, p$. Among the methods we explore are ones adapted from the Stokes-Darcy coupled system to the Biot system, and include BEFE [16, 18, 19, 1], see also [3] for other applications, BELF [12, 8], CNLF [11, 13], and ω-method [21, 20, 7].*

3. Partitioned numerical methods and the stability analysis. To discretize the Biot system in space, we use the finite element method, where the finite element spaces are denoted by

$$X_h^s \subset X^s, \text{ and } X_h^p \subset X^p,$$

based on a conforming FEM triangulation in Ω with maximum triangle diameter h. We assume that the mesh is such that the finite element spaces satisfy the usual inverse inequality

$$\|\nabla \boldsymbol{v}_h\| \leq C_{INV} h^{-1} \|\boldsymbol{v}_h\| \quad \forall \boldsymbol{v}_h \in X_h^{s/p}, \tag{3.1}$$

where C_{INV} depends on the element aspect ratio in the triangulation. We will make use of the following inequalities.

Poincaré - Friedrichs inequality:

$$\|\boldsymbol{v}\|_{L^2(\Omega)} \leq C_{PF} \left(\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} + \|\boldsymbol{v}\|_{L^2(\Gamma_D)} \right) \quad \forall \boldsymbol{v} \in X^{s/p}. \tag{3.2}$$

To reduce the volume of analysis we shall take $\eta_D = 0$, eliminating the second term on the right-hand side of (3.2). Provided $\eta_D \in L^2(\Gamma_D)$ the results easily extend to nonzero boundary conditions for all the methods.

Korn inequality:

$$\|\nabla \boldsymbol{v}\| \le C_K \|E(\boldsymbol{v})\| \quad \forall \boldsymbol{v} \in X^s. \tag{3.3}$$

Further note that

$$\|\nabla\cdot \boldsymbol{v}\| \le \sqrt{d}\|\nabla \boldsymbol{v}\| \quad \forall \boldsymbol{v} \in X^s, \tag{3.4}$$

d is the dimension of the space ($d \in \{2,3\}$). The various constants C_{PF}, C_T and C_K depend on the domain Ω. It will be useful to introduce the notation for the sum of discrete kinetic and elastic energy of the discrete Biot system:

$$\mathscr{E}^n = \frac{\rho}{2}\|\boldsymbol{u}_h^n\|^2_{L^2(\Omega)} + \mu\|E(\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla\cdot\boldsymbol{\eta}_h^n\|^2_{L^2(\Omega)} + \frac{s_0}{2}\|p_h^n\|^2_{L^2(\Omega)}. \tag{3.5}$$

We start by derivation of precise stability conditions for the classical drained split and fixed strain split partitioned strategies. After that, we propose other partitioned methods and give stability conditions for each method.

3.1. The drained split. *The drained split* method consists of solving the mechanics problem first, with the value of pressure given from the previous time step. After that, the flow problem is solved using the new values of the displacement. This method is known to have stability issues [9]. Here we derive a *sufficient* condition on model parameters under which this method is stable.

The discretization in time is done using the Backward Euler method, resulting in the following discrete problem: Given $t\in(0,T)$ and $(\boldsymbol{\eta}_h^n, \boldsymbol{u}_h^n, p_h^n)$, find $(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in X_h^s\times X_h^s\times X_h^p$, with $\boldsymbol{\eta}_h^{n+1} = \mathbf{0}$ on Γ_c, such that for all $(\boldsymbol{v}_h, \boldsymbol{w}_h, \psi_h) \in X_h^s\times X_h^s\times X_h^p$

$$\rho(\frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v}_h) + a_e(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{v}_h) = b(\boldsymbol{v}_h, p_h^n) + (\boldsymbol{f}^{n+1}, \boldsymbol{v}_h) + \int_{\Gamma_s} \boldsymbol{g}^{n+1}\cdot \boldsymbol{v}_h d\boldsymbol{x}, \tag{3.6a}$$

$$\rho(\frac{\boldsymbol{u}_h^{n+1}+\boldsymbol{u}_h^n}{2} - \frac{\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n}{\Delta t}, \boldsymbol{w}_h) = 0, \tag{3.6b}$$

$$s_0(\frac{p_h^{n+1}-p_h^n}{\Delta t}, \psi_h) + a_p(p_h^{n+1}, \psi_h) = -b(\frac{\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n}{\Delta t}, \psi_h) + (s^{n+1}, \psi_h). \tag{3.6c}$$

THEOREM 3.1. *Let $(\boldsymbol{\eta}_h^n, \boldsymbol{u}_h^n, p_h^n)_{0\le n\le N}$ be the solution of (3.6). Then under the condition*

$$\frac{\alpha^2}{\lambda s_0} < 1 \tag{3.7}$$

the following estimate holds:

$$\begin{aligned}
&\mathscr{E}^N + \frac{\mu}{2}\sum_{n=0}^{N-1}\|E(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{1}{2}(\lambda - \frac{\alpha^2}{s_0(1-\varepsilon)})\sum_{n=0}^{N-1}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{\varepsilon s_0}{2}\sum_{n=0}^{N-1}\|p_h^{n+1}-p_h^n\|^2_{L^2(\Omega)} \\
&\quad + \frac{\Delta t}{2}\sum_{n=0}^{N-1}\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^{n+1}\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\sqrt{(1-\varepsilon)s_0}\|p_h^{n+1}-p_h^n\|_{L^2(\Omega)} - \frac{\alpha}{\sqrt{(1-\varepsilon)s_0}}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|\Big)^2 \\
&\le \mathscr{E}^0 + \frac{C_{PF}^2C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{f}^{n+1}\|^2_{L^2(\Omega)} + \frac{\Delta t C_{PF}^2}{2k_{min}}\sum_{n=0}^{N-1}\|s^{n+1}\|^2_{L^2(\Omega)} + \frac{C_T^2C_{PF}C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{g}^{n+1}\|^2_{L^2(\Gamma_s)}.
\end{aligned} \tag{3.8}$$

Proof. To prove the energy estimate, we test the problem (3.6) with

$$(\boldsymbol{v}_h, \boldsymbol{w}_h, \psi_h) = (\frac{\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n}{\Delta t}, \frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n}{\Delta t}, p_h^{n+1}).$$

Then, after multiplying by Δt, and adding the equations (3.6a)-(3.6c), we get

$$\mathscr{E}^{n+1} + \mu \|E(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{s_0}{2}\|p_h^{n+1} - p_h^n\|^2_{L^2(\Omega)} + \Delta t \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^{n+1}\|^2_{L^2(\Omega)}$$
$$= \mathscr{E}^n - b(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n, p_h^{n+1} - p_h^n) + (\boldsymbol{f}^{n+1}, \boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n) + \int_{\Gamma_s} \boldsymbol{g}^{n+1}\cdot(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n) d\boldsymbol{x} + \Delta t(s, p_h^{n+1}).$$

To estimate the coupling term, we use (2.11b), Cauchy-Schwarz and the polarized identity

$$|b(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n, p_h^{n+1} - p_h^n)| \leq \frac{(1-\varepsilon)s_0}{2}\|p_h^{n+1} - p_h^n\|^2_{L^2(\Omega)} + \frac{\alpha^2}{2s_0(1-\varepsilon)}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)}$$
$$- \frac{1}{2}\Big(\sqrt{(1-\varepsilon)s_0}\|p_h^{n+1} - p_h^n\|_{L^2(\Omega)} - \frac{\alpha}{\sqrt{(1-\varepsilon)s_0}}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n)\|\Big)^2.$$

Stability follows provided

$$\frac{\alpha^2}{\lambda s_0} \leq 1 - \varepsilon.$$

The right-hand side is bounded in a standard way:

$$\Delta t(s^{n+1}, p_h^{n+1}) \leq \frac{\Delta t}{2}\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^{n+1}\|^2_{L^2(\Omega)} + \frac{\Delta t C_{PF}^2}{2k_{min}}\|s^{n+1}\|^2_{L^2(\Omega)}, \tag{3.9}$$

$$(\boldsymbol{f}^{n+1}, \boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n) \leq \frac{C_{PF}^2 C_K^2}{\mu}\|\boldsymbol{f}^{n+1}\|^2_{L^2(\Omega)} + \frac{\mu}{4}\|E(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)}, \tag{3.10}$$

$$\int_{\Gamma_s} \boldsymbol{g}^{n+1}\cdot(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n) d\boldsymbol{x} \leq \frac{C_T^2 C_{PF} C_K^2}{\mu}\|\boldsymbol{g}^{n+1}\|^2_{L^2(\Gamma_s)} + \frac{\mu}{4}\|E(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)}. \tag{3.11}$$

Summing over $0 \leq n \leq N$, we prove the stated estimate. □

3.2. The fixed strain split. *The fixed strain split* method consists of solving the flow problem first, with the value of the rate of displacement given from the previous time step. After that, the computed value of pressure is used to load the mechanics problem. This method is known to also have stability issues [10]. Here we derive a sufficient condition on model parameters or, alternatively, on the time step under which this method is conditionally stable.

The discretization in time is done using the Backward Euler method, resulting in the following discrete problem: Given $t \in (0,T)$ and $(\boldsymbol{\eta}_h^n, \boldsymbol{u}_h^n, p_h^n)$, find $(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in X_h^s \times X_h^s \times X_h^p$, with $\boldsymbol{\eta}_h^{n+1} = \boldsymbol{0}$ on Γ_c, such that for all $(\boldsymbol{v}_h, \boldsymbol{w}_h, \psi_h) \in X_h^s \times X_h^s \times X_h^p$

$$s_0\Big(\frac{p_h^{n+1} - p_h^n}{\Delta t}, \psi_h\Big) + a_p(p_h^{n+1}, \psi_h) = -b\Big(\frac{\boldsymbol{\eta}_h^n - \boldsymbol{\eta}_h^{n-1}}{\Delta t}, \psi_h\Big) + (s^{n+1}, \psi_h), \tag{3.12a}$$

$$\rho\Big(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v}_h\Big) + a_e(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{v}_h) = b(\boldsymbol{v}_h, p_h^{n+1}) + (\boldsymbol{f}^{n+1}, \boldsymbol{v}_h) + \int_{\Gamma_s} \boldsymbol{g}^{n+1}\cdot\boldsymbol{v}_h d\boldsymbol{x}, \tag{3.12b}$$

$$\rho\Big(\boldsymbol{u}_h^{n+1} - \frac{\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n}{\Delta t}, \boldsymbol{w}_h\Big) = 0. \tag{3.12c}$$

THEOREM 3.2. *Let $(\boldsymbol{\eta}_h^n, \boldsymbol{u}_h^n, p_h^n)_{0\leq n\leq N}$ be the solution of (3.12), and $\mathscr{E}^n$ defined as in (3.5). Assume either the problem parameters satisfy the condition*

$$\frac{\alpha^2}{\lambda s_0} < 1 \tag{3.13}$$

or Δt satisfies the time step condition

$$\Delta t < \frac{2\rho k_{min}}{\alpha^2 d C_{INV}^2 C_{PF}^2} h^2. \tag{3.14}$$

Then, the fixed strain split method is stable in time. If the condition (3.13) *on the problem parameters holds, we have*

$$\begin{aligned}
&\frac{\rho}{2}\|\boldsymbol{u}_h^N\|^2_{L^2(\Omega)} + \mu\|E(\boldsymbol{\eta}_h^N)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla\cdot\boldsymbol{\eta}_h^N\|^2_{L^2(\Omega)} + \frac{\varepsilon s_0}{2}\|p_h^N\|^2_{L^2(\Omega)} + \frac{\mu}{2}\sum_{n=0}^{N-1}\|E(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\big(\lambda - \frac{\alpha^2}{s_0(1-\varepsilon)}\big)\|\nabla\cdot(\boldsymbol{\eta}_h^N - \boldsymbol{\eta}_h^{N-1})\|^2_{L^2(\Omega)} + \frac{1}{2}\big(\lambda - \frac{\alpha^2}{s_0(1-\varepsilon)}\big)\sum_{n=0}^{N-1}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{\varepsilon s_0}{2}\sum_{n=0}^{N-1}\|p_h^{n+1} - p_h^n\|^2_{L^2(\Omega)} \\
&+\varepsilon\Delta t\sum_{n=0}^{N-1}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\frac{\alpha}{\sqrt{s_0(1-\varepsilon)}}\|\nabla\cdot(\eta^n - \eta^{n-1})\|_{L^2(\Omega)} - \sqrt{s_0(1-\varepsilon)}\|p^{n+1} - p^n\|_{L^2(\Omega)}\Big)^2 \\
&\le \mathscr{E}^0 + \frac{\lambda+\alpha^2}{2}\|\nabla\cdot(\boldsymbol{\eta}_h^1 - \boldsymbol{\eta}_h^0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|p_h^1\|^2_{L^2(\Omega)} \\
&\quad + \frac{C_{PF}^2 C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{f}^{n+1}\|^2_{L^2(\Omega)} + \frac{\Delta t C_{PF}^2}{4k_{min}(1-\varepsilon)}\sum_{n=0}^{N-1}\|s^{n+1}\|^2_{L^2(\Omega)} + \frac{C_T^2 C_{PF} C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{g}^{n+1}\|^2_{L^2(\Gamma_s)}.
\end{aligned} \tag{3.15}$$

Otherwise, if the time-step condition (3.14) *holds, we have the following estimate*

$$\begin{aligned}
&\mathscr{E}^N + \frac{\mu}{2}\sum_{n=0}^{N-1}\|E(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\sum_{n=0}^{N-1}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{s_0}{2}\sum_{n=0}^{N-1}\|p_h^{n+1}-p_h^n\|^2_{L^2(\Omega)} \\
&+\left(\frac{\rho}{2} - \frac{\Delta t\alpha^2 d C_{INV}^2 C_{PF}^2}{4h^2 k_{min}(1-\varepsilon)}\right)\sum_{n=0}^{N-1}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|^2_{L^2(\Omega)} + \varepsilon\Delta t\sum_{n=0}^{N-1}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|^2_{L^2(\Omega)} \\
&+\Delta t\sum_{n=0}^{N-1}\Big(\frac{\alpha\sqrt{d}C_{INV}C_{PF}}{2h\sqrt{k_{min}(1-\varepsilon)}}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)} - \sqrt{1-\varepsilon}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|_{L^2(\Omega)}\Big)^2 \\
&\quad\le \mathscr{E}^0 + \frac{C_{PF}^2 C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{f}^{n+1}\|^2_{L^2(\Omega)} + \frac{\Delta t C_{PF}^2}{4k_{min}(1-\varepsilon)}\sum_{n=0}^{N-1}\|s^{n+1}\|^2_{L^2(\Omega)} + \frac{C_T^2 C_{PF} C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{g}^{n+1}\|^2_{L^2(\Gamma_s)}.
\end{aligned} \tag{3.16}$$

Proof. To prove the energy estimate (3.15), we test the problem (3.12) with

$$(\boldsymbol{v}_h, \boldsymbol{w}_h, \psi_h) = (\frac{\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n}{\Delta t}, \frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n}{\Delta t}, p_h^{n+1}).$$

Then, after multiplying by Δt, and adding the equations (3.12a)-(3.12c), we get

$$\begin{aligned}
&\mathscr{E}^{n+1} + \mu\|E(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)} + \frac{s_0}{2}\|p_h^{n+1}-p_h^n\|^2_{L^2(\Omega)} \\
&\quad + \Delta t\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|^2_{L^2(\Omega)} + \frac{\rho}{2}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|^2_{L^2(\Omega)} \\
&= \mathscr{E}^n + b((\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)-(\boldsymbol{\eta}_h^n-\boldsymbol{\eta}_h^{n-1}), p_h^{n+1}) + (\boldsymbol{f}^{n+1}, \boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n) + \int_{\Gamma_s}\boldsymbol{g}^{n+1}\cdot(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)d\boldsymbol{x} + \Delta t(s, p_h^{n+1}).
\end{aligned} \tag{3.17}$$

We write the coupling term in the following way:

$$b((\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)-(\boldsymbol{\eta}_h^n-\boldsymbol{\eta}_h^{n-1}), p_h^{n+1}) = b(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n, p_h^{n+1}) - b(\boldsymbol{\eta}_h^n-\boldsymbol{\eta}_h^{n-1}, p_h^n) - b(\boldsymbol{\eta}_h^n-\boldsymbol{\eta}_h^{n-1}, p_h^{n+1}-p_h^n). \tag{3.18}$$

Furthermore, by adding and subtracting the term $\frac{\lambda}{2}\|\nabla\cdot(\boldsymbol{\eta}_h^n-\boldsymbol{\eta}_h^{n-1})\|^2_{L^2(\Omega)}$ from the left-hand side, we have

$$\begin{aligned}
&\mathscr{E}^{n+1}+\mathscr{C}^{n+1}+\mu\|\boldsymbol{E}(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)}+\frac{\lambda}{2}\|\nabla\cdot(\boldsymbol{\eta}_h^n-\boldsymbol{\eta}_h^{n-1})\|^2_{L^2(\Omega)}+\frac{s_0}{2}\|p_h^{n+1}-p_h^n\|^2_{L^2(\Omega)}\\
&\quad+\Delta t\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|^2_{L^2(\Omega)}+\frac{\rho}{2}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|^2_{L^2(\Omega)}\\
&=\mathscr{E}^n+\mathscr{C}^n-b(\boldsymbol{\eta}_h^n-\boldsymbol{\eta}_h^{n-1},p_h^{n+1}-p_h^n)+(\boldsymbol{f}^{n+1},\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)+\int_{\Gamma_s}\boldsymbol{g}^{n+1}\cdot(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)d\boldsymbol{x}+\Delta t(s,p_h^{n+1}),
\end{aligned}$$

where $\mathscr{C}^{n+1}=\frac{\lambda}{2}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)}-b(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n,p_h^{n+1})$. Now, using (2.11b), Cauchy-Schwarz and the polarized identity, we have

$$\begin{aligned}
|b(\boldsymbol{\eta}_h^n-\boldsymbol{\eta}_h^{n-1},p_h^{n+1}-p_h^n)|\le{}&\frac{\alpha^2}{2s_0(1-\varepsilon)}\|\nabla\cdot(\eta^n-\eta^{n-1})\|^2_{L^2(\Omega)}+\frac{s_0(1-\varepsilon)}{2}\|p^{n+1}-p^n\|^2_{L^2(\Omega)}\\
&-\frac{1}{2}\Big(\frac{\alpha}{\sqrt{s_0(1-\varepsilon)}}\|\nabla\cdot(\eta^n-\eta^{n-1})\|_{L^2(\Omega)}-\sqrt{s_0(1-\varepsilon)}\|p^{n+1}-p^n\|_{L^2(\Omega)}\Big)^2.
\end{aligned}$$

