LONG TIME STABILITY OF FOUR METHODS FOR SPLITTING THE EVOLUTIONARY STOKES-DARCY PROBLEM INTO STOKES AND DARCY SUBPROBLEMS

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We dedicate this paper to Professor Jan Verwer.

Abstract. This report analyzes the long time stability of four methods for non-iterative, sub-physics, uncoupling for the evolutionary Stokes-Darcy problem. The four methods uncouple each time step into separate Stokes and Darcy solves using ideas from splitting methods. Three methods uncouple sequentially while one is a parallel uncoupling method. We prove long time stability of four splitting based partitioned methods under time step restrictions depending on the problem parameters. The methods include ones stable uniformly in S_0, the storativity coefficient, for moderate $k_{\min}$, the minimum hydraulic conductivity, uniformly in $k_{\min}$ for moderate S_0 and with no coupling between the timestep and the spacial meshwidth.

1. Introduction

Many important applications such as coupled surfacewater groundwater flows require the accurate solution of multi-domain, multi-physics coupling of unobstructed flows with filtration or porous media flows (the Stokes-Darcy problem). There are large advantages in efficiency, storage, accuracy and programmer effort in using partitioned methods built from components optimized for the individual sub-processes. Partitioned methods for the evolutionary Stokes-Darcy problem confront several intrinsic difficulties which include:

- Values of the hydraulic conductivity k can be small, for example 10^{-12} for sands to 10^{-15} for clay, [B79].
- Values for the storativity coefficient S_0 range from 10^{-2} in unconfined aquifers to 10^{-5} in confined aquifers, [J67].
- The scale of the problem varies from large $L = diam(\Omega)$ for geophysics and small L for biomedical applications.
- Turnover times in aquifers can be large due to small hydraulic conductivity values and large domains. Thus accurate calculations are needed over long time intervals.
- Differences in flow rates in the Stokes and the Darcy regions can require different timesteps in the two domains for efficiency and accuracy.

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These features mean that stability is a primary issue for partitioned methods for the Stokes-Darcy problem. Uncoupling / partitioning necessarily induces a timestep restriction for long time stability. The severity of the restriction depends on the method chosen, the relaxation times of the individual subdomain problems and the strength of coupling of the underlying problem. We study herein stability vs the severity of the induced timestep restriction for small $k_{\min}$, S_0 and long time intervals for uncoupling by splitting methods. Since the Stokes-Darcy problem and the methods we consider are linear, their error satisfies the same equations as the approximate solution with the body force replaced by a consistency error. Thus, for errors also, stability over long time intervals for small S_0, k is the key to a method with good error behavior.

The four methods we analyze methods uncouple each time step into a separate Stokes flow problem and Darcy flow problem. The strength of the coupling between the two subdomains varies with different ranges of physical parameters and is reflected in restrictions on timesteps required for long time stability. Our estimates and tests suggest that these methods are stable for larger timesteps that the IMEX based partitioned methods in [MZ10], [LT11], [LTT11], [SZ11]. In particular, stability analysis and numerical tests herein indicate that splitting based partitioned methods are a very good option when either $k_{\min}$ or S_0 is small, Figures 1,2,3 in Section 5. Finding partitioned methods stable for large timesteps when both $k_{\min}$, S_0 are small is an open problem, Figures 4,5,6 in Section 5. Further, while the first order methods gave acceptable error levels, more accuracy is always desirable. Stable higher order partitioned methods for large timesteps and small parameters are also not yet known, e.g., Figure 7 Section 5.

1.1. **The Stokes-Darcy problem.** Let the two domains be Ω_f, Ω_p lie across an interface I from each other. The fluid velocity and porous media piezometric head (related to the Darcy pressure) satisfy

$$\begin{gathered}\rho u_t - \mu \triangle u + \nabla p = f_f, \text{ and } \nabla \cdot u = 0, \text{ in } \Omega_f, \\ S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) = f_p, \text{ in } \Omega_p, \\ \phi(x,0) = \phi_0, \text{ in } \Omega_p \text{ and } u(x,0) = u_0, \text{ in } \Omega_f, \\ \phi(x,t) = 0, \text{ in } \partial\Omega_p \backslash I \text{ and } u(x,t) = 0, \text{ in } \partial\Omega_f \backslash I, \\ + \text{ coupling conditions across } I.\end{gathered} \tag{1.1}$$

Let $\widehat{n}_{f/p}$ denote the indicated, outward pointing, unit normal vector on I. The coupling conditions are conservation of mass and balance of forces on I

$$\begin{gathered}u \cdot \widehat{n}_f - \mathcal{K} \nabla \phi \cdot \widehat{n}_p = 0, \text{ on I}, \\ p - \mu\, \widehat{n}_f \cdot \nabla u \cdot \widehat{n}_f = \rho g \phi \text{ on I}.\end{gathered}$$

The last condition needed is the Beavers-Joseph-Saffman (-Jones) condition

$$-\mu \nabla u \cdot \widehat{n}_f = \alpha \sqrt{\frac{\mu \rho g}{\widehat{\tau}_i \cdot \mathcal{K} \cdot \widehat{\tau}_i}} u \cdot \widehat{\tau}_i \equiv \chi u \cdot \widehat{\tau}_i, \text{ on } I \text{ for any } \widehat{\tau}_i \text{ tangent vector on } I,$$

see [BJ67], [S71], [JM00]. This is a simplification of the original and more physically realistic Beavers-Joseph conditions, in which $u \cdot \widehat{\tau}_i$ is replaced by $(u - u_p) \cdot \widehat{\tau}_i$, e.g., [CGHW08], [CGHWZ10]. Here ρ, g are the fluid density and gravitational acceleration constant and

$$\begin{aligned}
\phi &= \text{Darcy pressure + elevation induced pressure = piezometric head},\\
u_p &= -\mathcal{K}\nabla\phi = \text{ velocity in porous media region, } \Omega_p,\\
u &= \text{ velocity in Stokes region, } \Omega_f,\\
f_f,\ f_p &= \text{body forces in fluid region and source in porous media region},\\
\mathcal{K} &= \text{ hydraulic conductivity tensor with } min_{\Omega_p}\lambda_{\min}(\mathcal{K}) =: k_{\min} > 0,\\
\mu &= \text{ viscosity of fluid},\\
S_0 &= \text{ specific mass storativity coefficient.}
\end{aligned}$$

We assume that all material and fluid parameters are positive and the boundary conditions are simple Dirichlet conditions on the exterior boundaries (not including the interface I). While this is only one of several important boundary conditions, [B79], [PC06], *the algorithms herein and their numerical analysis can easily be extended to different combinations of exterior boundary conditions.*

Section 2 collects preliminaries and Section 3 presents four partitioned methods. Section 4 analyzes long time stability and derives the associated timestep restrictions. Section 5 gives numerical tests and Section 6 follows with conclusions and future prospects.

1.2. **Related Work.** Understanding of the equilibrium Stokes-Darcy problem is now advanced, e.g., [JM00], [LSY], [DMQ01], [PS98], [PSS99]. For the evolutionary problem, the monolithic approach (discretize the problem implicitly, assemble the fully coupled system at each time step, solve by an iterative method where uncoupling is attained by using a domain decomposition preconditioner) is an important complement to partitioned methods; it is developed in, e.g., [DMQ01], [CGHW11], [D04], [DQ], [DQ03], [HPV07], [CMX07], [MX07], [J09], [MQS03], [MX07], and [VY11]. Partitioned methods require neither access to a fully coupled system nor iteration at each time step, e.g., [LT11], [LTT11], [SZ11], [MZ10] (the first paper on partitioned methods for Stokes-Darcy), and [CGHW08], [CGHWZ10] (a interesting new approach and the first papers studying the Beavers-Joseph interface coupling). There is a very strong connection between application-specific partitioned methods and more general IMEX and splitting methods; see, e.g., [V09], [V80], [ARW95], [C80], [FHV96], [HV03], [V09], [M88], [M90], [Y71]. The idea used in CNsplit below to compute in parallel two approximations and then average occurs in the Dyakunov splitting method, e.g., [M88], [M90], [Y71], [HKLR10].