We bound the right hand side as in (3.9)-(3.11). Summing over $0\le n\le N-1$, we have

$$\begin{aligned}
&\mathscr{E}^N+\mathscr{C}^N+\frac{\mu}{2}\sum_{n=0}^{N-1}\|E(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)}+(\frac{\lambda}{2}-\frac{\alpha^2}{2s_0(1-\varepsilon)})\sum_{n=0}^{N-1}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|^2_{L^2(\Omega)}+\frac{\varepsilon s_0}{2}\sum_{n=0}^{N-1}\|p_h^{n+1}-p_h^n\|^2_{L^2(\Omega)}\\
&\quad+\frac{\rho}{2}\sum_{n=0}^{N-1}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|^2_{L^2(\Omega)}+\varepsilon\Delta t\sum_{n=0}^{N-1}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|^2_{L^2(\Omega)}\\
&\quad+\frac{1}{2}\sum_{n=0}^{N-1}\Big(\frac{\alpha}{\sqrt{s_0(1-\varepsilon)}}\|\nabla\cdot(\eta^n-\eta^{n-1})\|_{L^2(\Omega)}-\sqrt{s_0(1-\varepsilon)}\|p^{n+1}-p^n\|_{L^2(\Omega)}\Big)^2\\
&\le\mathscr{E}^0+\mathscr{C}^0+\frac{C_{PF}^2C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{f}^{n+1}\|^2_{L^2(\Omega)}+\frac{\Delta t C_{PF}^2}{4k_{min}(1-\varepsilon)}\sum_{n=0}^{N-1}\|s^{n+1}\|^2_{L^2(\Omega)}+\frac{C_T^2C_{PF}C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{g}^{n+1}\|^2_{L^2(\Gamma_s)}.
\end{aligned}$$

Stability and stated energy inequality thus follows provided

$$\frac{\alpha^2}{\lambda s_0}\le 1-\varepsilon\quad\text{and}\quad\mathscr{E}^N+\mathscr{C}^N\ge 0,\quad\text{for every } N.$$

Thus, using Young's inequality,

$$b(\boldsymbol{\eta}_h^N-\boldsymbol{\eta}_h^{N-1},p_h^N)\le\frac{\alpha^2}{2s_0(1-\varepsilon)}\|\nabla\cdot(\boldsymbol{\eta}_h^N-\boldsymbol{\eta}_h^{N-1})\|^2_{L^2(\Omega)}+\frac{s_0(1-\varepsilon)}{2}\|p_h^N\|^2_{L^2(\Omega)},$$

we have

$$\begin{aligned}
\mathscr{E}^N+\mathscr{C}^N\ge{}&\frac{\rho}{2}\|\boldsymbol{u}_h^N\|^2_{L^2(\Omega)}+\mu\|E(\boldsymbol{\eta}_h^N)\|^2_{L^2(\Omega)}+\frac{\lambda}{2}\|\nabla\cdot\boldsymbol{\eta}_h^N\|^2_{L^2(\Omega)}+\frac{\varepsilon s_0}{2}\|p_h^N\|^2_{L^2(\Omega)}\\
&+(\frac{\lambda}{2}-\frac{\alpha^2}{2s_0(1-\varepsilon)})\|\nabla\cdot(\boldsymbol{\eta}_h^N-\boldsymbol{\eta}_h^{N-1})\|^2_{L^2(\Omega)}\ge 0,
\end{aligned}$$

if the problem parameters condition (3.15) holds.

To prove the energy estimate (3.16), we handle the coupling term in equation (3.17) in the following way. First, note that from equation (3.12c) we have

$$\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n=\Delta t\boldsymbol{u}_h^{n+1}.$$

Thus, we can write the coupling term as

$$b((\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)-(\boldsymbol{\eta}_h^n-\boldsymbol{\eta}_h^{n-1}),p_h^{n+1})=\Delta t\, b(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n,p_h^{n+1}).$$

From here, using the polarized identity, divergence inequality, and Poincaré-Friedrichs inequality, we have

$$\begin{aligned}\Delta t\, b(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n,p_h^{n+1})\le&\frac{\Delta t\,\alpha^2 dC_{INV}^2C_{PF}^2}{4h^2k_{min}(1-\varepsilon)}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2+(1-\varepsilon)\Delta t\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|_{L^2(\Omega)}^2\\&-\Delta t\Big(\frac{\alpha\sqrt{d}C_{INV}C_{PF}}{2h\sqrt{k_{min}(1-\varepsilon)}}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)}-\sqrt{1-\varepsilon}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|_{L^2(\Omega)}\Big)^2.\end{aligned}$$

Bounding the right-hand side similar to (3.9)-(3.11) and summing over $0\le n\le N-1$, we prove the desired estimate. ∎

In addition to the classical schemes which sequentially decouple the system, we propose several partitioned schemes in which the partitioning is performed so that the mechanics problem can be solved at the same time as the flow problem in parallel.

3.3. Backward Euler-Forward Euler (BEFE). Let $t_n=n\Delta t$ and let superscripts denote the time level of the approximation. The BEFE partitioned approximations are: Given $t\in(0,T)$ and $(\boldsymbol{\eta}_h^n,\boldsymbol{u}_h^n,p_h^n)$, find $(\boldsymbol{\eta}_h^{n+1},\boldsymbol{u}_h^{n+1},p_h^{n+1})\in X_h^s\times X_h^s\times X_h^p$, with $\boldsymbol{\eta}_h^{n+1}=\boldsymbol{0}$ on Γ_c, such that for all $(\boldsymbol{v}_h,\boldsymbol{w}_h,\psi_h)\in X_h^s\times X_h^s\times X_h^p$

$$\rho(\frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n}{\Delta t},\boldsymbol{v}_h)+a_e(\boldsymbol{\eta}_h^{n+1},\boldsymbol{v}_h)-b(\boldsymbol{v}_h,p_h^n)=(\boldsymbol{f}^{n+1},\boldsymbol{v}_h)+\int_{\Gamma_s}\boldsymbol{g}^{n+1}\cdot\boldsymbol{v}_h d\boldsymbol{x}\quad \text{in }\Omega, \tag{3.19a}$$

$$\rho(\boldsymbol{u}_h^{n+1}-\frac{\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n}{\Delta t},\boldsymbol{w}_h)=0\quad \text{in }\Omega, \tag{3.19b}$$

$$s_0(\frac{p_h^{n+1}-p_h^n}{\Delta t},\psi_h)+b(\boldsymbol{u}_h^n,\psi_h)+a_p(p_h^{n+1},\psi_h)=(s^{n+1},\psi_h)\quad \text{in }\Omega. \tag{3.19c}$$

Note that this method differs from the drained split because in both equations the coupling terms are evaluated at the previous time step, leading to a scheme where the fluid and structure problems can be solved in parallel.

LEMMA 3.3. *Suppose* $(\boldsymbol{\eta}_h^{n+1},\boldsymbol{u}_h^{n+1},p_h^{n+1})$ *is solution to* (3.19). *Then* $\boldsymbol{u}_h^{n+1}=\frac{\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n}{\Delta t}$.

Proof. Let $\boldsymbol{w}_h=\boldsymbol{u}_h^{n+1}-\frac{\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n}{\Delta t}$ in (3.19b), we have

$$\left\|\boldsymbol{u}_h^{n+1}-\frac{\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n}{\Delta t}\right\|_{L^2(\Omega)}^2=0,$$

and the assertion follows. ∎

DEFINITION 3.4. *Define, via the Riesz representation theorem, the linear mapping* A_E^h *from* X_h^s *to* X_h^s *satisfying*

$$(\boldsymbol{u}_h,A_E^h\boldsymbol{\eta}_h)=-a_e(\boldsymbol{\eta}_h,\boldsymbol{u}_h)\ \ \forall\boldsymbol{\eta}_h,\,\boldsymbol{u}_h\in X_h^s. \tag{3.20}$$

We prove stability under two alternative conditions. The second is a realization (for this specific application) of results in [1], [8] that BEFE can be unconditionally stable if the part treated implicitly is larger than the components treated explicitly.

THEOREM 3.5. *Assume that we have either*

$$\frac{\alpha^2}{\lambda s_0}<1\quad \textit{and}\quad \Delta t\le\frac{\rho k_{min}}{\alpha^2 dC_{INV}^2C_{PF}^2}h^2, \tag{3.21}$$

or

$$\Delta t \le \min\left\{\frac{\rho k_{min}}{4\alpha^2 d C_{INV}^2 C_{PF}^2} h^2, \frac{\sqrt{\rho s_0}}{\alpha C_{INV}\sqrt{d}} h\right\}. \tag{3.22}$$

Then, BEFE method (3.19) *is stable. In particular, if* (3.21) *occurs, we have*

$$\begin{aligned}
&\mathscr{E}^N + \frac{\varepsilon\rho}{2}\sum_{n=0}^{N-1}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2 + \frac{\mu}{2}\sum_{n=0}^{N-1}\|\boldsymbol{E}(\boldsymbol{\eta}_h^{n+1}-\eta_h^n)\|_{L^2(\Omega)}^2 + \Big(\frac{\lambda}{2}-\frac{\alpha^2}{2s_0(1-\varepsilon)}\Big)\sum_{n=0}^{N-1}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1}-\eta_h^n)\|_{L^2(\Omega)}^2 \\
&\quad + \frac{\varepsilon s_0}{2}\sum_{n=0}^{N-1}\|p_h^{n+1}-p_h^n\|_{L^2(\Omega)}^2 + \Delta t\sum_{n=0}^{N-1}\Big(\frac{1-\varepsilon}{2}-\frac{\Delta t\alpha^2 C_{PF}^2 C_{INV}^2 d}{2\rho h^2 k_{min}(1-\varepsilon)}\Big)\|\boldsymbol{\kappa}^{1/2}p_h^{n+1}\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\sqrt{(1-\varepsilon)\rho}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)} - \frac{\Delta t\alpha C_{PF}C_{INV}\sqrt{d}}{\sqrt{\rho k_{min}(1-\varepsilon)}h}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|_{L^2(\Omega)}\Big)^2 \\
&\quad + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\frac{\Delta t\alpha}{\sqrt{s_0(1-\varepsilon)}}\|\nabla\cdot\boldsymbol{u}_h^{n+1}\|_{L^2(\Omega)} - \sqrt{s_0(1-\varepsilon)}\|p_h^{n+1}-p_h^n\|_{L^2(\Omega)}\Big)^2 \\
&\le \mathscr{E}^0 + \frac{C_{PF}^2 C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{f}^{n+1}\|_{L^2(\Omega)}^2 + \frac{C_T^2 C_{PF} C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{g}^{n+1}\|_{L^2(\Gamma_s)}^2 + \frac{\Delta t C_{PF}^2}{2k_{min}\varepsilon}\sum_{n=0}^{N-1}\|s^{n+1}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.23}$$

Otherwise, if (3.22) *occurs, we have*

$$\begin{aligned}
&\Big(\frac{\rho}{2}-\frac{\Delta t^2\alpha^2 d C_{INV}^2}{2s_0 h^2(1-\varepsilon)}\Big)\|\boldsymbol{u}_h^N\|_{L^2(\Omega)}^2 + \mu\|E(\boldsymbol{\eta}_h^N)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2}\|\nabla\cdot\eta_h^N\|_{L^2(\Omega)}^2 + \frac{\varepsilon s_0}{2}\|p_h^N\|_{L^2(\Omega)}^2 + \frac{\Delta t(1-\varepsilon)}{2}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^N\|_{L^2(\Omega)}^2 \\
&+\frac{\varepsilon\rho}{2}\sum_{n=0}^{N-1}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2 + \frac{\mu}{2}\sum_{n=0}^{N-1}\|\boldsymbol{E}(\boldsymbol{\eta}_h^{n+1}-\eta_h^n)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2}\sum_{n=0}^{N-1}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1}-\eta_h^n)\|_{L^2(\Omega)}^2 + \frac{s_0}{2}\sum_{n=0}^{N-1}\|p_h^{n+1}-p_h^n\|_{L^2(\Omega)}^2 \\
&+\frac{1}{2}\sum_{n=0}^{N-1}\Big(\sqrt{\rho(1-\varepsilon)}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)} - \frac{\Delta t\alpha C_{PF}C_{INV}\sqrt{d}}{\sqrt{\rho(1-\varepsilon)}h}\|\nabla(p_h^{n+1}+p_h^n)\|_{L^2(\Omega)}\Big)^2 \\
&\le \mathscr{E}^0 + \Delta t b(\boldsymbol{u}_h^0, p_h^0) + \frac{\Delta t}{4}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^0\|_{L^2(\Omega)}^2 \\
&\quad + \frac{C_{PF}^2 C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{f}^{n+1}\|_{L^2(\Omega)}^2 + \frac{C_T^2 C_{PF} C_K^2}{\mu}\sum_{n=0}^{N-1}\|\boldsymbol{g}^{n+1}\|_{L^2(\Gamma_s)}^2 + \Delta t\frac{C_{PF}^2}{2\varepsilon k_{min}}\sum_{n=0}^{N-1}\|s_h^{n+1}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.24}$$

Proof. In (3.19), by setting $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1}, \boldsymbol{w}_h = -\frac{1}{\rho}A_E^h\eta_h^{n+1}, \psi_h = p_h^{n+1}$, we obtain

$$\begin{aligned}
&\rho\Big(\frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{u}_h^{n+1}\Big) + a_e(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{u}_h^{n+1}) - b(\boldsymbol{u}_h^{n+1}, p_h^n) = (\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1}) + \int_{\Gamma_s}\boldsymbol{g}^{n+1}\cdot\boldsymbol{u}_h^{n+1}d\boldsymbol{x}, \\
&\Big(\boldsymbol{u}_h^{n+1} - \frac{\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n}{\Delta t}, A_B\eta_h^{n+1}\Big) = 0, \\
&s_0\Big(\frac{p_h^{n+1}-p_h^n}{\Delta t}, p_h^{n+1}\Big) + b(\boldsymbol{u}_h^n, p_h^{n+1}) + a_p(p_h^{n+1}, p_h^{n+1}) = (s^{n+1}, p_h^{n+1}).
\end{aligned}$$

Adding three equations above side by side gives

$$\begin{aligned}
&\frac{\rho}{2\Delta t}(\|\boldsymbol{u}_h^{n+1}\|_{L^2(\Omega)}^2 - \|\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2) + \frac{s_0}{2\Delta t}(\|p_h^{n+1}\|_{L^2(\Omega)}^2 - \|p_h^n\|_{L^2(\Omega)}^2 + \|p_h^{n+1}-p_h^n\|_{L^2(\Omega)}^2) \quad (3.26)\\
&\quad + \frac{1}{2\Delta t}(a_e(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{\eta}_h^{n+1}) - a_e(\boldsymbol{\eta}_h^n, \boldsymbol{\eta}_h^n) + a_e(\boldsymbol{\eta}_h^{n+1}-\eta_h^n, \boldsymbol{\eta}_h^{n+1}-\eta_h^n)) + a_p(p_h^{n+1}, p_h^{n+1}) - b(\boldsymbol{u}_h^{n+1}, p_h^n) + b(\boldsymbol{u}_h^n, p_h^{n+1})
\end{aligned}$$

$$= (\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1}) + \int_{\Gamma_s} \boldsymbol{g}^{n+1} \cdot \boldsymbol{u}_h^{n+1} d\boldsymbol{x} + (s^{n+1}, p_h^{n+1}).$$

Let $\mathscr{E}^n$ be defined as in (3.5). Multiplying (3.26) by Δt we get

$$\mathscr{E}^{n+1} - \mathscr{E}^n + \frac{\rho}{2}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2_{L^2(\Omega)} + \frac{s_0}{2}\|p_h^{n+1} - p_h^n\|^2_{L^2(\Omega)} + \frac{1}{2}a_e(\boldsymbol{\eta}_h^{n+1} - \eta_h^n, \boldsymbol{\eta}_h^{n+1} - \eta_h^n) + \Delta t a_p(p_h^{n+1}, p_h^{n+1})$$
$$- \Delta t b(\boldsymbol{u}_h^{n+1}, p_h^n) + \Delta t b(\boldsymbol{u}_h^n, p_h^{n+1}) = \Delta t(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1}) + \Delta t \int_{\Gamma_s} \boldsymbol{g}^{n+1} \cdot \boldsymbol{u}_h^{n+1} d\boldsymbol{x} + \Delta t(s^{n+1}, p_h^{n+1}). \tag{3.27}$$

To estimate the right hand side we proceed similar to (3.9)-(3.11). We then treat the term $-\Delta t b(\boldsymbol{u}_h^{n+1}, p_h^n) + \Delta t b(\boldsymbol{u}_h^n, p_h^{n+1}) = -\Delta t b(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, p_h^{n+1}) + \Delta t b(\boldsymbol{u}_h^{n+1}, p_h^{n+1} - p_h^n)$ as follows

$$|-\Delta t b(\boldsymbol{u}_h^{n+1}, p_h^n) + \Delta t b(\boldsymbol{u}_h^n, p_h^{n+1})| \leq \frac{(1-\varepsilon)\rho}{2}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2_{L^2(\Omega)} + \frac{\Delta t^2 \alpha^2 C_{PF}^2 C_{INV}^2 d}{2\rho h^2 k_{min}(1-\varepsilon)}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|^2_{L^2(\Omega)} \tag{3.28}$$
$$- \frac{1}{2}\Big(\sqrt{(1-\varepsilon)\rho}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{L^2(\Omega)} - \frac{\Delta t \alpha C_{PF} C_{INV}\sqrt{d}}{\sqrt{\rho k_{min}(1-\varepsilon)}h}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|_{L^2(\Omega)}\Big)^2 + \frac{\Delta t^2\alpha^2}{2s_0(1-\varepsilon)}\|\nabla \cdot \boldsymbol{u}_h^{n+1}\|^2_{L^2(\Omega)}$$
$$+ \frac{s_0(1-\varepsilon)}{2}\|p_h^{n+1} - p_h^n\|^2_{L^2(\Omega)} - \frac{1}{2}\Big(\frac{\Delta t \alpha}{\sqrt{s_0(1-\varepsilon)}}\|\nabla \cdot \boldsymbol{u}_h^{n+1}\|_{L^2(\Omega)} - \sqrt{s_0(1-\varepsilon)}\|p_h^{n+1} - p_h^n\|_{L^2(\Omega)}\Big)^2.$$

Combining (3.27)-(3.28), and taking into account $\boldsymbol{u}_h^{n+1} = \frac{\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n}{\Delta t}$ we have

$$\mathscr{E}^{n+1} - \mathscr{E}^n + \frac{\varepsilon\rho}{2}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2_{L^2(\Omega)} + \frac{\varepsilon s_0}{2}\|p_h^{n+1} - p_h^n\|^2_{L^2(\Omega)} + \frac{\mu}{2}\|\boldsymbol{E}(\boldsymbol{\eta}_h^{n+1} - \eta_h^n)\|^2_{L^2(\Omega)} \tag{3.29}$$
$$+ \big(\frac{\lambda}{2} - \frac{\alpha^2}{2s_0(1-\varepsilon)}\big)\|\nabla \cdot (\boldsymbol{\eta}_h^{n+1} - \eta_h^n)\|^2_{L^2(\Omega)} + \Delta t\big(\frac{1-\varepsilon}{2} - \frac{\Delta t \alpha^2 C_{PF}^2 C_{INV}^2 d}{2\rho h^2 k_{min}(1-\varepsilon)}\big)\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|^2_{L^2(\Omega)}$$
$$+ \frac{1}{2}\Big(\sqrt{(1-\varepsilon)\rho}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{L^2(\Omega)} - \frac{\Delta t \alpha C_{PF} C_{INV}\sqrt{d}}{\sqrt{\rho k_{min}(1-\varepsilon)}h}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|_{L^2(\Omega)}\Big)^2$$
$$+ \frac{1}{2}\Big(\frac{\Delta t \alpha}{\sqrt{s_0(1-\varepsilon)}}\|\nabla \cdot \boldsymbol{u}_h^{n+1}\|_{L^2(\Omega)} - \sqrt{s_0(1-\varepsilon)}\|p_h^{n+1} - p_h^n\|_{L^2(\Omega)}\Big)^2$$
$$\leq \frac{C_{PF}^2 C_K^2}{\mu}\|\boldsymbol{f}^{n+1}\|^2_{L^2(\Omega)} + \frac{C_T^2 C_{PF} C_K^2}{\mu}\|\boldsymbol{g}^{n+1}\|^2_{L^2(\Gamma_s)} + \frac{\Delta t C_{PF}^2}{2\varepsilon k_{min}}\|s^{n+1}\|^2_{L^2(\Omega)}.$$

Stability follows provided

$$\frac{\alpha^2}{s_0\lambda} \leq 1 - \varepsilon \quad \text{and} \quad \Delta t < \frac{\rho k_{min}(1-\varepsilon)^2}{\alpha^2 C_{PF}^2 C_{INV}^2 d}h^2.$$

Summing (3.29) from $n = 0$ to $N-1$ yields (3.23).