2. The continuous problem and semi-discrete approximation

We denote the $L^2(I)$ norm by $||\cdot||_I$ and the $L^2(\Omega_{f/p})$ norms by $||\cdot||_{f/p}$, respectively; the corresponding inner products are denoted by $(\cdot,\cdot)_{f/p}$. Let

$$\begin{aligned}
X_f &: = \{v \in \left(H^1(\Omega_f)\right)^d : v = 0 \text{ on } \partial\Omega_f\backslash I\},\\
X_p &: = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p\backslash I\},\\
Q_f &: = L^2_0(\Omega_f).
\end{aligned}$$

To discretize the Stokes-Darcy problem in space by the finite element method, we select conforming finite element spaces

$$\begin{aligned} \text{Velocity:} \quad & X_f^h \subset X_f = \{v \in \left(H^1(\Omega_f)\right)^d : v = 0 \text{ on } \partial\Omega_f \backslash I\}, \\ \text{Darcy pressure:} \quad & X_p^h \subset X_p = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p \backslash I\}, \\ \text{Stokes pressure:} \quad & Q_f^h \subset Q_f = L_0^2(\Omega_f). \end{aligned}$$

based on a conforming FEM triangulations in Ω_f, Ω_p with maximum triangle diameter "h". No mesh compatibility at or continuity across the interface I between the FEM meshes in the two subdomains is assumed. It is known that provided a minimum angle condition holds functions in piecewise polynomial finite element spaces including X_f^h, X_p^h and even Q_f^h (for the elementwise gradient) satisfy an inverse inequality[1]:

$$||\nabla v_h|| \le C_{INV} h^{-1} ||v_h||, \;\; h = \text{minimum meshwidth.} \tag{2.1}$$

The Stokes velocity-pressure FEM spaces (X_f^h, Q_f^h) are assumed to satisfy the usual discrete inf-sup / LBB^h condition for stability of the discrete pressure, e.g., [G89], [GR86], [L07]. We denote the discretely divergence free velocities by

$$V^h := X_f^h \cap \{v_h : (q_h, \nabla \cdot v_h)_f = 0, \text{ for all } q_h \in Q_f^h\}$$

The $H_{DIV}(\Omega_f)$ norm is given by

$$||u||_{DIV} := \sqrt{||u||_f^2 + ||\nabla \cdot u||_f^2}.$$

Note that if $d = \dim(\Omega_f)$, $||\nabla \cdot u||_f \le \sqrt{d}||\nabla u||_f$ and that the Poincaré - Friedrichs inequality holds in both domains:

$$||v||_{f/p} \le C_{PF}(\Omega_{f/p})||\nabla v||_{f/p}, \forall v \in X_{f/p}.$$

We use versions of the trace theorem on the interface I:

$$||\phi||_I \le C_p^* ||\phi||_p^{1/2} ||\nabla \phi||_p^{1/2} \text{ and } ||u||_I \le C_f^* ||u||_f^{1/2} ||\nabla u||_f^{1/2} \tag{2.2}$$

We shall assume that the domains $\Omega_{f/p}$ are such that the second trace inequality holds:

$$\left| \int_I \phi u \cdot \widehat{n} ds \right| \le C ||u||_{DIV} ||\phi||_{H^1(\Omega_p)}, \text{ for all } u \in X_f, \phi \in X_p. \tag{HDIV trace}$$

This inequality is standard if $\Omega_p = \Omega_f$ and $I = \partial\Omega_p$ and holds with $C = 1$ in that case, e.g., [GR86]. It also holds if Ω_p is contained in Ω_f and $I = \partial\Omega_p$ and visa versa. The most general domains and shared boundaries I which satisfy this inequality do not seem to be known. However, Moraiti [M11] shows that it holds in many cases directly (without extra assumptions like $\phi \in H_{00}^{1/2}(I)$) such as when one domain is an image under a smooth map of the other. For example, we have the following special case of Moraiti [M11].

Lemma 1. *Suppose $\Omega_{f/p}$ are open connected, regular sets in $\mathbb{R}^d$ sharing a boundary portion I which is an open connected set with $I \subset \{x = (x_1, \cdots, x_d) : x_d = 0\}$. Suppose Ω_p is the reflection of Ω_f across I, i.e., $(x_1, \cdots, x_d) \in \Omega_p$ if and only if $(x_1, \cdots, -x_d) \in \Omega_f$. Then (HDIV trace) holds with $C = 1$.*

[1] The constant C_{INV} depends upon the angles in the finite element mesh but not on the domain size. The analysis must either use $h_{\min}$ in stability restrictions and $h_{\max}$ in the interpolation inequalities or assume a quasi-uniform mesh. For notational simplicity we do the latter.

Proof. We have that $\phi(x_1, \cdots, x_d) \in X_p$ means $\phi^* := \phi(x_1, \cdots, -x_d)$ is a well defined function on Ω_f with the same regularity, norms and boundary conditions. Since $\phi^* = \phi$ on I we have

$$\begin{aligned}\int_I \phi u \cdot \widehat{n} ds &= \int_I \phi^* u \cdot \widehat{n} ds = \int_{\Omega_f} \nabla \cdot (u\phi^*)\, dx = \\ &= \int_{\Omega_f} (\nabla \cdot u)\, \phi^* dx + \int_{\Omega_f} u \cdot \nabla \phi^* dx.\end{aligned}$$

Thus, by the Cauchy-Schwarz inequality

$$\left| \int_I \phi u \cdot \widehat{n} ds \right| \leq ||u||_{DIV} ||\phi^*||_{H^1(\Omega_f)} = ||u||_{DIV} ||\phi||_{H^1(\Omega_p)}.$$

□

To present a convenient[2] variational formulation we first multiply the porous media equation through by ρg. Define the associated bilinear forms

$$\begin{aligned} a_f(u,v) &= (\mu \nabla u, \nabla v)_f + (\nabla \cdot u, \nabla \cdot v)_f + \sum_i \int_I \chi (u \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i) ds, \\ a_p(\phi, \psi) &= \rho g (\mathcal{K} \nabla \phi, \nabla \psi)_p, \quad \text{and} \\ c_I(u, \phi) &= \rho g \int_I \phi u \cdot \widehat{n}_f ds. \end{aligned}$$

A (monolithic) variational formulation of the coupled problem is to find $(u, p, \phi) : [0, \infty) \to X_f \times Q_f \times X_p$ satisfying the given initial conditions and, for all $v \in X_f, q \in Q_f, \psi \in X_p$

$$\begin{gathered} \rho(u_t, v)_f + a_f(u,v) - (p, \nabla \cdot v)_f + c_I(v, \phi) = (f_f, v)_f, \\ (q, \nabla \cdot u)_f = 0, \\ \rho g S_0 (\phi_t, \psi)_p + a_p(\phi, \psi) - c_I(u, \psi) = \rho g (f_p, \psi)_p. \end{gathered} \tag{2.3}$$

The bilinear forms $a_{f/p}(\cdot,\cdot)$ are symmetric, continuous and coercive. We include grad-div stabilization (the term $(\nabla \cdot u, \nabla \cdot v)_f$), an idea developed by [LO02], [OR02], [OR04], with coefficient (normally $O(1)$) chosen to be 1.

The key to the problem is the coupling term. The effect of the above pre-multiplications by ρg is to make the coupling exactly skew symmetric.

Lemma 2. *If (HDIV trace) holds we have for $u, \phi \in X_f, X_p$*

$$\begin{aligned} |c_I(u,\phi)| &\leq \frac{\mu}{2} ||\nabla u||_f^2 + \frac{\rho g k_{\min}}{2} ||\nabla \phi||_p^2 + \frac{(C_f^* C_p^*)^2 \, (\rho g)^{3/2}}{4\sqrt{\mu k_{\min}}} ||u||_f ||\phi||_p, \\ |c_I(u,\phi)| &\leq \frac{\mu}{2} ||\nabla u||_f^2 + \frac{\rho g k_{\min}}{2} ||\nabla \phi||_p^2 + \frac{\rho}{2} ||u||_f^2 + \frac{(C_f^* C_p^*)^4 \, (\rho g)^3}{32 \rho \mu k_{\min}} ||\phi||_p^2, \\ &\quad \textit{and} \\ |c_I(u,\phi)| &\leq \frac{\rho g k_{\min}}{2} ||\nabla \phi||_p^2 + \frac{\rho g (1 + C_{PF}^2(\Omega_p))}{2 k_{\min}} \left(||u||_j^2 + ||\nabla \cdot u||_f^2 \right). \end{aligned}$$

[2]Other variational formulations are possible. In (2.3) the volumetric porosity is implicit rather than explicit.

In the discrete case, if the inverse estimate (2.1) holds we have for all $u^h, \phi^h \in X_f^h, X_p^h$

$$|c_I(u^h,\phi^h)| \leq \rho g C_f^* C_p^* C_{INV} h^{-1}\left(\frac{1}{2}||u^h||_f^2 + \frac{1}{2}||\phi^h||_p^2\right).$$

Proof. Using (2.2) and the arithmetic geometric mean inequality twice we obtain

$$\begin{aligned} c_I(u,\phi) &= \rho g \int_I \phi u \cdot \widehat{n} ds \leq \rho g ||u||_I ||\phi||_I \\ &\leq \rho g C_f^* C_p^* ||\phi||_p^{1/2} ||\nabla \phi||_p^{1/2} ||u||_f^{1/2} ||\nabla u||_f^{1/2} \\ &\leq \frac{\mu}{2}||\nabla u||_f^2 + \frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \frac{(C_f^* C_p^*)^2 \, (\rho g)^{3/2}}{4\sqrt{\mu k_{\min}}}||u||_f ||\phi||_p. \end{aligned}$$

The second follows from the first by another application of the arithmetic-geometric mean inequality. For the third estimate we use (HDIV trace) and the Poincaré-Friedrichs inequality

$$\begin{aligned} |c_I(u,\phi)| &\leq \rho g ||u||_{DIV} ||\phi||_{H^1(\Omega_p)} \leq \rho g ||u||_{DIV}\sqrt{1 + C_{PF}^2(\Omega_p)}||\nabla\phi||_p \\ &\leq \frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \frac{\rho g(1 + C_{PF}^2(\Omega_p))}{2k_{\min}}||u||_{DIV}^2. \end{aligned}$$

The fourth follows similarly using the inverse estimate:

$$\begin{aligned} |c_I(u^h,\phi^h)| &\leq \rho g ||u^h||_I ||\phi^h||_I \leq \rho g C_f^* ||u||_f^{1/2} ||\nabla u||_f^{1/2} C_p^* ||\phi^h||_p^{1/2} ||\nabla\phi^h||_p^{1/2} \\ &\leq \rho g C_f^* C_p^* C_{INV} h^{-1}||u^h||_f ||\phi^h||_p \leq \rho g C_f^* C_p^* C_{INV} h^{-1}\left(\frac{1}{2}||u^h||_f^2 + \frac{1}{2}||\phi^h||_p^2\right). \end{aligned}$$

□

3. Four Splitting Based Partitioned Methods

Pick a time-step $\triangle t > 0$. Let $t^n := n\triangle t$, the (arbitrary) final time be $T = N\triangle t$ and let superscripts denote the time level of the approximation. We consider four uncoupling methods. BEsplit1 and 2 methods have superior stability properties in different cases of small physical parameters. The fourth method is second order accurate. The first method is a translation of the method from [V09] to the Stokes-Darcy problem.

Method 1: SDsplit = a Stokes-Darcy time-split method. SDsplit is a first order accurate, three sub-step method adapted from [V09]. The **SDsplit** approximations are: given (u_h^n, p_h^n, ϕ_h^n), find $(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1/2}) \in X_f^h \times Q_f^h \times X_p^h$ and $\phi_h^{n+1} \in X_p^h$ satisfying, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$:

$$\rho g S_0(\frac{\phi_h^{n+1/2} - \phi_h^n}{\triangle t}, \psi_h)_p + \frac{1}{2}a_p(\phi_h^{n+1/2}, \psi_h) - \frac{1}{2}c_I(u_h^n, \psi_h) = \frac{1}{2}\rho g(f_p^{n+1/2}, \psi_h)_p.$$

$$\rho(\frac{u_h^{n+1} - u_h^n}{\triangle t}, v_h)_f + a_f(u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla \cdot v_h)_f$$

$$+c_I(v_h, \phi_h^{n+1/2}) = (f_f^{n+1}, v_h)_f, \quad \text{and} \quad (q_h, \nabla \cdot u_h^{n+1})_f = 0, \tag{SDsplit}$$

$$\rho g S_0(\frac{\phi_h^{n+1} - \phi_h^{n+1/2}}{\triangle t}, \psi_h)_p + \frac{1}{2}a_p(\phi_h^{n+1}, \psi_h) - \frac{1}{2}c_I(u_h^{n+1}, \psi_h) = \frac{1}{2}\rho g(f_f^{n+1}, \psi_h)_p.$$

SDsplit is uncoupled but sequential: $u_h^n \to \phi_h^{n+1/2} \to u_h^{n+1} \to \phi_h^{n+1}$.

Method 2: BEsplit1 = a Backward Euler time-split method. The BEsplit approximations are: given (u_h^n, p_h^n, ϕ_h^n) find $(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ satisfying, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$,

$$\begin{aligned}
&\rho(\frac{u_h^{n+1}-u_h^n}{\triangle t}, v_h)_f + a_f(u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla\cdot v_h)_f + c_I(v_h, \phi_h^n) = (f_f^{n+1}, v_h)_f,\\
&(q_h, \nabla\cdot u_h^{n+1})_f = 0, \qquad \text{(BEsplit1)}\\
&\rho g S_0(\frac{\phi_h^{n+1}-\phi_h^n}{\triangle t}, \psi_h)_p + a_p(\phi_h^{n+1}, \psi_h) - c_I(u_h^{n+1}, \psi_h) = \rho g(f_p^{n+1}, \psi_h)_p.
\end{aligned}$$

The coupling term in the ϕ equation is evaluated at the newly computed value u_h^{n+1} so we compute $\phi_h^n \to u_h^{n+1} \to \phi_h^{n+1}$.

Method 3: BEsplit2. The order of cycling through the equations alters the computed results. **BEsplit2** is the previous method in the opposite order. It is given by: given (u_h^n, p_h^n, ϕ_h^n) find $(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ satisfying, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$,

$$\begin{aligned}
&\rho g S_0(\frac{\phi_h^{n+1}-\phi_h^n}{\triangle t}, \psi_h)_p + a_p(\phi_h^{n+1}, \psi_h) - c_I(u_h^n, \psi_h) = \rho g(f_p^{n+1}, \psi_h)_p\\
&\text{(BEsplit2)} \quad \rho(\frac{u_h^{n+1}-u_h^n}{\triangle t}, v_h)_f + \rho(\nabla\cdot\frac{u_h^{n+1}-u_h^n}{\triangle t}, \nabla\cdot v_h)_f + a_f(u_h^{n+1}, v_h)\\
&\qquad -(p_h^{n+1}, \nabla\cdot v_h)_f + c_I(v_h, \phi_h^{n+1}) = (f_f^{n+1}, v_h)_f,\\
&\qquad (q_h, \nabla\cdot u_h^{n+1})_f = 0.
\end{aligned}$$

Our initial analysis revealed that control was needed for a term $||u_h^{n+1} - u_h^n||_{DIV}$. This led to the idea of inserting the grad-div stabilization term $(\nabla\cdot\left(u_h^{n+1} - u_h^n\right)/\triangle t, \nabla\cdot v_h)_f$ acting on the time discretization of u_t. This term is exactly zero for the continuous problem so it does not increase the method's consistency error.