For the second stability inequality, rewrite $-b(\boldsymbol{u}_h^{n+1}, p_h^n) + b(\boldsymbol{u}_h^n, p_h^{n+1})$ as

$$-b(\boldsymbol{u}_h^{n+1}, p_h^n) + b(\boldsymbol{u}_h^n, p_h^{n+1}) = b(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) - b(\boldsymbol{u}_h^n, p_h^n) - b(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, p_h^{n+1} + p_h^n).$$

Using similar estimates to (3.9)-(3.11) and arranging terms in (3.27) in a different way than in the first part of the proof, we get

$$[\mathscr{E}^{n+1} + \Delta t b(\boldsymbol{u}_h^{n+1}, p_h^{n+1})] - [\mathscr{E}^n + \Delta t b(\boldsymbol{u}_h^n, p_h^n)] + \frac{\rho}{2}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2_{L^2(\Omega)} + \frac{s_0}{2}\|p_h^{n+1} - p_h^n\|^2_{L^2(\Omega)} \tag{3.30}$$
$$+ \frac{\mu}{2}\|\boldsymbol{E}(\boldsymbol{\eta}_h^{n+1} - \eta_h^n)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla \cdot (\boldsymbol{\eta}_h^{n+1} - \eta_h^n)\|^2_{L^2(\Omega)} + \frac{(1-\varepsilon)\Delta t}{2}\|\sqrt{\boldsymbol{\kappa}}\nabla p_h^{n+1}\|^2_{L^2(\Omega)} - \Delta t b(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, p_h^{n+1} + p_h^n)$$

$$\leq \frac{C_{PF}^2 C_K^2}{\mu}\|\boldsymbol{f}^{n+1}\|_{L^2(\Omega)}^2 + \frac{C_T^2 C_{PF} C_K^2}{\mu}\|\boldsymbol{g}^{n+1}\|_{L^2(\Gamma_s)}^2 + \frac{\Delta t C_{PF}^2}{2k_{min}\varepsilon}\|s^{n+1}\|_{L^2(\Omega)}^2.$$

We proceed to bound the term $\Delta t b(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n, p_h^{n+1}+p_h^n)$ as follows

$$\begin{aligned}
\Delta t|b(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n, p_h^{n+1}+p_h^n)| &\leq \frac{\rho(1-\varepsilon)}{2}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2 + \frac{\Delta t^2\alpha^2 C_{PF}^2 C_{INV}^2 d}{2\rho h^2(1-\varepsilon)}\|\nabla(p_h^{n+1}+p_h^n)\|_{L^2(\Omega)}^2 \\
&\quad -\frac{1}{2}\left(\sqrt{\rho(1-\varepsilon)}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)} - \frac{\Delta t\alpha C_{PF}C_{INV}\sqrt{d}}{\sqrt{\rho(1-\varepsilon)}h}\|\nabla(p_h^{n+1}+p_h^n)\|_{L^2(\Omega)}\right)^2 \\
&\leq \frac{\rho(1-\varepsilon)}{4}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2 + \frac{\Delta t^2\alpha^2 C_{PF}^2 C_{INV}^2 d}{\rho h^2(1-\varepsilon)}\|\nabla p_h^{n+1}\|_{L^2(\Omega)}^2 + \frac{\Delta t^2\alpha^2 C_{PF}^2 C_{INV}^2 d}{\rho h^2(1-\varepsilon)}\|\nabla p_h^n\|_{L^2(\Omega)}^2 \\
&\quad -\frac{1}{2}\left(\sqrt{\rho(1-\varepsilon)}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)} - \frac{\Delta t\alpha C_{PF}C_{INV}\sqrt{d}}{\sqrt{\rho(1-\varepsilon)}h}\|\nabla(p_h^{n+1}+p_h^n)\|_{L^2(\Omega)}\right)^2 \\
&\leq \frac{\rho(1-\varepsilon)}{2}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2 + \frac{\Delta t^2\alpha^2 C_{PF}^2 C_{INV}^2 d}{\rho h^2 k_{min}(1-\varepsilon)}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|_{L^2(\Omega)}^2 + \frac{\Delta t^2\alpha^2 C_{PF}^2 C_{INV}^2 d}{\rho h^2 k_{min}(1-\varepsilon)}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^n\|_{L^2(\Omega)}^2 \\
&\quad -\frac{1}{2}\left(\sqrt{\rho(1-\varepsilon)}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)} - \frac{\Delta t\alpha C_{PF}C_{INV}\sqrt{d}}{\sqrt{\rho(1-\varepsilon)}h}\|\nabla(p_h^{n+1}+p_h^n)\|_{L^2(\Omega)}\right)^2.
\end{aligned} \tag{3.31}$$

Assuming $\dfrac{\Delta t^2\alpha^2 C_{PF}^2 C_{INV}^2 d}{\rho h^2 k_{min}(1-\varepsilon)} \leq \dfrac{\Delta t(1-\varepsilon)}{4}$, and combining (3.30) and (3.31), we have

$$\begin{aligned}
&\left[\mathscr{E}^{n+1} + \Delta t b(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) + \frac{\Delta t(1-\varepsilon)}{4}\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^{n+1}\|_{L^2(\Omega)}^2\right] - \left[\mathscr{E}^n + \Delta t b(\boldsymbol{u}_h^n, p_h^n) + \frac{\Delta t(1-\varepsilon)}{4}\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^n\|_{L^2(\Omega)}^2\right] \\
&\quad + \frac{\varepsilon\rho}{2}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2 + \frac{s_0}{2}\|p_h^{n+1}-p_h^n\|_{L^2(\Omega)}^2 + \frac{\mu}{2}\|\boldsymbol{E}(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2}\|\nabla\cdot(\boldsymbol{\eta}_h^{n+1}-\boldsymbol{\eta}_h^n)\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{2}\left(\sqrt{\rho(1-\varepsilon)}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_{L^2(\Omega)} - \frac{\Delta t\alpha C_{PF}C_{INV}\sqrt{d}}{\sqrt{\rho(1-\varepsilon)}h}\|\nabla(p_h^{n+1}+p_h^n)\|_{L^2(\Omega)}\right)^2 \\
&\leq \frac{C_{PF}^2 C_K^2}{\mu}\|\boldsymbol{f}^{n+1}\|_{L^2(\Omega)}^2 + \frac{C_T^2 C_{PF} C_K^2}{\mu}\|\boldsymbol{g}^{n+1}\|_{L^2(\Gamma_s)}^2 + \frac{\Delta t C_{PF}^2}{2k_{min}\varepsilon}\|s^{n+1}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.32}$$

Finally, to prove stability we have to show that $\mathscr{E}^n + \Delta t b(\boldsymbol{u}_h^n, p_h^n) + \frac{\Delta t(1-\varepsilon)}{4}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^n\|_{L^2(\Omega)}^2 \geq 0$. We proceed as follows

$$\Delta t b(\boldsymbol{u}_h^n, p_h^n) \geq -\frac{\Delta t^2\alpha^2 d C_{INV}^2}{2s_0 h^2(1-\varepsilon)}\|\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2 - \frac{s_0(1-\varepsilon)}{2}\|p_h^n\|_{L^2(\Omega)}^2. \tag{3.33}$$

Therefore,

$$\begin{aligned}
&\mathscr{E}^n + \Delta t b(\boldsymbol{u}_h^n, p_h^n) + \frac{\Delta t(1-\varepsilon)}{4}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^n\|_{L^2(\Omega)}^2 \\
&\geq \left(\frac{\rho}{2} - \frac{\Delta t^2\alpha^2 d C_{INV}^2}{2s_0 h^2(1-\varepsilon)}\right)\|\boldsymbol{u}_h^n\|_{L^2(\Omega)}^2 + \mu\|E(\boldsymbol{\eta}_h^n)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2}\|\nabla\cdot\eta_h^n\|_{L^2(\Omega)}^2 + \frac{\varepsilon s_0}{2}\|p_h^n\|_{L^2(\Omega)}^2 + \frac{\Delta t(1-\varepsilon)}{4}\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^n\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.34}$$

Stability follows provided

$$\Delta t \leq \frac{\sqrt{\rho s_0(1-\varepsilon)}}{\alpha C_{INV}\sqrt{d}}h.$$

Summing equation (3.32) from $n=0$ to $N-1$ yields (3.24). □

3.4. Backward Euler - Leap Frog (BELF). Backward Euler - Leap Frog is a combination of the three level implicit method with the coupling terms treated by the explicit Leap-Frog method. On the surface, combining methods of different orders of accuracy can be questioned. However, in [12], BEFE was found to have the best stability properties for the Stokes-Darcy problem. The use of a higher order method for the (explicitly treated) coupling terms also reduces the penalty for uncoupling the system without increasing algorithmic complexity.

Approximations are needed at the first two time steps to begin. We shall suppose these are computed to appropriate accuracy, such as by BEFE (the first method above). We use the same time step, Δt, in both sub domains. The BELF partitioned approximations are : Given $t \in (0,T)$ and $(\boldsymbol{\eta}_h^n, \boldsymbol{u}_h^n, p_h^n)$ for $n \geq 2$, find $(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in X_h^s \times X_h^s \times X_h^p$, with $\boldsymbol{\eta}_h^{n+1} = \mathbf{0}$ on Γ_c, such that for all $(\boldsymbol{v}_h, \boldsymbol{w}_h, \psi_h) \in X_h^s \times X_h^s \times X_h^p$

$$\rho(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^{n-1}}{2\Delta t}, \boldsymbol{v}_h) + a_e\left(\frac{\boldsymbol{\eta}_h^{n+1} + \boldsymbol{\eta}_h^{n-1}}{2}, \boldsymbol{v}_h\right) - b(\boldsymbol{v}_h, p_h^n) = (\boldsymbol{f}^{n+1}, \boldsymbol{v}_h) + \int_{\Gamma_s} \boldsymbol{g}^{n+1} \cdot \boldsymbol{v}_h d\boldsymbol{x} \qquad \text{in } \Omega, \tag{3.35a}$$

$$\rho\left(\frac{\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}}{2} - \frac{\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^{n-1}}{2\Delta t}, \boldsymbol{w}_h\right) = 0 \qquad \text{in } \Omega, \tag{3.35b}$$

$$s_0(\frac{p_h^{n+1} - p_h^{n-1}}{2\Delta t}, \psi_h) + b(\boldsymbol{u}_h^n, \psi_h) + a_p(p_h^{n+1}, \psi_h) = (s^{n+1}, \psi_h) \qquad \text{in } \Omega. \tag{3.35c}$$

THEOREM 3.6. *Under the condition*

$$\Delta t \leq \frac{\rho h^2}{2\alpha^2 C_{INV}^2 d}\Big(\frac{k_{min}}{C_{PF}^2} + \Big(\frac{k_{min}^2}{C_{PF}^4} + \frac{4 s_0 \alpha^2 C_{INV}^2 d}{\rho h^2}\Big)^{1/2}\Big), \tag{3.36}$$

the following bound holds for the BELF method (3.35)*:*

$$\begin{aligned}
&\frac{\varepsilon\rho}{2}\big(\|\boldsymbol{u}_h^N\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_h^{N-1}\|_{L^2(\Omega)}^2\big) + \mu\big(\|E(\boldsymbol{\eta}_h^N)\|_{L^2(\Omega)}^2 + \|E(\boldsymbol{\eta}_h^{N-1})\|_{L^2(\Omega)}^2\big) + \frac{\lambda}{2}\big(\|\nabla\cdot\boldsymbol{\eta}_h^N\|_{L^2(\Omega)}^2 + \|\nabla\cdot\boldsymbol{\eta}_h^{N-1}\|_{L^2(\Omega)}^2\big)\\
&\quad + \frac{1}{2}\Big(s_0 + \frac{\Delta t k_{min}}{C_{PF}^2} - \frac{\alpha^2 \Delta t^2 C_{INV}^2 d}{\rho h^2(1-\varepsilon)}\Big)\big(\|p_h^N\|_{L^2(\Omega)}^2 + \|p_h^{N-1}\|_{L^2(\Omega)}^2\big) + \frac{\Delta t}{2}\sum_{n=1}^{N-1}\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^{n+1} + p_h^{n-1})\|_{L^2(\Omega)}^2\\
&\leq \mathscr{E}^1 + \mathscr{E}^0 + \frac{\Delta t}{2}\big(\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^1\|^2 + \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^0\|^2\big) + \Delta t\big(b(\boldsymbol{u}_h^0, p_h^1) - b(\boldsymbol{u}_h^1, p_h^0)\big)\\
&\quad + \Delta t\sum_{n=1}^{N-1}(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) + \Delta t\sum_{n=1}^{N-1}\int_{\Gamma_s}\boldsymbol{g}^{n+1}\cdot(\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1})d\boldsymbol{x} + \Delta t\sum_{n=1}^{N-1}(s^{n+1}, p_h^{n+1} + p_h^{n-1}).
\end{aligned}$$

Proof. In (3.35), set $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}, \boldsymbol{w}_h = -\frac{1}{\rho}A_E^h(\eta_h^{n+1} + \eta_h^{n-1}), \psi_h = p_h^{n+1} + p_h^{n-1}$ and add, we obtain

$$\begin{aligned}
&\frac{\rho}{2\Delta t}(\|\boldsymbol{u}_h^{n+1}\|_{L^2(\Omega)}^2 - \|\boldsymbol{u}_h^{n-1}\|_{L^2(\Omega)}^2) + \frac{1}{2\Delta t}(a_e(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{\eta}_h^{n+1}) - a_e(\boldsymbol{\eta}_h^{n-1}, \boldsymbol{\eta}_h^{n-1})) + \frac{s_0}{2\Delta t}(\|p_h^{n+1}\|_{L^2(\Omega)}^2 - \|p_h^{n-1}\|_{L^2(\Omega)}^2)\\
&+ \frac{1}{2}\big(a_p(p_h^{n+1}, p_h^{n+1}) - a_p(p_h^{n-1}, p_h^{n-1})\big) + \frac{1}{2}a_p(p_h^{n+1} + p_h^{n-1}, p_h^{n+1} + p_h^{n-1}) - b(\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}, p_h^n) + b(\boldsymbol{u}_h^n, p_h^{n+1} + p_h^{n-1})\\
&= (\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) + \int_{\Gamma_s}\boldsymbol{g}^{n+1}\cdot(\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1})d\boldsymbol{x} + (s^{n+1}, p_h^{n+1} + p_h^{n-1}).
\end{aligned} \tag{3.37}$$

Let the discrete energy $\mathscr{E}^n$ be denoted as in (3.5). Multiplying (3.37) by Δt and rearranging, we get

$$\begin{aligned}
&[\mathscr{E}^{n+1} + \frac{\Delta t}{2}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n+1}\|_{L^2(\Omega)}^2 + \Delta t b(\boldsymbol{u}_h^n, p_h^{n+1}) - \Delta t b(\boldsymbol{u}_h^{n+1}, p_h^n)]\\
&\quad - [\mathscr{E}^{n-1} + \frac{\Delta t}{2}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^{n-1}\|_{L^2(\Omega)}^2 + \Delta t b(\boldsymbol{u}_h^{n-1}, p_h^n) - \Delta t b(\boldsymbol{u}_h^n, p_h^{n-1})] + \frac{\Delta t}{2}\|\boldsymbol{\kappa}^{1/2}\nabla(p_h^{n+1} + p_h^{n-1})\|_{L^2(\Omega)}^2
\end{aligned}$$

$$= \Delta t(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) + \Delta t \int_{\Gamma_s} \boldsymbol{g}^{n+1} \cdot (\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) d\boldsymbol{x} + \Delta t(s^{n+1}, p_h^{n+1} + p_h^{n-1}).$$

Now add and subtract $\mathscr{E}^n + \frac{\Delta t}{2}\|\boldsymbol{\kappa}^{1/2}\nabla p_h^n\|^2$

$$[\mathscr{E}^{n+1} + \mathscr{E}^n + \frac{\Delta t}{2}\big(\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^{n+1}\|^2 + \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^n\|^2\big) + \Delta t\big(b(\boldsymbol{u}_h^n, p_h^{n+1}) - b(\boldsymbol{u}_h^{n+1}, p_h^n)\big)]$$
$$- [\mathscr{E}^n + \mathscr{E}^{n-1} + \frac{\Delta t}{2}\big(\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^n\|^2 + \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^{n-1}\|^2\big) + \Delta t\big(b(\boldsymbol{u}_h^{n-1}, p_h^n) - b(\boldsymbol{u}_h^n, p_h^{n-1})\big)] + \frac{\Delta t}{2}\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^{n+1} + p_h^{n-1})\|^2_{L^2(\Omega)}$$
$$= \Delta t(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) + \Delta t \int_{\Gamma_s} \boldsymbol{g}^{n+1} \cdot (\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) d\boldsymbol{x} + \Delta t(s^{n+1}, p_h^{n+1} + p_h^{n-1}),$$

and sum from $n = 1$ to $N-1$ to obtain

$$\mathscr{E}^N + \mathscr{E}^{N-1} + \frac{\Delta t}{2}\big(\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^N\|^2 + \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^{N-1}\|^2\big) + \Delta t\big(b(\boldsymbol{u}_h^{N-1}, p_h^N) - b(\boldsymbol{u}_h^N, p_h^{N-1})\big) + \frac{\Delta t}{2}\sum_{n=1}^{N-1}\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^{n+1} + p_h^{n-1})\|^2_{L^2(\Omega)}$$
$$= \mathscr{E}^1 + \mathscr{E}^0 + \frac{\Delta t}{2}\big(\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^1\|^2 + \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^0\|^2\big) + \Delta t\big(b(\boldsymbol{u}_h^0, p_h^1) - b(\boldsymbol{u}_h^1, p_h^0)\big)$$
$$+ \Delta t\sum_{n=1}^{N-1}(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) + \Delta t\sum_{n=1}^{N-1}\int_{\Gamma_s} \boldsymbol{g}^{n+1} \cdot (\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) d\boldsymbol{x} + \Delta t\sum_{n=1}^{N-1}(s^{n+1}, p_h^{n+1} + p_h^{n-1}).$$