Method 4: CNsplit= a Crank-Nicolson time-split method. CNsplit is second order accurate. It computes *in parallel*[3] two partitioned approximations $(\widehat{u}_h^{n+1}, \widehat{p}_h^{n+1}, \widehat{\phi}_h^{n+1})$ and $(\widetilde{u}_h^{n+1}, \widetilde{p}_h^{n+1}, \widetilde{\phi}_h^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ whereupon *the new approximation to each variable is the average of the two computed approximations*:

$$\text{(CNsplit)} \qquad (u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) = \frac{1}{2}[(\widehat{u}_h^{n+1}, \widehat{p}_h^{n+1}, \widehat{\phi}_h^{n+1}) + (\widetilde{u}_h^{n+1}, \widetilde{p}_h^{n+1}, \widetilde{\phi}_h^{n+1})].$$

The two individual approximations satisfy, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$

$$\begin{aligned}
&\rho(\frac{\widehat{u}_h^{n+1}-\widehat{u}_h^n}{\triangle t}, v_h)_f + a_f(\frac{\widehat{u}_h^{n+1}+\widehat{u}_h^n}{2}, v_h) - (\frac{\widehat{p}_h^{n+1}+\widehat{p}_h^n}{2}, \nabla\cdot v_h)_f\\
&\text{(CNsplit-a)} \qquad +c_I(v_h, \widehat{\phi}_h^n) = (f_f^{n+1/2}, v_h)_f, \text{ and } (q_h, \nabla\cdot\widehat{u}_h^{n+1})_f = 0,\\
&\rho g S_0(\frac{\widehat{\phi}_h^{n+1}-\widehat{\phi}_h^n}{\triangle t}, \psi_h)_p + a_p(\frac{\widehat{\phi}_h^{n+1}+\widehat{\phi}_h^n}{2}, \psi_h) - c_I(\widehat{u}_h^{n+1}, \psi_h) = \rho g(f_p^{n+1/2}, \psi_h)_p
\end{aligned}$$

[3]Two processors can be working simultaneously with waiting only due to the different speeds of solving the subdomain problems.

and

$$\rho g S_0(\frac{\widetilde{\phi}_h^{n+1} - \widetilde{\phi}_h^{n}}{\triangle t}, \psi_h)_p + a_p(\frac{\widetilde{\phi}_h^{n+1} + \widetilde{\phi}_h^{n}}{2}, \psi_h) - c_I(\widetilde{u}_h^n, \psi_h) = \rho g(f_p^{n+1/2}, \psi_h)_p.$$

$$\text{(CNsplit-b)} \qquad \rho(\frac{\widetilde{u}_h^{n+1} - \widetilde{u}_h^{n}}{\triangle t}, v_h)_f + a_f(\frac{\widetilde{u}_h^{n+1} + \widetilde{u}_h^{n}}{2}, v_h) - (\frac{\widetilde{p}_h^{n+1} + \widetilde{p}_h^{n}}{2}, \nabla \cdot v_h)_f$$
$$+c_I(v_h, \widetilde{\phi}_h^{n+1}) = (f_f^{n+1/2}, v_h)_f, \text{ and } (q_h, \nabla \cdot \widetilde{u}_h^{n+1})_f = 0.$$

The calculation can proceed as follows

Step 1: Pass previous values across the interface to the other domains

solve, in parallel for $\widehat{u}_h^{n+1}, \widetilde{\phi}_h^{n+1}$

Step 2: Pass each of $\widehat{u}_h^{n+1}, \widetilde{\phi}_h^{n+1}$ across the interface to the other domains

solve, in parallel, for $\widetilde{u}_h^{n+1}, \widehat{\phi}_h^{n+1}$.

Step 3: Average the two approximations on each domain

Averaging the equations of the two approximations shows that the averages u_h^n and ϕ_h^n satisfy

$$\text{(3.1)} \qquad \rho(\frac{u_h^{n+1} - u_h^{n}}{\triangle t}, v_h)_f + a_f(\frac{u_h^{n+1} + u_h^{n}}{2}, v_h) - (\frac{p_h^{n+1} + p_h^{n}}{2}, \nabla \cdot v_h)_f +$$
$$+c_I(v_h, \frac{\widetilde{\phi}_h^{n+1} + \widehat{\phi}_h^{n}}{2}) = (f_f^{n+1/2}, v_h)_f, \text{ and } (q_h, \nabla \cdot u_h^{n+1})_f = 0,$$
$$\rho g S_0(\frac{\phi_h^{n+1} - \phi_h^{n}}{\triangle t}, \psi_h)_p + a_p(\frac{\phi_h^{n+1} + \phi_h^{n}}{2}, \psi_h) - c_I(\frac{\widehat{u}_h^{n+1} + \widetilde{u}_h^{n}}{2}, \psi_h) = \rho g(f_p^{n+1/2}, \psi_h)_p$$

To assess consistency errors, the residual is estimated when the true solution $u(t), \phi(t)$ is inserted for all variables u, $\widetilde{u}$, $\widehat{u}$, ϕ, $\widetilde{\phi}$ and $\widehat{\phi}$ in (3.1). As this eliminates the differences between the "hat" and the "tilde" variables, it shows that CNsplit has the same consistency error as the (monolithic / fully coupled) Crank-Nicolson time discretization.

4. Analysis of Stability of SDsplit, BEsplit1/2 and CNsplit

Since the partitioned methods considered treat some variables in some steps explicitly, a timestep restriction for stability in unavoidable. This section gives a stability proof by energy methods in the form that implies stability over long time intervals and elucidates the timestep restriction required for the four methods.

4.1. **SDsplit Stability.**

We prove conditional stability (with a timestep restriction linked to the spacial meshwidth) of SDsplit in this subsection. The timestep restriction is of the form

$$\triangle t < C \min\{S_0, k_{min}\} h.$$

To be precise, define

$$\triangle T_0 := \frac{2}{\rho g (C_f^* C_p^*)^2 C_{INV}} \min\left\{\frac{S_0 \mu}{C_{PF}(\Omega_f)}, \frac{\rho k_{min}}{C_{PF}(\Omega_p)}\right\} h.$$

Theorem 1. *Suppose that for some α, $0<\alpha<1$,*

$$\triangle t \leq (1-\alpha)\triangle T_0. \tag{4.1}$$

Then SDsplit is stable:

$$\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^N||_f^2+\rho g S_0||\phi_h^N||_p^2\right]+\triangle t\sum_{n=0}^{N-1}\triangle t\frac{\rho g S_0}{2}||\frac{\phi_h^{n+1/2}-\phi_h^n}{\triangle t}||_p^2\\
&+\frac{\alpha\rho g S_0}{2}\triangle t\sum_{n=0}^{N-1}\triangle t||\frac{\phi_h^{n+1/2}-\phi_h^{n+1}}{\triangle t}||_p^2+\frac{\alpha\rho}{2}\triangle t\sum_{n=0}^{N-1}\triangle t||\frac{u_h^{n+1}-u_h^n}{\triangle t}||_f^2\\
&\leq\frac{1}{2}\left[\rho||u_h^0||_f^2+\rho g S_0||\phi_h^0||_p^2\right]+\frac{\rho g C_{PF}^2(\Omega_p)}{2k_{min}}\triangle t\sum_{n=0}^{N-1}||f_p^{n+1/2}||_p^2\\
&+\frac{C_{PF}^2(\Omega_f)}{2\mu}\triangle t\sum_{n=0}^{N-1}||f_f^{n+1}||_f^2+\frac{\rho g C_{PF}^2(\Omega_p)}{4k_{min}}\triangle t\sum_{n=0}^{N-1}||f_p^{n+1}||_p^2.
\end{aligned} \tag{4.2}$$

Proof. In the first 1/3 step of SDsplit, take $\psi=\triangle t\phi_h^{n+1/2}$. This gives

$$\begin{aligned}
&\frac{1}{2}\rho g S_0(||\phi_h^{n+1/2}||_p^2-||\phi_h^n||_p^2+||\phi_h^{n+1/2}-\phi_h^n||_p^2)+\frac{\triangle t}{2}a_p(\phi_h^{n+1/2},\phi_h^{n+1/2})\\
&\quad=\frac{\triangle t}{2}\rho g(f_p^{n+1/2},\phi_h^{n+1/2})_p+\frac{\triangle t}{2}c_I(u_h^n,\phi_h^{n+1/2}).
\end{aligned}$$

Take $v=\triangle t u_h^{n+1}, q=p_h^{n+1}$ in the 2/3 step and add. This gives

$$\begin{aligned}
&\frac{1}{2}\rho(||u_h^{n+1}||_f^2-||u_h^n||_f^2+||u_h^{n+1}-u_h^n||_f^2)+\triangle t a_f(u_h^{n+1},u_h^{n+1})\\
&\quad=\triangle t(f_f^{n+1},u_h^{n+1})_f-\triangle t c_I(u_h^{n+1},\phi_h^{n+1/2}).
\end{aligned}$$

In the 3/3 step, take $\psi=\triangle t\phi_h^{n+1}$:

$$\begin{aligned}
&\frac{1}{2}\rho g S_0(||\phi_h^{n+1}||_p^2-||\phi_h^{n+1/2}||_p^2+||\phi_h^{n+1}-\phi_h^{n+1/2}||_p^2)+\frac{\triangle t}{2}a_p(\phi_h^{n+1},\phi_h^{n+1})\\
&\quad=\frac{\triangle t}{2}\rho g(f_p^{n+1},\phi_h^{n+1})_p+\frac{\triangle t}{2}c_I(u_h^{n+1},\phi_h^{n+1}).
\end{aligned}$$

Adding, we obtain:

$$\begin{aligned}
&\frac{1}{2}\rho g S_0(||\phi_h^{n+1}||_p^2-||\phi_h^n||_p^2)+\frac{1}{2}\rho(||u_h^{n+1}||_f^2-||u_h^n||_f^2)\\
&+\frac{1}{2}\rho g S_0(||\phi_h^{n+1/2}-\phi_h^n||_p^2+||\phi_h^{n+1}-\phi_h^{n+1/2}||_p^2)+\frac{1}{2}\rho||u_h^{n+1}-u_h^n||_f^2\\
&+\frac{\triangle t}{2}a_p(\phi_h^{n+1/2},\phi_h^{n+1/2})+\frac{\triangle t}{2}a_p(\phi_h^{n+1},\phi_h^{n+1})+\triangle t a_f(u_h^{n+1},u_h^{n+1})\\
&=\frac{\triangle t}{2}\rho g(f_p^{n+1/2},\phi_h^{n+1/2})_p+\triangle t(f_f^{n+1},u_h^{n+1})_f+\frac{\triangle t}{2}\rho g(f_p^{n+1},\phi_h^{n+1})_p\\
&+\frac{\triangle t}{2}c_I(u_h^n,\phi_h^{n+1/2})-\triangle t c_I(u_h^{n+1},\phi_h^{n+1/2})+\frac{\triangle t}{2}c_I(u_h^{n+1},\phi_h^{n+1}).
\end{aligned}$$

Consider the interface terms (the last line):

$$\text{Interface Terms} = \frac{\triangle t}{2} c_I(u_h^n, \phi_h^{n+1/2}) - \triangle t c_I(u_h^{n+1}, \phi_h^{n+1/2}) + \frac{\triangle t}{2} c_I(u_h^{n+1}, \phi_h^{n+1}).$$

Rewrite the interface term as a difference by splitting the middle term. This gives

$$\begin{aligned}\text{Interface Terms} &= \frac{\triangle t}{2} c_I(u_h^n, \phi_h^{n+1/2}) - \frac{\triangle t}{2} c_I(u_h^{n+1}, \phi_h^{n+1/2}) \\ &\quad - \frac{\triangle t}{2} c_I(u_h^{n+1}, \phi_h^{n+1/2}) + \frac{\triangle t}{2} c_I(u_h^{n+1}, \phi_h^{n+1}) \\ &= \frac{\triangle t}{2} c_I(u_h^n - u_h^{n+1}, \phi_h^{n+1/2}) - \frac{\triangle t}{2} c_I(u_h^{n+1}, \phi_h^{n+1/2} - \phi_h^{n+1}).\end{aligned}$$

Lemma 2, the Poincaré-Friedrichs and inverse inequalities give the two bounds

$$\begin{aligned}&\frac{\triangle t}{2} |c_I(u^n - u^{n+1}, \phi^{n+1/2})| \leq \\ &\leq \frac{\rho g \triangle t}{4} ||\mathcal{K}^{1/2} \nabla \phi_h^{n+1/2}||_p^2 + \frac{\rho g (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_p) h^{-1} \triangle t}{4 k_{min}} ||u_h^n - u_h^{n+1}||_f^2.\end{aligned}$$

$$\begin{aligned}&\frac{\triangle t}{2} |c_I(u_h^{n+1}, \phi_h^{n+1/2} - \phi_h^{n+1})| \leq \\ &\leq \frac{\mu \triangle t}{4} ||\nabla u_h^{n+1}||_f^2 + \frac{\rho^2 g^2 (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_f) h^{-1} \triangle t}{4\mu} ||\phi_h^{n+1/2} - \phi_h^{n+1}||_p^2.\end{aligned}$$

Next, we bound the right-hand side in a standard way:

$$\begin{aligned}\frac{\triangle t}{2} \rho g (f_p^{n+1/2}, \phi_h^{n+1/2}) &\leq \frac{\rho g \triangle t}{8} ||K^{1/2} \nabla \phi_h^{n+1/2}||_p^2 + \frac{\rho g C_{PF}^2(\Omega_p) \triangle t}{2 k_{min}} ||f_p^{n+1/2}||_p^2, \\ \triangle t (f_f^{n+1}, u_h^{n+1}) &\leq \frac{C_{PF}^2(\Omega_f) \triangle t}{2\mu} ||f_f^{n+1}||_f^2 + \frac{\mu \triangle t}{2} ||\nabla u_h^{n+1}||_f^2, \\ \frac{\triangle t}{2} \rho g (f_p^{n+1}, \phi_h^{n+1}) &\leq \frac{\rho g \triangle t}{4} ||K^{1/2} \nabla \phi_h^{n+1}||_p^2 + \frac{\rho g C_{PF}^2(\Omega_p) \triangle t}{4 k_{min}} ||f_p^{n+1}||_p^2.\end{aligned}$$

For the left side. apply coercivity:

$$\begin{aligned}\frac{\triangle t}{2} a_p(\phi_h^{n+1/2}, \phi_h^{n+1/2}) &\geq \frac{\rho g \triangle t}{2} ||K^{1/2} \nabla \phi_h^{n+1/2}||_p^2, \\ \triangle t a_f(u_h^{n+1}, u_h^{n+1}) &\geq \mu \triangle t ||\nabla u_h^{n+1}||_f^2, \\ \frac{\triangle t}{2} a_p(\phi_h^{n+1}, \phi_h^{n+1}) &\geq \frac{\rho g \triangle t}{2} ||K^{1/2} \nabla \phi_h^{n+1}||_p^2.\end{aligned}$$

Combine, we arrive at:

$$\begin{aligned}
&\frac{1}{2}\rho g S_0(||\phi_h^{n+1}||_p^2 - ||\phi_h^n||_p^2) + \frac{1}{2}\rho(||u_h^{n+1}||_f^2 - ||u_h^n||_f^2) + \frac{1}{2}\rho g S_0||\phi_h^{n+1/2} - \phi_h^n||_p^2 \\
&\quad +(\frac{1}{2}\rho g S_0 - \frac{\rho^2 g^2 (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_f) h^{-1} \triangle t}{4\mu})||\phi_h^{n+1/2} - \phi_h^{n+1}||_p^2 \\
&\quad +(\frac{1}{2}\rho - \frac{\rho g (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_p) h^{-1} \triangle t}{4k_{min}})||u_h^{n+1} - u_h^n||_f^2 \\
&\le \frac{\rho g C_{PF}^2(\Omega_p)\triangle t}{2k_{min}}||f_p^{n+1/2}||_p^2 + \frac{C_{PF}^2(\Omega_f)\triangle t}{2\mu}||f_f^{n+1}||_f^2 + \frac{\rho g C_{PF}^2(\Omega_p)\triangle t}{4k_{min}}||f_p^{n+1}||_p^2.
\end{aligned}$$

Sum this over $n = 0, 1, \cdots, N-1$. We have:

$$\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^N||_f^2 + \rho g S_0||\phi_h^N||_p^2\right] + \frac{1}{2}\rho g S_0 \sum_{n=0}^{N-1}||\phi_h^{n+1/2} - \phi_h^n||_p^2 \\
&+(\frac{1}{2}\rho g S_0 - \frac{\rho^2 g^2 (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_f) h^{-1} \triangle t}{4\mu})\sum_{n=0}^{N-1}||\phi_h^{n+1/2} - \phi_h^{n+1}||_p^2 \\
&+(\frac{1}{2}\rho - \frac{\rho g (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_p) h^{-1} \triangle t}{4k_{min}})\sum_{n=0}^{N-1}||u_h^{n+1} - u_h^n||_f^2 \\
&\le \frac{1}{2}\left[\rho||u_h^0||_f^2 + \rho g S_0||\phi_h^0||_p^2\right] + \frac{\rho g C_{PF}^2(\Omega_p)\triangle t}{2k_{min}}\sum_{n=0}^{N-1}||f_p^{n+1/2}||_p^2 + \\
&+\frac{C_{PF}^2(\Omega_f)\triangle t}{2\mu}\sum_{n=0}^{N-1}||f_f^{n+1}||_f^2 + \frac{\rho g C_{PF}^2(\Omega_p)\triangle t}{4k_{min}}\sum_{n=0}^{N-1}||f_p^{n+1}||_p^2.
\end{aligned}$$