Note that

$$\Delta t\big(b(\boldsymbol{u}_h^{N-1}, p_h^N) - b(\boldsymbol{u}_h^N, p_h^{N-1})\big)$$
$$\geq -\frac{\rho(1-\varepsilon)}{2}\big(\|\boldsymbol{u}_h^N\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_h^{N-1}\|^2_{L^2(\Omega)}\big) - \frac{\alpha^2\Delta t^2 C_{INV}^2 d}{2\rho h^2(1-\varepsilon)}\big(\|p_h^N\|^2_{L^2(\Omega)} + \|p_h^{N-1}\|^2_{L^2(\Omega)}\big),$$

so using again the definition (3.5) and (3.2) we have

$$\frac{\varepsilon\rho}{2}\big(\|\boldsymbol{u}_h^N\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_h^{N-1}\|^2_{L^2(\Omega)}\big) + \mu\big(\|E(\boldsymbol{\eta}_h^N)\|^2_{L^2(\Omega)} + \|E(\boldsymbol{\eta}_h^{N-1})\|^2_{L^2(\Omega)}\big) + \frac{\lambda}{2}\big(\|\nabla\cdot\boldsymbol{\eta}_h^N\|^2_{L^2(\Omega)} + \|\nabla\cdot\boldsymbol{\eta}_h^{N-1}\|^2_{L^2(\Omega)}\big)$$
$$+ \big(\frac{s_0}{2} + \frac{\Delta t k_{min}}{2C_{PF}^2} - \frac{\alpha^2\Delta t^2 C_{INV}^2 d}{2\rho h^2(1-\varepsilon)}\big)\big(\|p_h^N\|^2_{L^2(\Omega)} + \|p_h^{N-1}\|^2_{L^2(\Omega)}\big) + \frac{\Delta t}{2}\sum_{n=1}^{N-1}\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla(p_h^{n+1} + p_h^{n-1})\|^2_{L^2(\Omega)}$$
$$\leq \mathscr{E}^1 + \mathscr{E}^0 + \frac{\Delta t}{2}\big(\|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^1\|^2 + \|\boldsymbol{\kappa}^{\frac{1}{2}}\nabla p_h^0\|^2\big) + \Delta t\big(b(\boldsymbol{u}_h^0, p_h^1) - b(\boldsymbol{u}_h^1, p_h^0)\big)$$
$$+ \Delta t\sum_{n=1}^{N-1}(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) + \Delta t\sum_{n=1}^{N-1}\int_{\Gamma_s} \boldsymbol{g}^{n+1} \cdot (\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}) d\boldsymbol{x} + \Delta t\sum_{n=1}^{N-1}(s^{n+1}, p_h^{n+1} + p_h^{n-1}).$$

Therefore stability holds provided Δt satisfies

$$0 \leq s_0 + \Delta t\frac{k_{min}}{C_{PF}^2} - \Delta t^2\frac{\alpha^2 C_{INV}^2 d}{\rho h^2(1-\varepsilon)},$$

which gives (3.36). ☐

REMARK 2. *Alternatively, we can discretize the problem using BELF method in the following way: Given* $t \in (0, T)$ *and* $(\boldsymbol{\eta}_h^n, \boldsymbol{u}_h^n, p_h^n)$ *for* $n \geq 2$*, find* $(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in X_h^s \times X_h^s \times X_h^p$*, with* $\boldsymbol{\eta}_h^{n+1} = \boldsymbol{0}$ *on* Γ_c*, such that for all* $(\boldsymbol{v}_h, \boldsymbol{w}_h, \psi_h) \in X_h^s \times X_h^s \times X_h^p$

$$\rho(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^{n-1}}{2\Delta t}, \boldsymbol{v}_h) + a_e(\boldsymbol{\eta}_h^n, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^n) = (\boldsymbol{f}^{n+1}, \boldsymbol{v}_h) + \int_{\Gamma_s} \boldsymbol{g}^{n+1} \cdot \boldsymbol{v}_h d\boldsymbol{x} \qquad in\ \Omega, \tag{3.38a}$$

$$\rho(\boldsymbol{u}_h^n - \frac{\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^{n-1}}{2\Delta t}, \boldsymbol{w}_h) = 0 \qquad in\ \Omega, \tag{3.38b}$$

$$s_0(\frac{p_h^{n+1} - p_h^{n-1}}{2\Delta t}, \psi_h) + b(\boldsymbol{u}_h^n, \psi_h) + a_p(p_h^{n+1}, \psi_h) = (s^{n+1}, \psi_h) \qquad in\ \Omega. \tag{3.38c}$$

It can be shown in a similar way as in the previous proof that this method is stable provided

$$\Delta t \leq \min\left\{ \frac{\rho k_{min}}{2\alpha^2 d C_{INV}^2 C_{PF}^2} h^2, \frac{\sqrt{\rho s_0}}{4\alpha C_{INV}\sqrt{d}} h \right\}. \tag{3.39}$$

3.5. ω-method. This is a three-level second order family of partitioned methods, containing Crank-Nicolson Leap-Frog (CNLF) and Backward Differentiation Formula 2 - Adams-Bashforth 2 (BDF2-AB2). Let $\omega \in [\frac{1}{2}, 1]$, and denote

$$\nu = \frac{1}{16(2\omega^2 - 3\omega + \frac{5}{4})} \in [\tfrac{1}{4}, \tfrac{1}{2}], \quad c_\omega = \frac{\omega^2}{(2\omega - 1)} \text{ for } \omega \in (\frac{1}{2}, 1], \quad \text{and} \quad c_{\frac{1}{2}} = 0.$$

Assuming the initial data $(\boldsymbol{\eta}_h^0, \boldsymbol{u}_h^0, p_h^0)$ is given and $(\boldsymbol{\eta}_h^1, \boldsymbol{u}_h^1, p_h^1)$ is computed with a second order accurate method, we consider the following method, which is a convex combination (via the ω variable) of BDF2-AB2 and CNLF:

$$\begin{aligned}
&\rho\Big(\frac{(2\omega - \frac{1}{2})\boldsymbol{u}_h^{n+1} + (-4\omega + 2)\boldsymbol{u}_h^n + (2\omega - \frac{3}{2})\boldsymbol{u}_h^{n-1}}{\Delta t}, \boldsymbol{v}_h\Big) + a_e(\omega\boldsymbol{\eta}^{n+1} + (1-\omega)\boldsymbol{\eta}^{n-1}, \boldsymbol{v}_h) \\
&\quad + \Delta t \frac{\alpha^2 c_\omega}{s_0}\big(\nabla\cdot(\omega\boldsymbol{u}_h^{n+1} + (1-\omega)\boldsymbol{u}_h^{n-1}), \nabla\cdot\boldsymbol{v}_h\big) \qquad (\textit{regularization term}) \\
&\quad - b(\boldsymbol{v}_h, 2\omega p^n + (-2\omega + 1)p^{n-1}) = \big(\boldsymbol{f}^{n+2\omega-1}, \boldsymbol{v}_h\big) + \int_{\Gamma_s} \boldsymbol{g}^{n+2\omega-1}\boldsymbol{v}_h\, d\sigma, \\
&\rho\big(\omega\boldsymbol{u}_h^{n+1} + (1-\omega)\boldsymbol{u}_h^{n-1} - \frac{(2\omega - \frac{1}{2})\boldsymbol{\eta}_h^{n+1} + (-4\omega + 2)\boldsymbol{\eta}_h^n + (2\omega - \frac{3}{2})\boldsymbol{\eta}_h^{n-1}}{\Delta t}, \boldsymbol{w}_h\big) = 0, \qquad (3.40) \\
&s_0\Big(\frac{(2\omega - \frac{1}{2})\boldsymbol{p}_h^{n+1} + (-4\omega + 2)\boldsymbol{p}_h^n + (2\omega - \frac{3}{2})\boldsymbol{p}_h^{n-1}}{\Delta t}, \psi_h\Big) + b(2\omega\boldsymbol{u}^n + (-2\omega + 1)\boldsymbol{u}^{n-1}, \psi_h) \\
&\quad + a_p(\omega p_h^{n+1} + (1-\omega)p_h^{n-1}, \psi_h) = (s^{n+2\omega-1}, \psi_h).
\end{aligned}$$

For $\omega = 1/2$ this method reduces to CNLF, and for $\omega = 1$ it becomes the IMEX method BDF2-AB2, e.g., [13, 21, 20, 7]. Let $0 \leq \varepsilon \ll 1$ and denote

$$\varepsilon_1 = \frac{\Delta t}{1-\varepsilon}\frac{\omega(1-\omega)}{\nu}\frac{\alpha d C_{INV}^2}{\rho h^2}, \quad \varepsilon_2 = s_0\nu + \Delta t\,\omega(2\omega - 1)\frac{k_{\min}}{C_{PF}^2} - \Delta t^2\omega^2\frac{\alpha^2 C_{INV}^2}{\rho h^2}\Big(\frac{(1-\omega)^2 d}{\nu(1-\varepsilon)} + 2\omega - 1\Big),$$

$$\Delta t_{CNLF} = \frac{\sqrt{\rho s_0}}{\alpha C_{INV}\sqrt{d}} h,$$

$$\Delta t_\omega = \frac{\rho h^2}{2\omega C_{inv}^2 C_{PF}^2 \alpha^2 (\frac{d(1-\omega)^2}{\nu} + 2\omega - 1)}\Big((2\omega - 1)k_{min} + \big((2\omega-1)^2 k_{min}^2 + 4 s_0 \nu \frac{C_{inv}^2 C_{PF}^4}{\rho h^2}\alpha^2(\frac{d(1-\omega)^2}{\nu} + 2\omega - 1)\big)^{\frac{1}{2}}\Big).$$

Note that under assumption (3.42), $\varepsilon_2 \geq 0$.

THEOREM 3.7. *Under the CFL conditions*

$$\Delta t < \frac{\sqrt{\rho s_0}}{\alpha C_{INV}\sqrt{d}} h, \tag{3.41}$$

when $\omega = \frac{1}{2}$, *and*

$$\Delta t < \min\Big\{ \frac{\nu}{\omega(1-\omega)} \Delta t_{CNLF}, \Delta t_{\omega} \Big\}, \tag{3.42}$$

for $\omega \in (\frac{1}{2}, 1]$, *the following energy inequality holds*

$$\begin{aligned}
&\varepsilon\rho\nu\big(\|\boldsymbol{u}_h^{N+1}\|^2 + \|\boldsymbol{u}_h^N\|^2\big) + \|(\boldsymbol{\eta}_h^{N+1}, \boldsymbol{\eta}_h^N)\|^2_{G,a_e} + \varepsilon_2\big(\|p_h^{N+1}\|^2 + \|p_h^N\|^2\big) \\
&\quad + \textit{Positive Terms} + \textit{Numerical Dissipation} \\
&\quad + (2\omega-1)\Delta t \sum_{n=2}^{N-1} \Big(\big((2\omega-1)\boldsymbol{\kappa} - \tfrac{\Delta t}{\rho}\alpha^2\omega^2 \frac{dC_{INV}^2}{h^2} C_{PF}^2 \boldsymbol{I}\big)\nabla p^n, \nabla p^n \Big) \\
&\quad + \omega(1-\omega)\Delta t \sum_{n=1}^{N} a_p\big(p_h^{n+1} + p_h^{n-1}, p_h^{n+1} + p_h^{n-1}\big) \\
&\leq \rho\|(\boldsymbol{u}_h^1, \boldsymbol{u}_h^0)\|_G^2 + s_0\|(p_h^1, p_h^0)\|_G^2 + \|(\boldsymbol{\eta}_h^1, \boldsymbol{\eta}_h^0)\|^2_{G,a_e} + 2\omega(1-\omega)\Delta t\big(b(\boldsymbol{u}^0, p^1) - b(\boldsymbol{u}^1, p^0)\big) \\
&\quad - (1-\omega)(2\omega-1)\Delta t\big(a_p(p^1, p^1) + a_p(p^0, p^0)\big) - (1-\omega)(2\omega-1)\Delta t^2 \frac{\alpha^2 c_\omega}{s_0}\big(\|\nabla\cdot\boldsymbol{u}_h^1\|^2 + \|\nabla\cdot\boldsymbol{u}_h^0\|^2\big) \\
&\quad + \Delta t \sum_{n=1}^{N} \Big(\big(\boldsymbol{f}^{n+2\omega-1}, \omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1}\big) + \int_{\Gamma_s} \boldsymbol{g}^{n+2\omega-1}\big(\omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1}\big)\, d\sigma \\
&\qquad + \big(s^{n+2\omega-1}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1}\big)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\textit{Positive Terms} :=& \rho\|a\boldsymbol{u}_h^{N+1} + b\boldsymbol{u}_h^N\|^2 + s_0\|ap_h^{N+1} + bp_h^N\|^2 + \omega^2(1-\omega)\frac{\alpha^2}{s_0}\Delta t^2\big(\|\nabla\cdot\boldsymbol{u}_h^{N+1}\|^2 + \|\nabla\cdot\boldsymbol{u}_h^N\|^2\big) \\
&+ \alpha\omega(1-\omega)\Delta t\Big(\big\|\sqrt{\varepsilon_1}p_h^{N+1} + \tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^N\big\|^2 + \big\|\sqrt{\varepsilon_1}p_h^N + \tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^{N+1}\big\|^2\Big),
\end{aligned}$$

and

$$\begin{aligned}
\textit{Numerical Dissipation} :=& \tfrac{2\omega-1}{4}\sum_{n=1}^{N} a_e\big(\boldsymbol{\eta}_h^{n+1} - 2\boldsymbol{\eta}_h^n + \boldsymbol{\eta}_h^{n-1}, \boldsymbol{\eta}_h^{n+1} - 2\boldsymbol{\eta}_h^n + \boldsymbol{\eta}_h^{n-1}\big)\Big) \\
&+ \Delta t\,\omega(1-\omega)\frac{\alpha^2 c_\omega}{s_0}\Delta t \sum_{n=1}^{N} \|\nabla\cdot(\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1})\|^2 \\
&+ (2\omega-1)\Delta t \sum_{n=1}^{N} \Big(\Big\|\sqrt{\tfrac{\rho}{4\Delta t}}p^{n+1} + \sqrt{\tfrac{\Delta t}{\rho}}\alpha\omega\nabla\cdot(\boldsymbol{u}^{n+1} - 2\boldsymbol{u}^n + \boldsymbol{u}^{n-1})\Big\|^2\Big) \\
&+ (2\omega-1)\Delta t \sum_{n=1}^{N} \Big(\Big\|\sqrt{\tfrac{s_0}{4\Delta t}}(p^{n+1} - 2p^n + p^{n-1}) + \sqrt{\tfrac{\Delta t}{s_0}}\alpha\omega\,\nabla\cdot\boldsymbol{u}^{n+1}\Big\|^2\Big).
\end{aligned}$$

REMARK 3. *When* $\omega = \frac{1}{2}$, *the regularizing third term in the first equation in* (3.40) *vanishes, and* (3.40) *reduces to the Crank Nicolson-Leap Frog (CNLF) scheme: Given* $(\boldsymbol{u}_h^{n-1}, \boldsymbol{\eta}_h^{n-1}, p_h^{n-1}), (\boldsymbol{u}_h^n, \boldsymbol{\eta}_h^n, p_h^n) \in X_h^s \times X_h^s \times X_h^p$ *find* $(\boldsymbol{u}_h^{n+1}, \boldsymbol{\eta}_h^{n+1}, p_h^{n+1}) \in X_h^s \times X_h^s \times X_h^p$ *such that* $\forall(\boldsymbol{v}_h, \boldsymbol{w}_h, \psi_h) \in X_h^s \times X_h^s \times X_h^p$:

$$\rho\left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^{n-1}}{2\Delta t}, \boldsymbol{v}_h\right) + a_e\left(\frac{\boldsymbol{\eta}_h^{n+1} + \boldsymbol{\eta}_h^{n-1}}{2}, \boldsymbol{v}_h\right) - b(\boldsymbol{v}_h, p_h^n) = (\boldsymbol{f}^n, \boldsymbol{v}_h) + \int_{\Gamma_s} \boldsymbol{g}^n \cdot \boldsymbol{v}_h \, d\boldsymbol{x}, \tag{3.43}$$

$$\rho\left(\frac{\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n-1}}{2} - \frac{\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^{n-1}}{2\Delta t}, \boldsymbol{w}_h\right) = 0, \tag{3.44}$$

$$s_0\left(\frac{p_h^{n+1}-p_h^{n-1}}{2\Delta t},\psi_h\right)+b(\boldsymbol{u}_h^n,\psi_h)+a_p\left(\frac{p_h^{n+1}+p_h^{n-1}}{2},\psi_h\right)=(s^n,\psi_h). \tag{3.45}$$

The stability condition for CNLF is similar to the one for BELF. The proof of stability follows as in Section 3.4, ignoring the contribution of the molecular diffusion term $a_p(p_h,p_h)$ in the energy balance.

The proof of the general case is based on energy-type estimates, it involves the G-stability methodology [4, 6], and produces the time-step restriction for stability by balancing the contribution from the coupling term with both the numerical and molecular dissipation (see e.g.[20]). For the reader's convenience, we include the proof in the Appendix 6. [1]

The CNLF time-step restriction (3.41) and the first term in (3.42) are related to the second part of the condition in (3.36) in BELF, and inverse related to conditions (3.7) in the *drained split* method (3.6) and also to (3.13) in the *fixed strain split* method (3.12). The second term in (3.42) is proportional to condition (3.14) in the *fixed strain split* method, the first condition in (3.21) and (3.22) in BEFE, and the first part of the condition in (3.36) in BELF.