Stability follows under the two conditions below, which are equivalent to the time step restriction $\triangle t \le (1-\alpha)\triangle T_0$:

$$\begin{aligned}
\frac{1}{2}\rho g S_0 - \frac{\rho^2 g^2 (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_f) h^{-1} \triangle t}{4\mu} &\ge \alpha\frac{\rho g S_0}{2}, \\
\frac{1}{2}\rho - \frac{\rho g (C_f^* C_p^*)^2 C_{INV} C_{PF}(\Omega_P) h^{-1} \triangle t}{4k_{min}} &\le \alpha\frac{\rho}{2}.
\end{aligned}$$

□

4.2. **BEsplit1 Stability.** Define

$$\begin{aligned}
\triangle T_1 &:= 2\min\{\mu k_{\min} S_0 \frac{16\rho}{(C_f^* C_p^*)^4 (\rho g)^2}, 1\}, \\
\triangle T_2 &:= \frac{2\min\{1, g S_0\}}{g C_f^* C_p^* C_{INV}} h, \\
\triangle T_3 &= 2\rho g S_0 \mu h \left(\rho g C_f^* C_p^*\right)^{-2} (C_{INV} C_{PF}(\Omega_f))^{-1} \\
\triangle T_4 &= \frac{2\min\{1, \rho\}}{\rho g (1 + C_{PF}^2(\Omega_p))} k_{\min}, \\
Parameters &:= (1 + C_{PF}^2(\Omega_p))(C_{PF}^2(\Omega_f) + d)\frac{\rho g}{k_{\min}\mu}.
\end{aligned}$$

Note that $\triangle T_1$ and $\triangle T_4$ are independent of h but depend on $k_{\min}$ and S_0 as $\triangle T_1 \simeq S_0 k_{\min}$ and $\triangle T_4 \simeq k_{\min}$. $\triangle T_2$ and $\triangle T_3$ are independent of $k_{\min}$ but depend on h and S_0 as $\triangle T_{2/3} \simeq S_0 h$. The combination of physical parameters $Parameters$ is independent of h and S_0 but depends on all the other physical parameters. When $\mu = O(1)$, the meshwidth h in the porous medium is moderate and $k_{\min}$, S_0 are small the above restrictions mean

$$\text{either } \triangle t \leq C \max\{k_{\min}, S_0 k_{\min}, S_0 h\} \text{ or } C\sqrt{\mu k_{\min}} \geq 1.$$

Theorem 2 (Uniform in time stability of BEsplit1)**.** *Suppose either the problem parameters satisfy*

$$Parameters \leq 1,$$

or there is an $0 < \alpha < 1$ such that $\triangle t$ satisfies the time step restriction

$$\triangle t \leq (1-\alpha)\max\{\triangle T_1, \triangle T_2, \triangle T_3, \triangle T_4\}$$

Then, (BEsplit1) is stable uniformly in time. Specifically, if the timestep restriction with $\triangle T_3$ is active then:

$$\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^N||_f^2 + \rho g S_0||\phi_h^N||_p^2\right] + \\
&+\triangle t \sum_{n=0}^{N-1}[\frac{\triangle t}{2}\rho||\frac{u_h^{n+1}-u_h^n}{\triangle t}||_f^2 \\
&+\alpha a_f(u_h^{n+1}, u_h^{n+1}) + a_p(\phi_h^{n+1}, \phi_h^{n+1})] \leq \frac{1}{2}\left[\rho||u_h^0||_f^2 + \rho g S_0||\phi_h^0||_p^2\right] \\
&+\triangle t \sum_{n=0}^{N-1}\left[(f_f^{n+1}, u_h^{n+1})_f + \rho g(f_p^{n+1}, \phi_h^{n+1})_p\right].
\end{aligned}$$

If any of the other timestep restrictions are active then for any $N > 0$, there holds

$$\begin{aligned}
&\alpha\left[\rho||u_h^N||_f^2 + \rho g S_0||\phi_h^N||_p^2\right] + \\
&+\frac{\triangle t}{2}\sum_{n=0}^{N-1}[a_f(u_h^{n+1}+u_h^n, u_h^{n+1}+u_h^n) + a_p(\phi_h^{n+1}+\phi_h^n, \phi_h^{n+1}+\phi_h^n)] \\
&\leq \alpha\left[\rho||u_h^0||_f^2 + \rho g S_0||\phi_h^0||_p^2\right] + \\
&+\triangle t \sum_{n=0}^{N-1}\left[(f_f^{n+1}, u_h^{n+1}+u_h^n)_f + \rho g(f_p^{n+1}, \phi_h^{n+1}+\phi_h^n)_p\right].
\end{aligned}$$

Proof. In (BEsplit1) set $v_h = u_h^{n+1} + u_h^n, q_h = p_h^{n+1}$, average the incompressibility condition at successive time levels and add. We use

$$\begin{aligned}
a_f(u_h^{n+1}, u_h^{n+1}+u_h^n) = \frac{1}{2}a_f(u_h^{n+1}, u_h^{n+1}) - \frac{1}{2}a_f(u_h^n, u_h^n) + \\
+\frac{1}{2}a_f(u_h^{n+1}+u_h^n, u_h^{n+1}+u_h^n).
\end{aligned} \tag{4.3}$$

This gives:

$$\begin{aligned}
&\frac{1}{2}\left[2\rho||u_h^{n+1}||_f^2 + \triangle t a_f(u_h^{n+1}, u_h^{n+1})\right] - \frac{1}{2}\left[2\rho||u_h^n||_f^2 + \triangle t a_f(u_h^n, u_h^n)\right] + \\
&+\frac{\triangle t}{2}a_f(u_h^{n+1}+u_h^n, u_h^{n+1}+u_h^n) + \triangle t c_I(\phi_h^n, u_h^{n+1}+u_h^n) = \triangle t(f_f^{n+1}, u_h^{n+1}+u_h^n)_f.
\end{aligned} \tag{4.4}$$

Similarly, in the porous media equation, set $\psi_h = \phi_h^{n+1} + \phi_h^n$. We use here

$$a_p(\phi_h^{n+1}, \phi_h^{n+1} + \phi_h^n) = \frac{1}{2} a_p(\phi_h^{n+1}, \phi_h^{n+1}) - \frac{1}{2} a_p(\phi_h^n, \phi_h^n) + \\ + \frac{1}{2} a_p(\phi_h^{n+1} + \phi_h^n, \phi_h^{n+1} + \phi_h^n).$$

This gives

$$\frac{1}{2}\left[2\rho g S_0 ||\phi_h^{n+1}||_p^2 + \triangle t a_p(\phi_h^{n+1}, \phi_h^{n+1})\right] - \frac{1}{2}\left[2\rho g S_0 ||\phi_h^n||_p^2 + \triangle t a_p(\phi_h^n, \phi_h^n)\right] \\ + \frac{\triangle t}{2} a_p(\phi_h^{n+1} + \phi_h^n, \phi_h^{n+1} + \phi_h^n) - \triangle t c_I(\phi_h^{n+1} + \phi_h^n, u_h^{n+1}) = \triangle t \rho g (f_p^{n+1}, \phi_h^{n+1} + \phi_h^n)_p. \tag{4.5}$$

Add (4.4) and (4.5). Consider the sum of the two coupling terms that results

$$\begin{aligned} Coupling &= \triangle t \left[c_I(\phi_h^n, u_h^{n+1} + u_h^n) - c_I(\phi_h^{n+1} + \phi_h^n, u_h^{n+1})\right] = \\ &= \triangle t \left[c_I(\phi_h^n, u_h^n) - c_I(\phi_h^{n+1}, u_h^{n+1})\right]. \end{aligned}$$

Let us denote $C^n = c_I(\phi_h^n, u_h^n)$ and

$$E^n = \frac{1}{2}\left[2\rho ||u_h^n||_f^2 + 2\rho g S_0 ||\phi_h^n||_p^2 + \triangle t a_f(u_h^n, u_h^n) + \triangle t a_p(\phi_h^n, \phi_h^n)\right],$$

$$D^n = \frac{1}{2} a_f(u_h^{n+1} + u_h^n, u_h^{n+1} + u_h^n) + \frac{1}{2} a_p(\phi_h^{n+1} + \phi_h^n, \phi_h^{n+1} + \phi_h^n).$$

Adding the two energy estimates and using the above reduction of the coupling term reduces the total energy estimate to

$$\left[E^{n+1} - \triangle t C^{n+1}\right] - \left[E^n - \triangle t C^n\right] + \\ + \triangle t D^n = \triangle t \Big((f_f^{n+1}, u_h^{n+1} + u_h^n)_f + \rho g (f_p^{n+1}, \phi_h^{n+1} + \phi_h^n)_p \Big).$$

Summing this up from $n = 0$ to $n = N - 1$ results in

$$\left[E^N - \triangle t C^N\right] + \triangle t \sum_{n=0}^{N-1} D^n = \left[E^0 - \triangle t C^0\right] + \\ + \triangle t \sum_{n=0}^{N-1} \left[(f_f^{n+1}, u_h^{n+1} + u_h^n)_f + \rho g (f_p^{n+1}, \phi_h^{n+1} + \phi_h^n)_p\right].$$

Stability and the stated energy inequality thus follows provided

$$E^N - \triangle t C^N > 0 \text{ for every } N.$$

We have already shown that

$$\begin{aligned} D^n &\geq \frac{\mu}{2} ||\nabla\left(u_h^{n+1} + u_h^n\right)||_f^2 + \frac{\rho g k_{\min}}{2} ||\nabla\left(\phi_h^{n+1} + \phi_h^n\right)||_p^2, \\ |C^n| &\leq \frac{\mu}{2} ||\nabla u_h^n||_f^2 + \frac{\rho g k_{\min}}{2} ||\nabla \phi_h^n||_p^2 + \frac{\rho}{2} ||u_h^n||_f^2 + \frac{(C_f^* C_p^*)^4 (\rho g)^3}{32 \rho \mu k_{\min}} ||\phi_h^n||_p^2. \end{aligned}$$

Thus,

$$E^n - \triangle t C^n \geq \rho ||u_h^n||_f^2 + \rho g S_0 ||\phi_h^n||_p^2 + \frac{\triangle t}{2}\left(\mu ||\nabla u_h^n||_f^2 + \rho g k_{\min} ||\nabla \phi_h^n||_p^2\right) \\ - \triangle t [\frac{\mu}{2} ||\nabla u_h^n||_f^2 + \frac{\rho g k_{\min}}{2} ||\nabla \phi_h^n||_p^2 + \frac{\rho}{2} ||u_h^n||_f^2 + \frac{(C_f^* C_p^*)^4 (\rho g)^3}{32 \rho \mu k_{\min}} ||\phi_h^n||_p^2]. \tag{4.6}$$

Thus stability follows provided

$$\triangle t\frac{(C_f^*C_p^*)^4\,(\rho g)^3}{32\rho\mu k_{\min}} \leq (1-\alpha)\rho g S_0, \text{ or}$$
$$\triangle t \leq (1-\alpha)\mu k_{\min} S_0 \frac{32\rho}{(C_f^*C_p^*)^4\,(\rho g)^2} \equiv (1-\alpha)\triangle T_1.$$

Alternate conditions are obtained using different estimates of the coupling / interface term. Indeed, using Lemma 2

$$|C^n| = |c_I(u_h^n, \phi_h^n)| \leq \rho g C_f^* C_p^* C_{INV} h^{-1}\left(\frac{1}{2}||u_h^n||_f^2 + \frac{1}{2}||\phi_h^n||_p^2\right).$$

Thus stability follows provided

$$\begin{aligned}\frac{\triangle t}{h}\rho g C_f^* C_p^* C_{INV} &\leq 2(1-\alpha)\min\{\rho, \rho g S_0\}, \text{ or}\\ \triangle t &\leq (1-\alpha)\frac{2\min\{1, gS_0\}}{gC_f^*C_p^*C_{INV}}h \equiv (1-\alpha)\triangle T_2,\end{aligned}$$

which is the second condition.

For the condition $Parameters \leq 1$, that by Lemma 2

$$\begin{aligned}|C^n| &\leq \frac{\rho g k_{\min}}{2}||\nabla\phi_h^n||_p^2 + \frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}}||u_h^n||_{DIV}^2\\ &\leq \frac{\rho g k_{\min}}{2}||\nabla\phi_h^n||_p^2 + \frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}}(||u_h^n||_f^2 + d||\nabla u_h^n||_f^2)\\ &\leq \frac{\rho g k_{\min}}{2}||\nabla\phi_h^n||_p^2 + \frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}}(C_{PF}^2(\Omega_f) + d)||\nabla u_h^n||_f^2\end{aligned}$$

Thus the method is also stable if the problem data satisfies

$$\frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}}(C_{PF}^2(\Omega_f)+d) \leq \frac{\mu}{2} \text{ or}$$
$$Parameters = (1+C_{PF}^2(\Omega_p))(C_{PF}^2(\Omega_f)+d)\frac{\rho g}{k_{\min}\mu} \leq 1$$

The condition involving $\triangle T_3$ requires a separate stability proof. In (BEsplit1) set $v_h = u_h^{n+1}, q_h = p_h^{n+1}$ and add. We use

$$(u_h^{n+1} - u_h^n, u_h^{n+1})_f = \frac{1}{2}\left[||u_h^{n+1}||_f^2 - ||u_h^n||_f^2\right] + \frac{1}{2}||u_h^{n+1} - u_h^n||_f^2,$$

and similarly for ϕ. This gives:

$$\begin{aligned}&\frac{\rho}{2}\left[||u_h^{n+1}||_f^2 - ||u_h^n||_f^2\right] + \frac{\rho}{2}||u_h^{n+1} - u_h^n||_f^2 + \triangle t a_f(u_h^{n+1}, u_h^{n+1}) +\\ &\qquad + \triangle t c_I(\phi_h^n, u_h^{n+1}) = \triangle t(f_f^{n+1}, u_h^{n+1})_f.\end{aligned}$$

Similarly, in the porous media equation, set $\psi_h = \phi_h^{n+1}$, we get

$$\begin{aligned}&\frac{1}{2}\left[\rho g S_0||\phi_h^{n+1}||_p^2 - \rho g S_0||\phi_h^n||_p^2 + \rho g S_0||\phi_h^{n+1} - \phi_h^n||_p^2\right] + \triangle t a_p(\phi_h^{n+1}, \phi_h^{n+1})\\ &\qquad - \triangle t c_I(\phi_h^{n+1}, u_h^{n+1}) = \triangle t \rho g(f_p^{n+1}, \phi_h^{n+1})_p.\end{aligned}$$

Add these two equations and consider the sum of the two coupling terms that result:

$$|Coupling| = \triangle t|c_I(\phi_h^n, u_h^{n+1}) - c_I(\phi_h^{n+1}, u_h^{n+1})| = \triangle t|c_I(\phi_h^{n+1} - \phi_h^n, u_h^{n+1})|.$$

The following bound holds by an analogous proof as that of in Lemma 2:

$$|Coupling| \leq \frac{\rho g S_0}{2}||\phi_h^{n+1}-\phi_h^n||_p^2 +$$
$$+\triangle t\left[\frac{\triangle t}{2\rho g S_0}\left(\rho g C_f^* C_p^*\right)^2 C_{INV} h^{-1}||u_h^{n+1}||_f||\nabla u_h^{n+1}||_f\right]$$
$$\leq \frac{\rho g S_0}{2}||\phi_h^{n+1}-\phi_h^n||_p^2 +$$
$$+\triangle t\left[\frac{\triangle t}{2\rho g S_0 \mu}\left(\rho g C_f^* C_p^*\right)^2 C_{INV} h^{-1} C_{PF}(\Omega_f) a_f(u_h^{n+1}, u_h^{n+1})\right].$$