4. Numerical examples. In this section, we numerically investigate the stability properties of the methods presented in Section 3. As a benchmark problem, we consider the cantilever bracket problem, studied previously for poroelastic systems in [14, 22, 17]. Let domain Ω to be a square $[0,1]\times[0,1]$, and denote by Γ_1 and Γ_3 the bottom and top boundaries of Ω, and by Γ_2 and Γ_4 the right and left boundaries of Ω, respectively, so that $\partial\Omega=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$. We prescribe the following boundary conditions for the elasticity problem

$$\begin{aligned}\boldsymbol{\eta}&=0 \quad \text{on } \Gamma_4,\\ \boldsymbol{\sigma}^p\boldsymbol{n}&=0 \quad \text{on } \Gamma_1\cup\Gamma_2,\\ \boldsymbol{\sigma}^p\boldsymbol{n}&=(0,-1) \quad \text{on } \Gamma_3,\end{aligned}$$

and a Dirichlet boundary condition for the flow problem

$$p=20 \text{ on } \partial\Omega.$$

This problem reaches the steady state at $T=50s$. The structure material properties are given by $\rho=2g/cm^2$, $\mu=3.57\times10^3 dyne/cm^2$ and $\lambda=1.4\times10^4 dyne/cm^2$ (corresponding to the values $\nu=0.4dyne/cm^2$ and $E=10^4 dyne/cm^2$). The source terms are $\boldsymbol{f}=0$, $s=0$, and the Biot-Willis constant that determines the strength of the coupling between the fluid and structure is $\alpha=1$. To numerically test the stability properties of the proposed algorithms, we first solve the problem using a monolithic solver until the steady state is reached. We will refer to this solution as a *reference solution*. Then, we solve the same problem using each partitioned scheme and measure the relative error between the obtained solution and the reference solution. Since parameters s_0 and $\boldsymbol{\kappa}$ are frequently very small in applications, we test the problem with different values of s_0 and $\boldsymbol{\kappa}$, and the time step size Δt. In all the test cases, each side of the boundary is equally divided by 20 grid points. Figure 4.1 shows the reference displacement and pressure obtained at $T=50s$ using $\Delta t=1$ and values of the parameters $\boldsymbol{\kappa}=10^{-7}\boldsymbol{I}$ and $s_0=10^{-5}$.

4.1. Drained split. We numerically test the stability of the drained split method in three cases: $\boldsymbol{\kappa}=10^{-7}\boldsymbol{I}$, $s_0=5\times10^{-5}$; $\boldsymbol{\kappa}=10^{-7}\boldsymbol{I}$, $s_0=10^{-4}$; and $\boldsymbol{\kappa}=10^{-7}\boldsymbol{I}$, $s_0=5\cdot10^{-4}$. In each case we use the time step $\Delta t=0.1$. We observe that drained split method is unstable in the first ($\alpha^2/\lambda s_0=1.4$) case, and stable in the second ($\alpha^2/\lambda s_0=0.7$) and third case ($\alpha^2/\lambda s_0=0.14$). *This is in good agreement with the theory, in which we proved that the method is stable with $\alpha^2/\lambda s_0<1$.*

4.2. Fixed strain split. The stability condition for this method to be stable is either

$$\frac{\alpha^2}{\lambda s_0}<1 \quad \text{or} \quad \Delta t<\frac{2\rho k_{\min}}{\alpha^2 d C_{PF}^2 C_{INV}^2}h^2.$$

We test fixed strain split for the following cases:

[1] An extended version of this report with more details in the proof is available at http://www.mathematics.pitt.edu/research/technical-reports.php

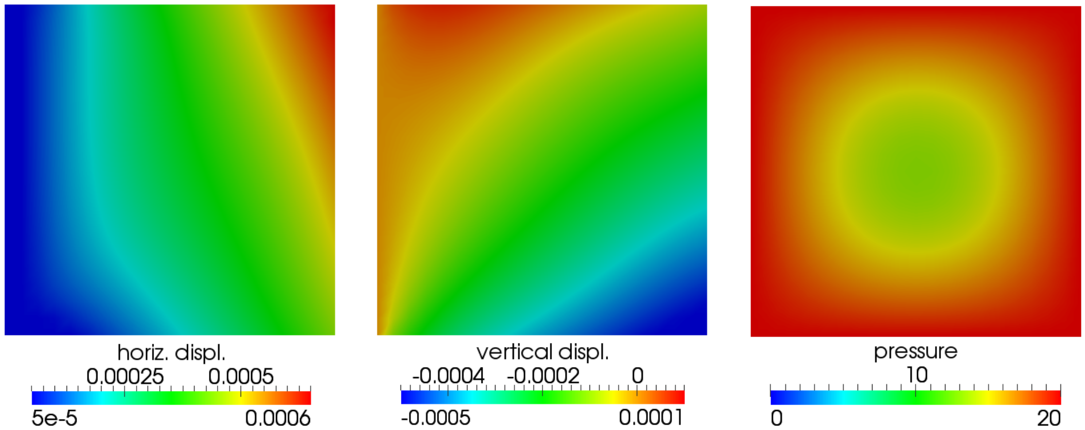

FIG. 4.1. *The reference solution at $T = 50s$. Left: Horizontal displacement. Middle: Vertical displacement. Right: Pressure.*

1.) $\boldsymbol{\kappa} = 10^{-5}\boldsymbol{I}$, $s_0 = 5\times 10^{-5}$, $\Delta t = 0.1$: In this case

$$\frac{\alpha^2}{\lambda s_0} = 1.4 \quad \text{or} \quad \Delta t \leq \frac{5\cdot 10^{-8}}{C_{INV}^2 C_{PF}^2}.$$

Even thought the first quantity is larger than 1, and the time step is larger than the one dictated by the time step condition, this method is stable. Note that we used the same parameters as for the drained split method, which was in this case unstable. *This indicates that it is more favorable to solve the fluid problem first, and the structure mechanics problem second.*

2.) $\boldsymbol{\kappa} = 10^{-5}\boldsymbol{I}$, $s_0 = 10^{-5}$, $\Delta t = 0.1$: In this case

$$\frac{\alpha^2}{\lambda s_0} = 7 \quad \text{or} \quad \Delta t \leq \frac{5\cdot 10^{-8}}{C_{INV}^2 C_{PF}^2}.$$

Numerical experiments show that using $\Delta t = 0.1$ the method is unstable. The method becomes stable with the choice of $\Delta t = 10^{-4}$. Moreover, if we decrease $\boldsymbol{\kappa}$ to $\boldsymbol{\kappa} = 10^{-6}\boldsymbol{I}$, in order to achieve stability we have to take $\Delta t = 10^{-5}$, *which confirms the linear relationship between Δt and $\boldsymbol{\kappa}$, for fixed h.*

3.) $\boldsymbol{\kappa} = 10^{-7}\boldsymbol{I}$, $s_0 = 10^{-4}$: We have

$$\frac{\alpha^2}{\lambda s_0} = 0.7 \quad \text{or} \quad \Delta t \leq \frac{5\cdot 10^{-10}}{C_{INV}^2 C_{PF}^2}.$$

If we try to take $\boldsymbol{\kappa} = 10^{-7}\boldsymbol{I}$ in the first test case, the scheme exhibits instabilities. However, in this case the first quantity is less than 1, so the theory predicts unconditional stability (no time step restrictions). *The numerical experiments show that fixed strain split in this case is stable for virtually every time step size, confirming the predictions.*

4.3. BEFE. The stability condition for this method is either

$$\frac{\alpha^2}{\lambda s_0} < 1 \quad \text{and} \quad \Delta t \leq \frac{\rho k_{\min}}{\alpha^2 C_{PF}^2 C_{INV}^2 d} h^2, \qquad \text{or} \qquad \Delta t \leq \min\left\{ \frac{\rho k_{\min}}{4\alpha^2 C_{PF}^2 C_{INV}^2 d} h^2, \frac{\sqrt{\rho s_0}}{\alpha C_{INV}\sqrt{d}} h \right\}.$$

Note that the first set of conditions is more restrictive than conditions in fixed strain split. This indicates that if one prefers to solve the problem using BEFE and a parallel solver, the price to pay is stronger conditions on the problem parameters and the time step. In particular, to solve the last test case in fixed strain split using BEFE, one would need a time step that is less than $\Delta t = 10^{-10}$.

4.4. BELF. The stability condition for BELF is

$$\Delta t \leq \max\left\{\frac{\rho k_{\min}}{\alpha^2 C_{PF}^2 C_{INV}^2 d} h^2, \frac{\sqrt{\rho s_0}}{\alpha C_{INV}\sqrt{d}} h\right\} \leq \frac{\rho h^2}{2\alpha^2 C_{INV}^2 C_{PF}^2 d}\left(k_{min} + \left(k_{min}^2 + \frac{4 s_0 \alpha^2 C_{INV}^2 C_{PF}^4 d}{\rho h^2}\right)^{1/2}\right).$$

If $s_0 \ll 1$ (is negligible), the method of choice can be BELF, ω-method (with $\omega > \frac{1}{2}$), or fixed strain split. *However, note that BELF and ω-method allow to solve the elasticity and the fluid problem in parallel, using a smaller time step than the one needed for the fixed strain split. In the case when the $k_{min} \ll 1$ is very small (as is often in applications), BELF offers a CFL condition depending on the structure density ρ and the storage coefficient s_0.* The same time step condition is theoretically needed for the CNLF method ($\omega = \frac{1}{2}$). Note that the alternative condition in fixed strain split method is a condition on the parameters of the problem, independent of Δt.

To test this scheme we choose $\boldsymbol{\kappa} = 10^{-6}\boldsymbol{I}$ and $s_0 = 10^{-5}$. The theoretical condition becomes

$$\Delta t \leq \frac{1.3 \cdot 10^{-3}}{C_{INV}^2 C_{PF}^2}\left(10^{-6} + \left(10^{-12} + 1.6 \cdot 10^{-2} C_{INV}^2 C_{PF}^4\right)^{1/2}\right).$$

The numerical experiments shows that this method is stable for $\Delta t = 3 \cdot 10^{-5}$. Note that in this case $\frac{\alpha^2}{\lambda s_0} = 7$, and the time step needed to solve this test case with fixed strain split method was $\Delta t = 10^{-5}$.

4.5. ω-method. The ω-method is a second order method that solves the fluid and the mechanics problem in parallel. However, in the cases when $\omega > \frac{1}{2}$ the scheme contains a penalty term which improves the stability properties, but may affect the accuracy. In the cases when k_{min} is small, CNLF method ($\omega = \frac{1}{2}$) offers an appealing time step if values of s_0 are large enough. The time step conditions is the same as the one for the BELF method in the cases with dominant s_0. Indeed, if $\boldsymbol{\kappa} = 10^{-6}\boldsymbol{I}$ and $s_0 = 10^{-5}$, the time step restriction for CNLF is

$$\Delta t < \frac{1.6 \cdot 10^{-4}}{C_{INV}}.$$

Numerical experiments show that CNLF is stable for $\Delta t = 3 \cdot 10^{-5}$.

If $\theta \neq \frac{1}{2}$, the time step restriction for the ω method is given by

$$\Delta t < \min\left\{\frac{\nu}{\omega(1-\omega)}\Delta t_{CNLF}, \Delta t_{\omega}\right\},$$

where

$$\Delta t_{\omega} = \frac{\rho h^2}{2\omega C_{inv}^2 C_{PF}^2 \alpha^2 (\frac{d(1-\omega)^2}{\nu} + 2\omega - 1)}\Big((2\omega-1)k_{min} + \big((2\omega-1)^2 k_{min}^2 + 4 s_0 \nu \frac{C_{inv}^2 C_{PF}^4}{\rho h^2}\alpha^2(\frac{d(1-\omega)^2}{\nu} + 2\omega - 1)\big)^{\frac{1}{2}}\Big).$$

A popular choice of ω is $\omega = 1$, in which case the ω-method is known as BDF2-AB2. With the values of parameters $\boldsymbol{\kappa} = 10^{-6}\boldsymbol{I}$ and $s_0 = 10^{-5}$, the time step conditions becomes

$$\Delta t < \frac{\rho h^2}{2 C_{inv}^2 C_{PF}^2 \alpha^2}\Big(k_{min} + (k_{min}^2 + 4 s_0 \nu \frac{C_{inv}^2 C_{PF}^4}{\rho h^2}\alpha^2)^{\frac{1}{2}}\Big) = \frac{2.5 \cdot 10^{-3}}{C_{inv}^2 C_{PF}^2}\Big(10^{-6} + (10^{-12} + 2 \cdot 10^{-3} C_{inv}^2 C_{PF}^4)^{\frac{1}{2}}\Big)$$

The numerical experiments show that this method is stable when $\Delta t = 10^{-5}$.

5. Conclusions. We have expanded the number of tools (partitioned methods) available for solving the coupled Biot system using methods optimized for individual sub-physics problems. We have also given a comprehensive stability analysis of the methods based on energy methods. These results give sufficient conditions for the given (non periodic) boundary conditions, include the effects of non-constant physical parameters and have carefully tracked the dependence on physical parameters in the results. In our experiments, the sufficient conditions were found to

Effect	Definition
Elastic wave speed	$\mathbf{c}_E = \sqrt{\dfrac{\lambda}{\rho}}$
derived Elastic time scale	$\tau_E = \dfrac{L}{c_E} = L\sqrt{\dfrac{\rho}{\lambda}}$
——	—-
Darcy relaxation time	$\tau_D = \dfrac{L^2 s_0}{\kappa}$
derived Darcy relaxation speed	$\mathbf{c}_D = \dfrac{\kappa}{L s_0}$
Coupling Strength	$\mathbf{\Lambda} = \sqrt{\dfrac{\alpha^2}{\rho s_0}}$

TABLE 5.1
Characteristic quantities for the coupled Biot system.

be quite sharp. The stability results seem essentially complex and thus hard to use to choose a good method for a given physical setting. However, at least for constant physical parameters, the results can be (perhaps surprisingly) simplified greatly and into a useful table (below). We conclude this paper by giving this analysis of the Biot system and the methods studied.

The coupled Biot system has three competing effects: Wave propagation in the elastic components, with key parameter **wave-speed**

$$\mathbf{c}_E = \sqrt{\frac{\lambda}{\rho}},$$

porous media relaxation in Darcy, with key parameter **relaxation time**

$$\tau_D = \frac{L^2 s_0}{\kappa},$$

where L is the global length scale, and the system coupling, with key parameter a measure of coupling strength. For the relaxation time, we form the **relaxation speed**

$$\mathbf{c}_D = \frac{\kappa}{L s_0}.$$

Our intuition is that the essential steps in the numerical analysis involve interaction of these effects and thus should be phrased in terms of parameters measuring their magnitude. To this end, we define (for constant physical parameters[2]) the three key parameters and two derived parameters for each subproblem and the system coupling in Table 5.1.

For the coupling strengths, the natural measures of it in each equation are the **elastic coupling strength**

$$\mathbf{\Lambda}_E = \frac{\alpha}{\rho},$$

[2]The detailed stability theorems are stated in terms of the correct values needed in the variable case.

Parameter condition		
$\frac{\alpha^2}{\lambda s_0} < 1$	$\frac{\mathbf{\Lambda}^2}{c_E^2} < 1$	
Time step condition	**Time scale τ_E**	**Time scale τ_D**
$\Delta t \leq C^{\bigstar} \left(\frac{h}{L}\right)^2 \frac{\rho\kappa}{\alpha^2}$	$\frac{\triangle t}{\tau_E} \leq C^{\bigstar} \left(\frac{h}{L}\right)^2 \frac{c_D \cdot c_E}{\mathbf{\Lambda}^2}$	$\frac{\triangle t}{\tau_D} \leq C^{\bigstar} \left(\frac{h}{L}\right)^2 \frac{c_D^2}{\mathbf{\Lambda}^2}$
$\Delta t \leq C^{\bigstar} \left(\frac{h}{L}\right) \frac{L\sqrt{\rho s_0}}{\alpha}$	$\frac{\triangle t}{\tau_E} \leq C^{\bigstar} \left(\frac{h}{L}\right) \frac{1}{\mathbf{\Lambda}}$	$\frac{\triangle t}{\tau_D} \leq C^{\bigstar} \left(\frac{h}{L}\right) \frac{c_D}{\mathbf{\Lambda}}$
$\Delta t \leq \frac{\rho\kappa}{\alpha^2}$	$\frac{\triangle t}{\tau_E} \leq \frac{c_D \cdot c_E}{\mathbf{\Lambda}^2}$	$\frac{\triangle t}{\tau_D} \leq \frac{c_D^2}{\mathbf{\Lambda}^2}$

TABLE 5.2

Characteristic quantities for the coupled Biot system.

the **Darcy coupling strength**

$$\mathbf{\Lambda}_D = \frac{\alpha}{S_0},$$

and their geometric average

$$\mathbf{\Lambda} = \sqrt{\frac{\alpha^2}{\rho s_0}}.$$

The choice of the global length scale L (such as $L = diam(\Omega)$) is arbitrary at this level of generality. For the two constants of analysis in the stability theorems, note that C_{INV} depends only on the minimum angle and is unit free while C_{PF} has units $[C_{PF}] = L$. Let $C^{\bigstar}$ denote an absolute (unit free) constant. Four conditions occur often in the stability analysis. Rearranging these in the stability theorems and choosing time scale to be τ_E or τ_D gives the equivalent conditions presented in Table 5.2. We observe that the key combination of the parameters is

$$B := \frac{\textit{Coupling Strength}}{\textit{Speed of Propagation}}.$$

Therefore, we define the following quantities.

DEFINITION 5.1. *With $\mathbf{\Lambda}, c_D, c_E$ defined in Table 5.1, set*

$$B_E \ : \ = \frac{\mathbf{\Lambda}}{c_E}\ , \ B_D := \frac{\mathbf{\Lambda}}{c_D}, \textit{ and}$$
$$B_{Biot} = B := \sqrt{B_D \cdot B_E} = \frac{\mathbf{\Lambda}}{\sqrt{c_D \cdot c_E}}.$$

To make precise estimates, denote by $C = C_{INV}\sqrt{d}$, and let $\tilde{c}_{PF}$ be the dimensionless part of C_{PF}. Table 5.3 gives a summary comparison of the methods with respect to *Coupling Strength / Speed of Propagation*.

From qualitative behavior shown in Figure 5.1 we conclude that when the speed of propagation is dominated by the coupling strength, the optimal method from the ω family (3.40) is CNLF. Also we notice that there is a rapid transition between the two regimes/methods (CNLF and BDF2-AB2) as the ratio of the speed of propagation and

Method	Stability Condition
DS	$B_E^2 < 1$
FSS	$B_E^2 < 1 \quad \text{or} \quad \dfrac{\Delta t}{\tau_D} < 2\dfrac{1}{\tilde{c}_{PF}^2}\big(\dfrac{h}{L}\big)^2(CB_D)^{-2}$
BEFE	$B_E^2 < 1 \quad \text{and} \quad \dfrac{\Delta t}{\tau_D} < \dfrac{1}{\tilde{c}_{PF}^2}\big(\dfrac{h}{L}\big)^2(CB_D)^{-2}$
or	$\dfrac{\triangle t}{\tau_D} \leq \min\big\{\dfrac{1}{4}\dfrac{1}{\tilde{c}_{PF}^2}\big(\dfrac{h}{L}\big)^2(CB_D)^{-2},\ \dfrac{h}{L}(CB_D)^{-1}\big\}$
BELF	$\dfrac{\triangle t}{\tau_D} \leq \max\big\{\dfrac{1}{\tilde{c}_{PF}^2}\big(\dfrac{h}{L}\big)^2(CB_D)^{-2},\ \dfrac{h}{L}(CB_D)^{-1}\big\}$
BA($\theta = \frac{1}{2}$)	$\dfrac{\Delta t}{\tau_D} \leq \dfrac{h}{L}(CB_D)^{-1}$
BA($\theta \neq \frac{1}{2}$)	$\dfrac{\triangle t}{\tau_D} \leq \min\big\{\dfrac{1}{\tilde{c}_{PF}^2}\dfrac{2\theta-1}{\theta^2}\big(\dfrac{h}{L}\big)^2(CB_D)^{-2},\ \dfrac{\nu}{\theta(1-\theta)}\dfrac{h}{L}(CB_D)^{-1},\ \dfrac{C^2}{\theta}(CB_D)^{-2}\big\}$

TABLE 5.3

Comparison of stability properties of the partitioned methods. The abbreviations in the first column are: ***DS****= Drained Split,* ***FSS****= Fixed strain split and* ***BA*** *= ω.*

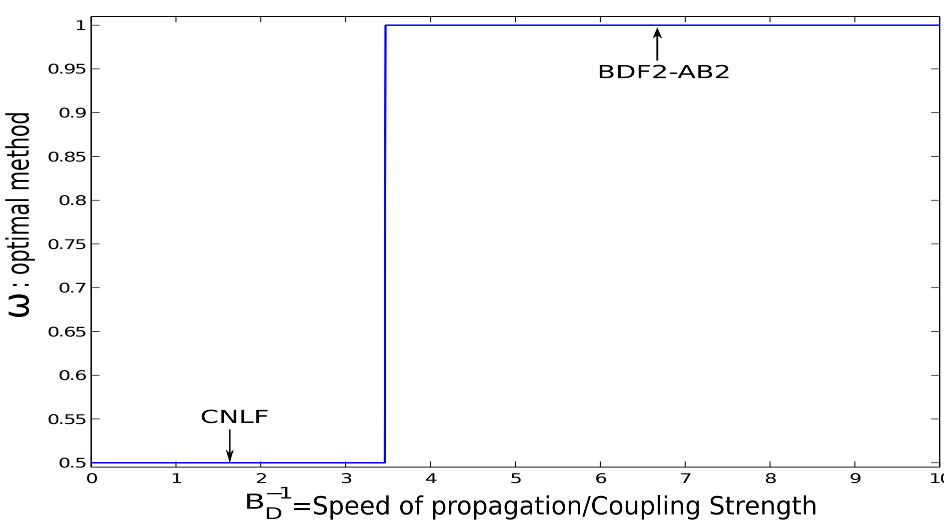

FIG. 5.1. *The transition between CNLF and BDF2-AB2 regimes of the ω family (3.40), as a function of the ratio between the coupling strength and the speed of propagation.*

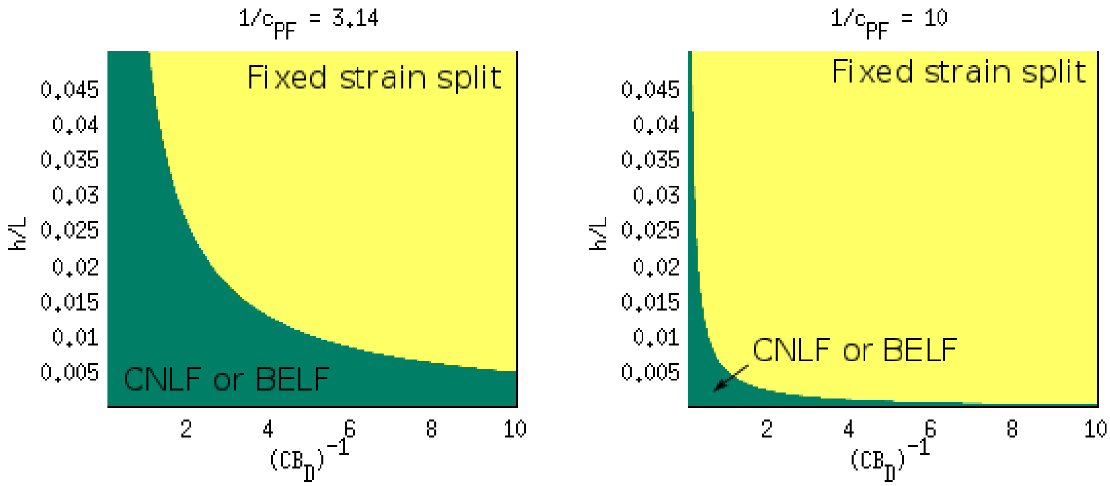

FIG. 5.2. *The choice of method which allows the largest time step in the case when* $\frac{1}{\tilde{c}_{PF}} = 3.14$ *(left), and* $\frac{1}{\tilde{c}_{PF}} = 10$ *(right).*

the coupling strength increases. In the cases with high speeds of propagation, the optimal method, with highest allowable time step, is BDF2-AB2.

To summarize, we find the method that allows the largest time step with respect to the relative mesh step $\frac{h}{L}$ and the parameters of the problem CB_D. We distinguish two cases, $B_E^2 < 1$ and $B_E^2 > 1$. When $B_E^2 < 1$, the method of choice could be drained split or fixed strain split. If $B_E^2 > 1$, our results are given as follows. In order to estimate the size of $\tilde{c}_{PF}$, we used the following result from [5] for the domain and boundary conditions in the numerical tests:

$$\frac{1}{\tilde{c}_{PF}} \geq \pi.$$

Figure 5.2 left shows the method that allows the largest time step in the case when $\frac{1}{\tilde{c}_{PF}} = 3.14$. The case when $\frac{1}{\tilde{c}_{PF}} = 10$ is shown in Figure 5.2 right.

6. Appendix. In order to prove the energy estimate in Theorem 3.7, we will use Dalhquist's G-stability methodology [4, 6, 20]. Let define the positive definite matrix

$$G = \begin{pmatrix} 2\omega^2 - \omega + \frac{1}{4} & -\frac{(2\omega-1)^2}{2} \\ -\frac{(2\omega-1)^2}{2} & 2\omega^2 - 3\omega + \frac{5}{4} \end{pmatrix}, \quad \omega \in \left[\tfrac{1}{2}, 1\right], \tag{6.1}$$

and notice that

$$\begin{aligned} &\left((2\omega - \tfrac{1}{2})\boldsymbol{u}_h^{n+1} + (-4\omega + 2)\boldsymbol{u}_h^n + (2\omega - \tfrac{3}{2})\boldsymbol{u}_h^{n-1}, \omega\boldsymbol{u}_h^{n+1} + (1-\omega)\boldsymbol{u}_h^{n-1}\right) \\ &\quad = \|(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^n)\|_G^2 - \|(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n-1})\|_G^2 + \tfrac{2\omega-1}{4}\|\boldsymbol{u}_h^{n+1} - 2\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}\|^2, \end{aligned}$$

similarly

$$\begin{aligned} &\left((2\omega - \tfrac{1}{2})p_h^{n+1} + (-4\omega + 2)p_h^n + (2\omega - \tfrac{3}{2})p_h^{n-1}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1}\right) \\ &\quad = \|(p_h^{n+1}, p_h^n)\|_G^2 - \|(p_h^n, p_h^{n-1})\|_G^2 + \tfrac{2\omega-1}{4}\|p_h^{n+1} - 2p_h^n + p_h^{n-1}\|^2, \end{aligned}$$

and

$$a_e\left(\omega\boldsymbol{\eta}_h^{n+1} + (1-\omega)\boldsymbol{\eta}_h^{n-1}, (2\omega - \tfrac{1}{2})\boldsymbol{\eta}_h^{n+1} + (-4\omega + 2)\boldsymbol{\eta}_h^n + (2\omega - \tfrac{3}{2})\boldsymbol{\eta}_h^{n-1}\right)$$

$$= \|(\boldsymbol{\eta}_h^{n+1},\boldsymbol{\eta}_h^{n})\|^2_{G,a_e} - \|(\boldsymbol{\eta}_h^{n},\boldsymbol{\eta}_h^{n-1})\|^2_{G,a_e} + \tfrac{2\omega-1}{4} a_e\big(\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1},\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1}\big),$$

where

$$\begin{aligned}\|(\boldsymbol{\eta}_h^{n+1},\boldsymbol{\eta}_h^{n})\|^2_{G,a_e} :=& (2\omega^2-\omega+\tfrac{1}{2})a_e(\boldsymbol{\eta}_h^{n+1},\boldsymbol{\eta}_h^{n+1}) - (2\omega-1)^2 a_e(\boldsymbol{\eta}_h^{n+1},\boldsymbol{\eta}_h^{n})\\ &+ (2\omega^2-3\omega+\tfrac{5}{4})a_e(\boldsymbol{\eta}_h^{n},\boldsymbol{\eta}_h^{n}).\end{aligned}$$

For the proof of Theorem 3.7 we need the following preliminary results, which follow by algebraic manipulation.

LEMMA 6.1. $\|(u^{n+1},u^n)\|^2_G = \nu\big(\|u^{n+1}\|^2+\|u^n\|^2\big) + \|au^{n+1}+bu^n\|^2$, *where*

$$\begin{aligned}&\nu = \frac{1}{16(2\omega^2-3\omega+\frac{5}{4})} \in \big[\tfrac{1}{4},\tfrac{1}{2}\big],\\ &a = -\sqrt{\tfrac{2\omega-1}{2}\big(\sqrt{1+(2\omega-1)^2}+1\big)},\ b = \sqrt{\tfrac{2\omega-1}{2}\big(\sqrt{1+(2\omega-1)^2}-1\big)}.\end{aligned}$$

LEMMA 6.2. *The contribution of the coupling terms to the energy equation is*

$$\begin{aligned}&\sum_{n=1}^{N}\Big(-b(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1},2\omega p^n+(-2\omega+1)p^{n-1})\\ &\qquad + b(2\omega\boldsymbol{u}^{n}+(-2\omega+1)\boldsymbol{u}^{n-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\Big)\\ &= 2\omega(1-\omega)\big(b(\boldsymbol{u}^N,p^{N+1})-b(\boldsymbol{u}^{N+1},p^N)\big)\\ &\qquad +\omega(1-2\omega)\sum_{n=1}^{N}\big(b(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1},p^{n+1})-b(\boldsymbol{u}^{n+1},p^{n+1}-2p^n+p^{n-1})\big)\\ &\qquad -2\omega(1-\omega)\big(b(\boldsymbol{u}^0,p^1)-b(\boldsymbol{u}^1,p^0)\big).\end{aligned}$$

LEMMA 6.3. *The dissipative term in pressure writes as*

$$\begin{aligned}&\sum_{n=1}^{N} a_p(\omega p^{n+1}+(1-\omega)p^{n-1},\omega p^{n+1}+(1-\omega)p^{n-1})\\ &= \omega(2\omega-1)\Big(a_p(p^{N+1},p^{N+1})+a_p(p^N,p^N)\Big) + (2\omega-1)^2\sum_{n=2}^{N-1}a_p(p^n,p^n)\\ &\qquad -(1-\omega)(2\omega-1)\big(a_p(p^1,p^1)+a_p(p^0,p^0)\big)\\ &\qquad +\omega(1-\omega)\sum_{n=1}^{N}a_p(p^{n+1}+p^{n-1},p^{n+1}+p^{n-1}),\end{aligned}$$

and similarly the stabilizing term

$$\begin{aligned}&\sum_{n=1}^{N}\|\nabla\cdot(\omega\boldsymbol{u}_h^{n+1}+(1-\omega)\boldsymbol{u}_h^{n-1})\|^2\\ &= \omega(2\omega-1)\big(\|\nabla\cdot\boldsymbol{u}_h^{N+1}\|^2+\|\nabla\cdot\boldsymbol{u}_h^{N}\|^2\big) + (2\omega-1)^2\sum_{n=2}^{N-1}\|\nabla\cdot\boldsymbol{u}_h^{n}\|^2\\ &\qquad -(1-\omega)(2\omega-1)(\|\nabla\cdot\boldsymbol{u}_h^{1}\|^2+\|\nabla\cdot\boldsymbol{u}_h^{0}\|^2)\\ &\qquad +\omega(1-\omega)\sum_{n=1}^{N}\|\nabla\cdot(\boldsymbol{u}_h^{n+1}+\boldsymbol{u}_h^{n-1})\|^2.\end{aligned}$$

Proof. [Proof of Theorem 3.7] Using the following test functions in (3.40)

$$\begin{aligned}
&\boldsymbol{v}_h = \omega \boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1},\\
&\boldsymbol{w}_h = -\tfrac{1}{\rho}\nabla\cdot\boldsymbol{\sigma}^E(\omega\boldsymbol{\eta}^{n+1} + (1-\omega)\boldsymbol{\eta}^{n-1})\\
&\psi_h = \omega p_h^{n+1} + (1-\omega)p_h^{n-1}
\end{aligned}$$

after summation we obtain

$$\begin{aligned}
&\rho\big(\tfrac{(2\omega-\frac{1}{2})\boldsymbol{u}_h^{n+1}+(-4\omega+2)\boldsymbol{u}_h^{n}+(2\omega-\frac{3}{2})\boldsymbol{u}_h^{n-1}}{\Delta t}, \omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1}\big)\\
&\quad + a_e(\omega\boldsymbol{\eta}^{n+1} + (1-\omega)\boldsymbol{\eta}^{n-1}, \omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1})\\
&\quad + \Delta t\frac{\alpha^2 c_\omega}{s_0}\big(\nabla\cdot(\omega\boldsymbol{u}_h^{n+1} + (1-\omega)\boldsymbol{u}_h^{n-1}), \nabla\cdot(\omega\boldsymbol{u}_h^{n+1} + (1-\omega)\boldsymbol{u}_h^{n-1})\big)\\
&\quad - b(\omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1}, 2\omega p^n + (-2\omega+1)p^{n-1})\\
&- a_e\big(\omega\boldsymbol{u}_h^{n+1} + (1-\omega)\boldsymbol{u}_h^{n-1}, \omega\boldsymbol{\eta}^{n+1} + (1-\omega)\boldsymbol{\eta}^{n-1}\big)\\
&\quad + a_e\big(\tfrac{(2\omega-\frac{1}{2})\boldsymbol{\eta}_h^{n+1}+(-4\omega+2)\boldsymbol{\eta}_h^{n}+(2\omega-\frac{3}{2})\boldsymbol{\eta}_h^{n-1}}{\Delta t}, \omega\boldsymbol{\eta}^{n+1} + (1-\omega)\boldsymbol{\eta}^{n-1}\big)\\
&+ s_0\big(\tfrac{(2\omega-\frac{1}{2})\boldsymbol{p}_h^{n+1}+(-4\omega+2)\boldsymbol{p}_h^{n}+(2\omega-\frac{3}{2})\boldsymbol{p}_h^{n-1}}{\Delta t}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1}\big)\\
&\quad + b(2\omega\boldsymbol{u}^n + (-2\omega+1)\boldsymbol{u}^{n-1}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1})\\
&\quad + a_p(\omega p_h^{n+1} + (1-\omega)p_h^{n-1}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1})\\
&= \big(\boldsymbol{f}^{n+2\omega-1}, \omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1}\big) + \int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1})\,d\sigma\\
&+ (s^{n+2\omega-1}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1}).
\end{aligned}$$

By rearranging terms,

$$\begin{aligned}
&\rho\big(\tfrac{(2\omega-\frac{1}{2})\boldsymbol{u}_h^{n+1}+(-4\omega+2)\boldsymbol{u}_h^{n}+(2\omega-\frac{3}{2})\boldsymbol{u}_h^{n-1}}{\Delta t}, \omega\boldsymbol{u}_h^{n+1} + (1-\omega)\boldsymbol{u}_h^{n-1}\big)\\
&+ s_0\big(\tfrac{(2\omega-\frac{1}{2})p_h^{n+1}+(-4\omega+2)p_h^{n}+(2\omega-\frac{3}{2})p_h^{n-1}}{\Delta t}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1}\big)\\
&+ a_e(\omega\boldsymbol{\eta}^{n+1} + (1-\omega)\boldsymbol{\eta}^{n-1}, \tfrac{(2\omega-\frac{1}{2})\boldsymbol{\eta}_h^{n+1}+(-4\omega+2)\boldsymbol{\eta}_h^{n}+(2\omega-\frac{3}{2})\boldsymbol{\eta}_h^{n-1}}{\Delta t})\\
&\quad + \Delta t\frac{\alpha^2 c_\omega}{s_0}\big(\nabla\cdot(\omega\boldsymbol{u}_h^{n+1} + (1-\omega)\boldsymbol{u}_h^{n-1}), \nabla\cdot(\omega\boldsymbol{u}_h^{n+1} + (1-\omega)\boldsymbol{u}_h^{n-1})\big)\\
&\quad + a_p(\omega p_h^{n+1} + (1-\omega)p_h^{n-1}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1})\\
&- b(\omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1}, 2\omega p^n + (-2\omega+1)p^{n-1})\\
&\quad + b(2\omega\boldsymbol{u}^n + (-2\omega+1)\boldsymbol{u}^{n-1}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1})\\
&= \big(\boldsymbol{f}^{n+2\omega-1}, \omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1}\big) + \int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1} + (1-\omega)\boldsymbol{u}^{n-1})\,d\sigma\\
&+ (s^{n+2\omega-1}, \omega p_h^{n+1} + (1-\omega)p_h^{n-1}).
\end{aligned}$$

using the G-matrix (6.1), the above estimate becomes

$$\begin{aligned}
&\frac{\rho}{\Delta t}\big(\|(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n})\|_G^2 - \|(\boldsymbol{u}_h^{n}, \boldsymbol{u}_h^{n-1})\|_G^2 + \tfrac{2\omega-1}{4}\|\boldsymbol{u}_h^{n+1} - 2\boldsymbol{u}_h^{n} + \boldsymbol{u}_h^{n-1}\|^2\big)\\
&+ \frac{s_0}{\Delta t}\big(\|(p_h^{n+1}, p_h^{n})\|_G^2 - \|(p_h^{n}, p_h^{n-1})\|_G^2 + \tfrac{2\omega-1}{4}\|p_h^{n+1} - 2p_h^{n} + p_h^{n-1}\|^2\big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Delta t}\Big(\|(\boldsymbol{\eta}_h^{n+1},\boldsymbol{\eta}_h^{n})\|_{G,a_e}^2-\|(\boldsymbol{\eta}_h^{n},\boldsymbol{\eta}_h^{n-1})\|_{G,a_e}^2+\tfrac{2\omega-1}{4}a_e\big(\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1},\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1}\big)\Big)\\
&\quad+\Delta t\frac{\alpha^2c_\omega}{s_0}\big(\nabla\cdot(\omega\boldsymbol{u}_h^{n+1}+(1-\omega)\boldsymbol{u}_h^{n-1}),\nabla\cdot(\omega\boldsymbol{u}_h^{n+1}+(1-\omega)\boldsymbol{u}_h^{n-1})\big)\\
&\quad+a_p(\omega p_h^{n+1}+(1-\omega)p_h^{n-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\\
&-b(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1},2\omega p^{n}+(-2\omega+1)p^{n-1})\\
&\quad+b(2\omega\boldsymbol{u}^{n}+(-2\omega+1)\boldsymbol{u}^{n-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\\
&=\big(\boldsymbol{f}^{n+2\omega-1},\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1}\big)+\int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1})\,d\sigma\\
&+(s^{n+2\omega-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1}),
\end{aligned}
$$

and summing from $n=1$ to N, we obtain

$$
\begin{aligned}
&\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^{N+1},\boldsymbol{u}_h^{N})\|_G^2+\tfrac{\rho}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}\|\boldsymbol{u}_h^{n+1}-2\boldsymbol{u}_h^{n}+\boldsymbol{u}_h^{n-1}\|^2\\
&+\frac{s_0}{\Delta t}\|(p_h^{N+1},p_h^{N})\|_G^2+\tfrac{s_0}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}\|p_h^{n+1}-2p_h^{n}+p_h^{n-1}\|^2\\
&+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^{N+1},\boldsymbol{\eta}_h^{N})\|_{G,a_e}^2+\tfrac{1}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}a_e\big(\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1},\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1}\big)\Big)\\
&\quad+\Delta t\frac{\alpha^2c_\omega}{s_0}\sum_{n=1}^{N}\|\nabla\cdot(\omega\boldsymbol{u}_h^{n+1}+(1-\omega)\boldsymbol{u}_h^{n-1})\|^2\\
&\quad+\sum_{n=1}^{N}a_p(\omega p_h^{n+1}+(1-\omega)p_h^{n-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\\
&+\sum_{n=1}^{N}\Big(-b(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1},2\omega p^{n}+(-2\omega+1)p^{n-1})\\
&\quad+b(2\omega\boldsymbol{u}^{n}+(-2\omega+1)\boldsymbol{u}^{n-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\Big)\\
&=\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^{1},\boldsymbol{u}_h^{0})\|_G^2+\frac{s_0}{\Delta t}\|(p_h^{1},p_h^{0})\|_G^2+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^{1},\boldsymbol{\eta}_h^{0})\|_{G,a_e}^2\\
&\quad+\sum_{n=1}^{N}\Big(\big(\boldsymbol{f}^{n+2\omega-1},\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1}\big)+\int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1})\,d\sigma\\
&\quad+(s^{n+2\omega-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\Big).
\end{aligned}
$$

Using by Lemma 6.2 the energy balance writes

$$
\begin{aligned}
&\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^{N+1},\boldsymbol{u}_h^{N})\|_G^2+\tfrac{\rho}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}\|\boldsymbol{u}_h^{n+1}-2\boldsymbol{u}_h^{n}+\boldsymbol{u}_h^{n-1}\|^2\\
&+\frac{s_0}{\Delta t}\|(p_h^{N+1},p_h^{N})\|_G^2+\tfrac{s_0}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}\|p_h^{n+1}-2p_h^{n}+p_h^{n-1}\|^2\\
&+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^{N+1},\boldsymbol{\eta}_h^{N})\|_{G,a_e}^2+\tfrac{1}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}a_e\big(\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1},\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1}\big)\Big)\\
&\quad+\Delta t\frac{\alpha^2c_\omega}{s_0}\sum_{n=1}^{N}\|\nabla\cdot(\omega\boldsymbol{u}_h^{n+1}+(1-\omega)\boldsymbol{u}_h^{n-1})\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{N} a_p(\omega p_h^{n+1}+(1-\omega)p_h^{n-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\\
&+2\omega(1-\omega)\big(b(\boldsymbol{u}^N,p^{N+1})-b(\boldsymbol{u}^{N+1},p^N)\big)\\
&+\omega(1-2\omega)\sum_{n=1}^{N}\big(b(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1},p^{n+1})-b(\boldsymbol{u}^{n+1},p^{n+1}-2p^n+p^{n-1})\big)\\
&=\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^1,\boldsymbol{u}_h^0)\|_G^2+\frac{s_0}{\Delta t}\|(p_h^1,p_h^0)\|_G^2+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^1,\boldsymbol{\eta}_h^0)\|_{G,a_e}^2+2\omega(1-\omega)\big(b(\boldsymbol{u}^0,p^1)-b(\boldsymbol{u}^1,p^0)\big)\\
&+\sum_{n=1}^{N}\Big(\big(\boldsymbol{f}^{n+2\omega-1},\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1}\big)+\int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1})\,d\sigma\\
&+(s^{n+2\omega-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\Big).
\end{aligned}
$$

Using (2.11b), the polarized identity, we write the coupling terms

$$
\begin{aligned}
&\omega(1-2\omega)\sum_{n=1}^{N}\big(b(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1},p^{n+1})-b(\boldsymbol{u}^{n+1},p^{n+1}-2p^n+p^{n-1})\big)\\
&=\omega(1-2\omega)\alpha\sum_{n=1}^{N}\big((p^{n+1},\nabla\cdot(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1}))-(p^{n+1}-2p^n+p^{n-1},\nabla\cdot\boldsymbol{u}^{n+1})\big)\\
&=-(2\omega-1)\sum_{n=1}^{N}\big(\frac{\rho}{4\Delta t}\frac{h^2}{dC_{inv}^2}\|\nabla\cdot(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1})\|^2+\frac{\Delta t}{\rho}\alpha^2\omega^2\frac{dC_{inv}^2}{h^2}\|p^{n+1}\|^2\big)\\
&-(2\omega-1)\sum_{n=1}^{N}\big(\frac{s_0}{4\Delta t}\|p^{n+1}-2p^n+p^{n-1}\|^2+\frac{\Delta t}{s_0}\alpha^2\omega^2\|\nabla\cdot\boldsymbol{u}^{n+1}\|^2\big)\\
&+(2\omega-1)\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{\rho}{4\Delta t}}p^{n+1}+\sqrt{\tfrac{\Delta t}{\rho}}\alpha\omega\nabla\cdot(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1})\big\|^2\big)\\
&+(2\omega-1)\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{s_0}{4\Delta t}}(p^{n+1}-2p^n+p^{n-1})+\sqrt{\tfrac{\Delta t}{s_0}}\alpha\omega\,\nabla\cdot\boldsymbol{u}^{n+1}\big\|^2\big),
\end{aligned}
$$

and also (using the notation $\varepsilon_1=\frac{\Delta t}{1-\varepsilon}\frac{\omega(1-\omega)}{\nu}\frac{\alpha dC_{inv}^2}{\rho h^2}$)

$$
\begin{aligned}
&2\omega(1-\omega)\big(b(\boldsymbol{u}_h^N,p_h^{N+1})-b(\boldsymbol{u}_h^{N+1},p_h^N)\big)\\
&=\varepsilon_1\alpha\omega(1-\omega)\big(\|p_h^{N+1}\|^2+\|p_h^N\|^2\big)+\tfrac{1}{\varepsilon_1}\alpha\omega(1-\omega)\big(\|\nabla\cdot\boldsymbol{u}_h^{N+1}\|^2+\|\nabla\cdot\boldsymbol{u}_h^N\|^2\big)\\
&+\alpha\omega(1-\omega)\big(\big\|\sqrt{\varepsilon_1}p_h^{N+1}+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^N\big\|^2+\big\|\sqrt{\varepsilon_1}p_h^N+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^{N+1}\big\|^2\big).
\end{aligned}
$$

Then substituting above we obtain

$$
\begin{aligned}
&\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^{N+1},\boldsymbol{u}_h^N)\|_G^2+\tfrac{\rho}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}\|\boldsymbol{u}_h^{n+1}-2\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1}\|^2\\
&+\frac{s_0}{\Delta t}\|(p_h^{N+1},p_h^N)\|_G^2+\tfrac{s_0}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}\|p_h^{n+1}-2p_h^n+p_h^{n-1}\|^2\\
&+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^{N+1},\boldsymbol{\eta}_h^N)\|_{G,a_e}^2+\tfrac{1}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}a_e\big(\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^n+\boldsymbol{\eta}_h^{n-1},\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^n+\boldsymbol{\eta}_h^{n-1}\big)\Big)\\
&+\Delta t\frac{\alpha^2c_\omega}{s_0}\sum_{n=1}^{N}\|\nabla\cdot(\omega\boldsymbol{u}_h^{n+1}+(1-\omega)\boldsymbol{u}_h^{n-1})\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{N} a_p(\omega p_h^{n+1}+(1-\omega)p_h^{n-1}, \omega p_h^{n+1}+(1-\omega)p_h^{n-1}) \\
&-\varepsilon_1\alpha\omega(1-\omega)\big(\|p^{N+1}\|^2+\|p^N\|^2\big)-\tfrac{1}{\varepsilon_1}\alpha\omega(1-\omega)\big(\|\nabla\cdot\boldsymbol{u}^{N+1}\|^2+\|\nabla\cdot\boldsymbol{u}^N\|^2\big) \\
&-(2\omega-1)\sum_{n=1}^{N}\big(\frac{\rho}{4\Delta t}\frac{h^2}{dC_{inv}^2}\|\nabla\cdot(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1})\|^2+\frac{\Delta t}{\rho}\alpha^2\omega^2\frac{dC_{inv}^2}{h^2}\|p^{n+1}\|^2\big) \\
&-(2\omega-1)\sum_{n=1}^{N}\big(\frac{s_0}{4\Delta t}\|p^{n+1}-2p^n+p^{n-1}\|^2+\frac{\Delta t}{s_0}\alpha^2\omega^2\|\nabla\cdot\boldsymbol{u}^{n+1}\|^2\big) \\
&+\alpha\omega(1-\omega)\big(\big\|\sqrt{\varepsilon_1}p_h^{N+1}+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^N\big\|^2+\big\|\sqrt{\varepsilon_1}p_h^N+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^{N+1}\big\|^2\big) \\
&+(2\omega-1)\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{\rho}{4\Delta t}}p^{n+1}+\sqrt{\tfrac{\Delta t}{\rho}}\alpha\omega\nabla\cdot(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1})\big\|^2\big) \\
&+(2\omega-1)\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{s_0}{4\Delta t}}(p^{n+1}-2p^n+p^{n-1})+\sqrt{\tfrac{\Delta t}{s_0}}\alpha\omega\,\nabla\cdot\boldsymbol{u}^{n+1}\big\|^2\big) \\
&=\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^1,\boldsymbol{u}_h^0)\|_G^2+\frac{s_0}{\Delta t}\|(p_h^1,p_h^0)\|_G^2+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^1,\boldsymbol{\eta}_h^0)\|_{G,a_e}^2+2\omega(1-\omega)\big(b(\boldsymbol{u}^0,p^1)-b(\boldsymbol{u}^1,p^0)\big) \\
&+\sum_{n=1}^{N}\Big(\big(\boldsymbol{f}^{n+2\omega-1},\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1}\big)+\int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1})\,d\sigma \\
&+(s^{n+2\omega-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\Big).
\end{aligned}
$$

Using the inverse and divergence inequalities (3.1) and (3.4), canceling out terms yields

$$
\begin{aligned}
&\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^{N+1},\boldsymbol{u}_h^N)\|_G^2-\tfrac{1}{\varepsilon_1}\alpha\omega(1-\omega)\big(\|\nabla\cdot\boldsymbol{u}^{N+1}\|^2+\|\nabla\cdot\boldsymbol{u}^N\|^2\big) \\
&+\frac{s_0}{\Delta t}\|(p_h^{N+1},p_h^N)\|_G^2+\tfrac{s_0}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}\|p_h^{n+1}-2p_h^n+p_h^{n-1}\|^2 \\
&+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^{N+1},\boldsymbol{\eta}_h^N)\|_{G,a_e}^2+\tfrac{1}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}a_e\big(\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^n+\boldsymbol{\eta}_h^{n-1},\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^n+\boldsymbol{\eta}_h^{n-1})\Big) \\
&+\Delta t\frac{\alpha^2c_\omega}{s_0}\sum_{n=1}^{N}\|\nabla\cdot(\omega\boldsymbol{u}_h^{n+1}+(1-\omega)\boldsymbol{u}_h^{n-1})\|^2 \\
&+\sum_{n=1}^{N}a_p(\omega p_h^{n+1}+(1-\omega)p_h^{n-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})-(2\omega-1)\frac{\Delta t}{\rho}\alpha^2\omega^2\frac{dC_{inv}^2}{h^2}\sum_{n=1}^{N}\|p^{n+1}\|^2 \\
&-\varepsilon_1\alpha\omega(1-\omega)\big(\|p^{N+1}\|^2+\|p^N\|^2\big) \\
&-(2\omega-1)\sum_{n=1}^{N}\big(\frac{s_0}{4\Delta t}\|p^{n+1}-2p^n+p^{n-1}\|^2+\frac{\Delta t}{s_0}\alpha^2\omega^2\|\nabla\cdot\boldsymbol{u}^{n+1}\|^2\big) \\
&+\alpha\omega(1-\omega)\big(\big\|\sqrt{\varepsilon_1}p_h^{N+1}+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^N\big\|^2+\big\|\sqrt{\varepsilon_1}p_h^N+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^{N+1}\big\|^2\big) \\
&+(2\omega-1)\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{\rho}{4\Delta t}}p^{n+1}+\sqrt{\tfrac{\Delta t}{\rho}}\alpha\omega\nabla\cdot(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1})\big\|^2\big) \\
&+(2\omega-1)\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{s_0}{4\Delta t}}(p^{n+1}-2p^n+p^{n-1})+\sqrt{\tfrac{\Delta t}{s_0}}\alpha\omega\,\nabla\cdot\boldsymbol{u}^{n+1}\big\|^2\big) \\
&\le\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^1,\boldsymbol{u}_h^0)\|_G^2+\frac{s_0}{\Delta t}\|(p_h^1,p_h^0)\|_G^2+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^1,\boldsymbol{\eta}_h^0)\|_{G,a_e}^2+2\omega(1-\omega)\big(b(\boldsymbol{u}^0,p^1)-b(\boldsymbol{u}^1,p^0)\big)
\end{aligned}
$$

$$+\sum_{n=1}^{N}\Big(\big(\boldsymbol{f}^{n+2\omega-1},\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1}\big)+\int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1})\,d\sigma$$
$$+(s^{n+2\omega-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\Big).$$

Using the inverse, Poincaré-Friedrichs inequalities (3.1), (3.2), Lemmata 6.1 and 6.3, we have

$$\underbrace{\Big(\frac{\rho\nu}{\Delta t}-\tfrac{1}{\varepsilon_1}\alpha\omega(1-\omega)\frac{dC_{inv}^2}{h^2}\Big)}_{\equiv\varepsilon\frac{\rho\nu}{\Delta t}}\big(\|\boldsymbol{u}_h^{N+1}\|^2+\|\boldsymbol{u}_h^N\|^2\big)+\frac{\rho}{\Delta t}\|a\boldsymbol{u}_h^{N+1}+b\boldsymbol{u}_h^N\|^2$$
$$+\frac{s_0}{\Delta t}\nu\big(\|p_h^{N+1}\|^2+\|p_h^N)\|^2\big)+\frac{s_0}{\Delta t}\|ap_h^{N+1}+bp_h^N\|^2+\tfrac{s_0}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}\|p_h^{n+1}-2p_h^n+p_h^{n-1}\|^2$$
$$+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^{N+1},\boldsymbol{\eta}_h^N)\|_{G,a_e}^2+\tfrac{1}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}a_e\big(\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^n+\boldsymbol{\eta}_h^{n-1},\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^n+\boldsymbol{\eta}_h^{n-1}\big)$$
$$+\omega(2\omega-1)\Delta t\frac{\alpha^2c_\omega}{s_0}\big(\|\nabla\cdot\boldsymbol{u}_h^{N+1}\|^2+\|\nabla\cdot\boldsymbol{u}_h^N\|^2\big)+(2\omega-1)^2\Delta t\frac{\alpha^2c_\omega}{s_0}\sum_{n=2}^{N-1}\|\nabla\cdot\boldsymbol{u}_h^n\|^2$$
$$-(1-\omega)(2\omega-1)\Delta t\frac{\alpha^2c_\omega}{s_0}\big(\|\nabla\cdot\boldsymbol{u}_h^1\|^2+\|\nabla\cdot\boldsymbol{u}_h^0\|^2\big)$$
$$+\omega(1-\omega)\Delta t\frac{\alpha^2c_\omega}{s_0}\sum_{n=1}^{N}\|\nabla\cdot(\boldsymbol{u}_h^{n+1}+\boldsymbol{u}_h^{n-1})\|^2$$
$$+\omega(2\omega-1)a_p(p^{N+1},p^{N+1})-(2\omega-1)\tfrac{\Delta t}{\rho}\alpha^2\omega^2\|\nabla p^{N+1}\|^2-\varepsilon_1\alpha\omega(1-\omega)\big)\|p^{N+1}\|^2$$
$$+\omega(2\omega-1)a_p(p^N,p^N)-(2\omega-1)\tfrac{\Delta t}{\rho}\alpha^2\omega^2\|\nabla p^N\|^2-\varepsilon_1\alpha\omega(1-\omega)\big)\|p^N\|^2$$
$$+(2\omega-1)\sum_{n=2}^{N-1}\Big((2\omega-1)a_p(p^n,p^n)-\tfrac{\Delta t}{\rho}\alpha^2\omega^2\frac{dC_{inv}^2}{h^2}C_{PF}\|\nabla p^n\|^2\Big)$$
$$+\omega(1-\omega)\sum_{n=1}^{N}a_p(p^{n+1}+p^{n-1},p^{n+1}+p^{n-1})$$
$$-(2\omega-1)\sum_{n=1}^{N}\Big(\frac{s_0}{4\Delta t}\|p^{n+1}-2p^n+p^{n-1}\|^2+\frac{\Delta t}{s_0}\alpha^2\omega^2\|\nabla\cdot\boldsymbol{u}^{n+1}\|^2\Big)$$
$$+\alpha\omega(1-\omega)\Big(\big\|\sqrt{\varepsilon_1}p_h^{N+1}+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^N\big\|^2+\big\|\sqrt{\varepsilon_1}p_h^N+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^{N+1}\big\|^2\Big)$$
$$+(2\omega-1)\sum_{n=1}^{N}\Big(\Big\|\sqrt{\tfrac{\rho}{4\Delta t}}p^{n+1}+\sqrt{\tfrac{\Delta t}{\rho}}\alpha\omega\nabla\cdot(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^n+\boldsymbol{u}^{n-1})\Big\|^2\Big)$$
$$+(2\omega-1)\sum_{n=1}^{N}\Big(\Big\|\sqrt{\tfrac{s_0}{4\Delta t}}(p^{n+1}-2p^n+p^{n-1})+\sqrt{\tfrac{\Delta t}{s_0}}\alpha\omega\,\nabla\cdot\boldsymbol{u}^{n+1}\Big\|^2\Big)$$
$$\le\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^1,\boldsymbol{u}_h^0)\|_G^2+\frac{s_0}{\Delta t}\|(p_h^1,p_h^0)\|_G^2+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^1,\boldsymbol{\eta}_h^0)\|_{G,a_e}^2+2\omega(1-\omega)\big(b(\boldsymbol{u}^0,p^1)-b(\boldsymbol{u}^1,p^0)\big)$$
$$-(1-\omega)(2\omega-1)\big(a_p(p^1,p^1)+a_p(p^0,p^0)\big)$$
$$+\sum_{n=1}^{N}\Big(\big(\boldsymbol{f}^{n+2\omega-1},\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1}\big)+\int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1})\,d\sigma$$
$$+(s^{n+2\omega-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\Big).$$

Rearranging terms and using the definition (2.11c), yields

$$
\begin{aligned}
&\varepsilon\frac{\rho\nu}{\Delta t}\big(\|\boldsymbol{u}_h^{N+1}\|^2+\|\boldsymbol{u}_h^{N}\|^2\big)+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^{N+1},\boldsymbol{\eta}_h^{N})\|^2_{G,a_e}\\
&+\big(\frac{s_0}{\Delta t}\nu-\varepsilon_1\alpha\omega(1-\omega)\big)\|p^{N+1}\|^2+\omega(2\omega-1)\Big(\big(\boldsymbol{\kappa}-\tfrac{\Delta t}{\rho}\alpha^2\omega\boldsymbol{I}\big)\nabla p^{N+1},\nabla p^{N+1}\Big)\\
&+\big(\frac{s_0}{\Delta t}\nu-\varepsilon_1\alpha\omega(1-\omega)\big)\|p^{N}\|^2+\omega(2\omega-1)\Big(\big(\boldsymbol{\kappa}-\tfrac{\Delta t}{\rho}\alpha^2\omega\boldsymbol{I}\big)\nabla p^{N},\nabla p^{N}\Big)\\
&+\frac{\rho}{\Delta t}\|a\boldsymbol{u}_h^{N+1}+b\boldsymbol{u}_h^{N}\|^2+\frac{s_0}{\Delta t}\|ap_h^{N+1}+bp_h^{N}\|^2+\omega(2\omega-1)\Delta t\frac{\alpha^2c_\omega}{s_0}\big(\|\nabla\cdot\boldsymbol{u}_h^{N+1}\|^2+\|\nabla\cdot\boldsymbol{u}_h^{N}\|^2\big)\\
&\quad+\tfrac{1}{\Delta t}\tfrac{2\omega-1}{4}\sum_{n=1}^{N}a_e\big(\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1},\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1}\big)\Big)\\
&\quad+(2\omega-1)^2\Delta t\frac{\alpha^2c_\omega}{s_0}\sum_{n=2}^{N-1}\|\nabla\cdot\boldsymbol{u}_h^{n}\|^2\\
&\quad+\omega(1-\omega)\Delta t\frac{\alpha^2c_\omega}{s_0}\sum_{n=1}^{N}\|\nabla\cdot(\boldsymbol{u}_h^{n+1}+\boldsymbol{u}_h^{n-1})\|^2\\
&\quad+(2\omega-1)\sum_{n=2}^{N-1}\Big(\big((2\omega-1)\boldsymbol{\kappa}-\tfrac{\Delta t}{\rho}\alpha^2\omega^2\frac{dC_{inv}^2}{h^2}C_{PF}^2\boldsymbol{I}\big)\nabla p^n,\nabla p^n\Big)\\
&\quad+\omega(1-\omega)\sum_{n=1}^{N}a_p(p^{n+1}+p^{n-1},p^{n+1}+p^{n-1})\\
&\quad-(2\omega-1)\frac{\Delta t}{s_0}\alpha^2\omega^2\sum_{n=1}^{N}\|\nabla\cdot\boldsymbol{u}^{n+1}\|^2\\
&\quad+\alpha\omega(1-\omega)\big(\big\|\sqrt{\varepsilon_1}p_h^{N+1}+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^{N}\big\|^2+\big\|\sqrt{\varepsilon_1}p_h^{N}+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot\boldsymbol{u}_h^{N+1}\big\|^2\big)\\
&\quad+(2\omega-1)\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{\rho}{4\Delta t}}p^{n+1}+\sqrt{\tfrac{\Delta t}{\rho}}\alpha\omega\nabla\cdot(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^{n}+\boldsymbol{u}^{n-1})\big\|^2\big)\\
&\quad+(2\omega-1)\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{s_0}{4\Delta t}}(p^{n+1}-2p^{n}+p^{n-1})+\sqrt{\tfrac{\Delta t}{s_0}}\alpha\omega\,\nabla\cdot\boldsymbol{u}^{n+1}\big\|^2\big)\\
&\le\frac{\rho}{\Delta t}\|(\boldsymbol{u}_h^1,\boldsymbol{u}_h^0)\|^2_G+\frac{s_0}{\Delta t}\|(p_h^1,p_h^0)\|^2_G+\frac{1}{\Delta t}\|(\boldsymbol{\eta}_h^1,\boldsymbol{\eta}_h^0)\|^2_{G,a_e}+2\omega(1-\omega)\big(b(\boldsymbol{u}^0,p^1)-b(\boldsymbol{u}^1,p^0)\big)\\
&\quad-(1-\omega)(2\omega-1)\big(a_p(p^1,p^1)+a_p(p^0,p^0)\big)-(1-\omega)(2\omega-1)\Delta t\frac{\alpha^2c_\omega}{s_0}\big(\|\nabla\cdot\boldsymbol{u}_h^1\|^2+\|\nabla\cdot\boldsymbol{u}_h^0\|^2\big)\\
&\;+\sum_{n=1}^{N}\Big(\big(\boldsymbol{f}^{n+2\omega-1},\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1}\big)+\int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1})\,d\sigma\\
&\quad+(s^{n+2\omega-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\Big),
\end{aligned}
$$

and multiply by Δt, after some calculations involving (3.2) and (3.1) we get

$$
\begin{aligned}
&\varepsilon\rho\nu\big(\|\boldsymbol{u}_h^{N+1}\|^2+\|\boldsymbol{u}_h^{N}\|^2\big)+\|(\boldsymbol{\eta}_h^{N+1},\boldsymbol{\eta}_h^{N})\|^2_{G,a_e}\\
&\quad+\Big(s_0\nu+\Delta t\omega(2\omega-1)\frac{k_{min}}{C_{PF}^2}-\Delta t^2\omega^2\frac{\alpha^2C_{inv}^2}{\rho h^2}\big(\frac{(1-\omega)^2d}{\nu(1-\varepsilon)}+2\omega-1\big)\Big)\big(\|p_h^{N+1}\|^2+\|p_h^{N}\|^2\big)\\
&\quad+\rho\|a\boldsymbol{u}_h^{N+1}+b\boldsymbol{u}_h^{N}\|^2+s_0\|ap_h^{N+1}+bp_h^{N}\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Delta t \underbrace{\left(\omega(2\omega-1)\Delta t\frac{\alpha^2 c_\omega}{s_0}-(2\omega-1)\frac{\Delta t}{s_0}\alpha^2\omega^2\right)}_{\omega^2(1-\omega)\Delta t\frac{\alpha^2}{s_0}\equiv O(\Delta t),\text{ and }\equiv 0\text{ when }\omega=\frac{1}{2}}\left(\|\nabla\cdot \boldsymbol{u}_h^{N+1}\|^2+\|\nabla\cdot \boldsymbol{u}_h^{N}\|^2\right)\\
&+\tfrac{2\omega-1}{4}\sum_{n=1}^{N} a_e\big(\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1},\boldsymbol{\eta}_h^{n+1}-2\boldsymbol{\eta}_h^{n}+\boldsymbol{\eta}_h^{n-1})\big)\\
&+\Delta t \underbrace{\left((2\omega-1)^2\Delta t\frac{\alpha^2 c_\omega}{s_0}-(2\omega-1)\frac{\Delta t}{s_0}\alpha^2\omega^2\right)}_{=0}\sum_{n=2}^{N-1}\|\nabla\cdot \boldsymbol{u}_h^{n}\|^2\\
&+\Delta t\,\omega(1-\omega)\frac{\alpha^2 c_\omega}{s_0}\Delta t\sum_{n=1}^{N}\|\nabla\cdot(\boldsymbol{u}_h^{n+1}+\boldsymbol{u}_h^{n-1})\|^2\\
&+(2\omega-1)\Delta t\sum_{n=2}^{N-1}\Big(\big((2\omega-1)\boldsymbol{\kappa}-\tfrac{\Delta t}{\rho}\alpha^2\omega^2\frac{dC_{inv}^2}{h^2}C_{PF}^2\boldsymbol{I}\big)\nabla p^n,\nabla p^n\Big)\\
&+\omega(1-\omega)\Delta t\sum_{n=1}^{N} a_p(p_h^{n+1}+p_h^{n-1},p_h^{n+1}+p_h^{n-1})\\
&+\alpha\omega(1-\omega)\Delta t\big(\big\|\sqrt{\varepsilon_1}p_h^{N+1}+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot \boldsymbol{u}_h^{N}\big\|^2+\big\|\sqrt{\varepsilon_1}p_h^{N}+\tfrac{1}{\sqrt{\varepsilon_1}}\nabla\cdot \boldsymbol{u}_h^{N+1}\big\|^2\big)\\
&+(2\omega-1)\Delta t\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{\rho}{4\Delta t}}p^{n+1}+\sqrt{\tfrac{\Delta t}{\rho}}\alpha\omega\nabla\cdot(\boldsymbol{u}^{n+1}-2\boldsymbol{u}^{n}+\boldsymbol{u}^{n-1})\big\|^2\big)\\
&+(2\omega-1)\Delta t\sum_{n=1}^{N}\big(\big\|\sqrt{\tfrac{s_0}{4\Delta t}}(p^{n+1}-2p^{n}+p^{n-1})+\sqrt{\tfrac{\Delta t}{s_0}}\alpha\omega\,\nabla\cdot \boldsymbol{u}^{n+1}\big\|^2\big)\\
\le{}&\rho\|(\boldsymbol{u}_h^1,\boldsymbol{u}_h^0)\|_G^2+s_0\|(p_h^1,p_h^0)\|_G^2+\|(\boldsymbol{\eta}_h^1,\boldsymbol{\eta}_h^0)\|_{G,a_e}^2+2\omega(1-\omega)\Delta t\big(b(\boldsymbol{u}^0,p^1)-b(\boldsymbol{u}^1,p^0)\big)\\
&-(1-\omega)(2\omega-1)\Delta t\big(a_p(p^1,p^1)+a_p(p^0,p^0)\big)-(1-\omega)(2\omega-1)\Delta t^2\frac{\alpha^2 c_\omega}{s_0}\big(\|\nabla\cdot \boldsymbol{u}_h^1\|^2+\|\nabla\cdot \boldsymbol{u}_h^0\|^2\big)\\
&+\Delta t\sum_{n=1}^{N}\Big(\big(\boldsymbol{f}^{n+2\omega-1},\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1}\big)+\int_{\Gamma_s}\boldsymbol{g}^{n+2\omega-1}(\omega\boldsymbol{u}^{n+1}+(1-\omega)\boldsymbol{u}^{n-1})\,d\sigma\\
&\qquad+(s^{n+2\omega-1},\omega p_h^{n+1}+(1-\omega)p_h^{n-1})\Big).
\end{aligned}
$$

Therefore we have a stability result by energy methods provided

$$
\begin{aligned}
&0\le s_0\nu+\Delta t\,\omega(2\omega-1)\frac{k_{min}}{C_{PF}^2}-\Delta t^2\omega^2\frac{\alpha^2C_{inv}^2}{\rho h^2}\big(\frac{(1-\omega)^2d}{\nu(1-\varepsilon)}+2\omega-1\big),\\
&0\le(2\omega-1)\big((2\omega-1)k_{min}-\tfrac{\alpha^2\omega^2}{\rho}\frac{dC_{inv}^2}{h^2}C_{PF}^2\Delta t\big),
\end{aligned}
$$

namely the time-step satisfies the following CFL conditions:

$$
\Delta t\le\sqrt{\frac{\rho s_0}{d}}\frac{h}{\alpha C_{INV}}\sqrt{1-\varepsilon},
$$

when $\omega=\frac{1}{2}$, and

$$
\Delta t\le\min\Big\{\frac{\nu}{\omega(1-\omega)}\sqrt{\frac{\rho s_0}{d}}\frac{h}{\alpha C_{INV}}\sqrt{1-\varepsilon},
$$

$$\frac{\rho h^2}{2\omega C_{inv}^2 C_{PF}^2 \alpha^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}\Big((2\omega-1)k_{min}+\big((2\omega-1)^2k_{min}^2+4s_0\nu\frac{C_{inv}^2C_{PF}^4}{\rho h^2}\alpha^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)\big)^{\frac{1}{2}}\Big)\Big\}$$

for $\omega \in (\frac{1}{2}, 1]$, respectively. □

We now transform this time-step restrictions in term of the characteristic parameters defined in Table 5.1. The CNLF ($\omega = \frac{1}{2}$) condition writes

$$\frac{\Delta t}{\tau_D} \leq \frac{k_{min}}{L^2 s_0}\sqrt{\frac{\rho s_0}{d}}\frac{h}{\alpha C_{INV}}\sqrt{1-\varepsilon} = \frac{h}{L}\frac{k_{min}\sqrt{\rho}}{\alpha L\sqrt{s_0}}\frac{1}{C_{INV}\sqrt{d}}\sqrt{1-\varepsilon} = \frac{h}{L}B_D^{-1}\frac{1}{C_{INV}\sqrt{d}}\sqrt{1-\varepsilon} \leq \frac{h}{L}(CB_D)^{-1}$$

For $\omega \in (\frac{1}{2}, 1]$ we have similarly

$$\frac{\Delta t}{\tau_D} \leq \min\Big\{\frac{\nu}{\omega(1-\omega)}\frac{h}{L}(CB_D)^{-1}, \omega\Big\}$$

where

$$\begin{aligned}
\omega &= \frac{k_{min}}{L^2 s_0}\frac{\rho h^2}{2\omega C_{inv}^2 C_{PF}^2 \alpha^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}\Big((2\omega-1)k_{min}+\big((2\omega-1)^2k_{min}^2+4s_0\nu\frac{C_{inv}^2C_{PF}^4}{\rho h^2}\alpha^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)\big)^{\frac{1}{2}}\Big)\\
&= \frac{k_{min}}{L^2 s_0}\frac{\rho h^2}{2\omega C_{inv}^2 C_{PF}^2 \alpha^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}(2\omega-1)k_{min}\\
&\quad+\frac{k_{min}}{L^2 s_0}\frac{\rho h^2}{2\omega C_{inv}^2 C_{PF}^2 \alpha^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}\sqrt{(2\omega-1)^2k_{min}^2+4s_0\nu\frac{C_{inv}^2C_{PF}^4}{\rho h^2}\alpha^2\Big(\frac{d(1-\omega)^2}{\nu}+2\omega-1\Big)}\\
&= \frac{\rho k_{min}^2}{L^2 s_0 \alpha^2}\frac{h^2}{C_{inv}^2 C_{PF}^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}\frac{2\omega-1}{2\omega}\\
&\quad+\frac{\rho k_{min}^2}{L^2 s_0 \alpha^2}\frac{h^2}{2\omega C_{inv}^2 C_{PF}^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}\sqrt{(2\omega-1)^2+4s_0\nu\frac{C_{inv}^2C_{PF}^4}{\rho h^2 k_{min}^2}\alpha^2\Big(\frac{d(1-\omega)^2}{\nu}+2\omega-1\Big)}\\
&= B_D^{-2}\frac{h^2}{L^2}\frac{1}{C_{inv}^2(C_{PF}/L)^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}\frac{2\omega-1}{2\omega}\\
&\quad+B_D^{-2}\frac{h^2}{L^2}\frac{1}{2\omega C_{inv}^2(C_{PF}/L)^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}\sqrt{(2\omega-1)^2+4s_0\nu\frac{C_{inv}^2L^4(C_{PF}/L)^4}{\rho h^2 k_{min}^2}\alpha^2\Big(\frac{d(1-\omega)^2}{\nu}+2\omega-1\Big)}\\
&= B_D^{-2}\frac{h^2}{L^2}\frac{1}{C_{inv}^2\tilde{c}_{PF}^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}\frac{2\omega-1}{2\omega}\\
&\quad+B_D^{-2}\frac{h^2}{L^2}\frac{1}{2\omega C_{inv}^2\tilde{c}_{PF}^2(\frac{d(1-\omega)^2}{\nu}+2\omega-1)}\sqrt{(2\omega-1)^2+4s_0\nu\frac{C_{inv}^2L^4\tilde{c}_{PF}^4}{\rho h^2 k_{min}^2}\alpha^2\Big(\frac{d(1-\omega)^2}{\nu}+2\omega-1\Big)}\\
&= B_D^{-2}\frac{h^2}{L^2}\frac{1}{dC_{inv}^2\tilde{c}_{PF}^2(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d})}\frac{2\omega-1}{2\omega}\\
&\quad+B_D^{-2}\frac{h^2}{L^2}\frac{1}{2\omega dC_{inv}^2\tilde{c}_{PF}^2(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d})}\sqrt{(2\omega-1)^2+4s_0\nu\frac{dC_{inv}^2L^4\tilde{c}_{PF}^4}{\rho h^2 k_{min}^2}\alpha^2\Big(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d}\Big)}\\
&= (CB_D)^{-2}\frac{h^2}{L^2}\frac{1}{\tilde{c}_{PF}^2(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d})}\frac{2\omega-1}{2\omega}
\end{aligned}$$

$$+ (CB_D)^{-2}\frac{h^2}{L^2}\frac{1}{2\omega\widetilde{c}_{PF}^2(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d})}\sqrt{(2\omega-1)^2+4C^2\widetilde{c}_{PF}^4\nu\frac{L^2}{h^2}\frac{L^2 s_0\alpha^2}{\rho k_{min}^2}\Big(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d}\Big)}$$

$$= \frac{1}{2\omega\widetilde{c}_{PF}^2(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d})}(CB_D)^{-2}\Big(\frac{h}{L}\Big)^2\Big((2\omega-1)+\sqrt{(2\omega-1)^2+4\widetilde{c}_{PF}^4\nu\Big(\frac{h}{L}\Big)^{-2}(CB_D)^2\Big(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d}\Big)}\Big)$$

$$= \frac{1}{2\omega\widetilde{c}_{PF}^2(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d})}\Big((2\omega-1)(CB_D)^{-2}\Big(\frac{h}{L}\Big)^2+\sqrt{(2\omega-1)^2(CB_D)^{-4}\Big(\frac{h}{L}\Big)^4+4\widetilde{c}_{PF}^4\nu\Big(\frac{h}{L}\Big)^2(CB_D)^{-2}\Big(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d}\Big)}\Big).$$

Therefore

$$\frac{\Delta t}{\tau_D} \le \min\Big\{\frac{\nu}{\omega(1-\omega)}\frac{h}{L}(CB_D)^{-1},$$

$$\frac{1}{2\omega\widetilde{c}_{PF}^2(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d})}\Big((2\omega-1)(CB_D)^{-2}\Big(\frac{h}{L}\Big)^2+\sqrt{(2\omega-1)^2(CB_D)^{-4}\Big(\frac{h}{L}\Big)^4+4\widetilde{c}_{PF}^4\nu\Big(\frac{h}{L}\Big)^2(CB_D)^{-2}\Big(\frac{(1-\omega)^2}{\nu}+\frac{2\omega-1}{d}\Big)}\Big)\Big\}.$$

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