The remainder of the proof follows the above pattern and is complete, provided

$$\frac{\triangle t}{2\rho g S_0 \mu}\left(\rho g C_f^* C_p^*\right)^2 C_{INV} h^{-1} C_{PF}(\Omega_f) \leq 1-\alpha, \text{ or}$$
$$\triangle t < (1-\alpha)\frac{2\rho g S_0 \mu}{\left(\rho g C_f^* C_p^*\right)^2 C_{INV} C_{PF}(\Omega_f)} h \equiv (1-\alpha)\triangle T_3.$$

For the $\triangle T_4$ condition, we exploit the added grad-div stabilization. By the third inequality of Lemma 2

$$|Coupling| \leq \triangle t\frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \triangle t\frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}}||u||^2 + \triangle t\frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}}||\nabla\cdot u||^2.$$

The last term can be subsumed into the grad-div stabilization term provided

$$\triangle t\frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}} \leq 1.$$

The other two terms are subsumed into the system energy. Stability thus follows provided

$$\rho||u_h^n||_f^2 + \rho g S_0||\phi_h^n||_p^2 + \frac{\triangle t}{2}\left(\mu||\nabla u_h^n||_f^2 + \rho g k_{\min}||\nabla\phi_h^n||_p^2\right)$$
$$-\left[\triangle t\frac{\rho g k_{\min}}{2}||\nabla\phi||_p^2 + \triangle t\frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}}||u||^2\right] > 0.$$

This requires

$$\triangle t\frac{\rho g(1+C_{PF}^2(\Omega_p))}{2k_{\min}} \leq \rho$$

Thus, stability follows under these two conditions, i.e., if

$$\triangle t \leq \min\{1,\rho\}\frac{2k_{\min}}{\rho g(1+C_{PF}^2(\Omega_p))} = \triangle T_4.$$

The rest of the proof follows by summing. □

4.3. **BEsplit2 stability.** Due to the similarity of the analysis for BEsplit2 to BEsplit1, we present the aspects of the proof that differ only. Define

$$\triangle T_5 \quad : \quad = \frac{2k_{\min}h}{g(C_f^* C_p^*)^2 C_{PF}(\Omega_p) C_{INV}}$$
$$\triangle T_6 \quad : \quad = \frac{2}{g\left(1+C_{PF}^2(\Omega_p)\right)}k_{\min}.$$

We prove uniform in time stability under a time step restriction of the form that occurred in BEsplit1 with $\triangle T_3$ replaced by $\triangle T_5$ and $\triangle T_4$ replaced by $\triangle T_6$. Thus, for small S_0 the active constraint is expected to be

$$\triangle t < \triangle T_6 \simeq C k_{\min}$$

which is independent of both h and S_0. Thus, BEsplit1/2 are promising for the quasi-static approximation and for problems with very small S_0 and moderate $k_{\min}$.

Theorem 3 (Uniform in time and S_0 stability)**.** *Consider the method (BEsplit2). Suppose that there is an $\alpha, 0 < \alpha < 1$, such that either the problem parameters satisfy*

$$Parameters \leq 1 - \alpha,$$

or $\triangle t$ satisfies the time step restriction

$$\triangle t \leq (1-\alpha)\max\{\triangle T_1, \triangle T_2, \triangle T_5, \triangle T_6\}.$$

Then, BEsplit2 is stable uniformly in time and uniformly in S_0. Specifically, for any $N > 0$ we have the energy inequality (which also proves stability)

$$\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^N||_f^2 + \rho||\nabla \cdot u_h^N||_f^2 + \rho g S_0||\phi_h^N||_p^2\right] + \\
&+\triangle t \sum_{n=0}^{N-1}\left[\frac{\triangle t}{2}\rho g S_0||\frac{\phi_h^{n+1}-\phi_h^n}{\triangle t}||_p^2 + a_f(u_h^{n+1}, u_h^{n+1}) + \alpha a_p(\phi_h^{n+1}, \phi_h^{n+1})\right] \\
&\leq \frac{1}{2}\left[\rho||u_h^0||_f^2 + \rho||\nabla \cdot u_h^0||_f^2 + \rho g S_0||\phi_h^0||_p^2\right] + \triangle t \sum_{n=0}^{N-1}\left[(f_f^{n+1}, u_h^{n+1})_f + \rho g(f_p^{n+1}, \phi_h^{n+1})_p\right].
\end{aligned}$$

Proof. The derivation of the stability conditions involving $Parameters$ and $\triangle T_1$, $\triangle T_2$ is very similar to the case of BEsplit1. We therefore move to the condition involving $\triangle T_5$ and T_6.
In (BEsplit2) set $\psi_h = \phi_h^{n+1}, v_h = u_h^{n+1}, q_h = p_h^{n+1}$, and add. We use

$$-(u_h^n, u_h^{n+1})_f = -\frac{1}{2}(u_h^n, u_h^n)_f - \frac{1}{2}(u_h^{n+1}, u_h^{n+1})_f + \frac{1}{2}(u_h^{n+1} - u_h^n, u_h^{n+1} - u_h^n)_f,$$

and similarly for the $(\nabla \cdot u_h^n, \nabla \cdot u_h^{n+1})_f$ terms and the analogous terms in the ϕ equation. This gives:

$$\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^{n+1}||_f^2 + \rho||\nabla \cdot u_h^{n+1}||_f^2 + \rho g S_0||\phi_h^{n+1}||_p^2\right] - \frac{1}{2}\left[\rho||u_h^n||_f^2 + \rho||\nabla \cdot u_h^n||_f^2 + \rho g S_0||\phi_h^n||_p^2\right] + \\
&+\frac{1}{2}\left[\rho||u_h^{n+1} - u_h^n||_f^2 + \rho||\nabla \cdot (u_h^{n+1} - u_h^n)||_f^2 + \rho g S_0||\phi_h^{n+1} - \phi_h^n||_p^2\right] \\
&+\triangle t\left[a_f(u_h^{n+1}, u_h^{n+1}) + a_p(\phi_h^{n+1}, \phi_h^{n+1})\right] \\
&+\triangle t c_I(\phi_h^{n+1}, u_h^{n+1} - u_h^n) = \triangle t(f_f^{n+1}, u_h^{n+1})_f + \triangle t \rho g(f_p^{n+1}, \phi_h^{n+1})_p.
\end{aligned}$$

Consider the sum of the two coupling terms

$$Coupling = \triangle t c_I(\phi_h^{n+1}, u_h^{n+1} - u_h^n).$$

For the condition involving $\triangle T_5$,

$$
\begin{aligned}
|Coupling| &\leq \triangle t\rho g C_f^* C_p^* C_{PF}^{\frac{1}{2}}(\Omega_p)(C_{INV}h^{-1})^{\frac{1}{2}}||\nabla\phi_h^{n+1}||_p||u_h^{n+1}-u_h^n||_f \\
&\leq \frac{1}{2}\rho||u_h^{n+1}-u_h^n||_f^2 + \frac{g(C_f^*C_p^*)^2 C_{PF}(\Omega_p)C_{INV}h^{-1}\triangle t^2}{2k_{min}} a_p(\phi_h^{n+1},\phi_h^{n+1})
\end{aligned}
$$

Subsuming the above two terms in the obvious places, the method is stable if

$$
\triangle t \leq \frac{2k_{min}h}{g(C_f^*C_p^*)^2 C_{PF}(\Omega_p)C_{INV}} = \triangle T_5.
$$

For the stability condition involving $\triangle T_6$, we have, using Lemma 2 and $a_p(\phi_h^{n+1},\phi_h^{n+1}) \geq \rho g k_{\min}||\nabla\phi_h^{n+1}||_p$,

$$
\begin{aligned}
|Coupling| &\leq \triangle t\,(\rho g)\,||\phi_h^{n+1}||_{H^1(\Omega_p)}||u_h^{n+1}-u_h^n||_{DIV} \\
&\leq \triangle t\,(\rho g)\sqrt{1+C_{PF}^2(\Omega_p)}||\nabla\phi_h^{n+1}||_p||u_h^{n+1}-u_h^n||_{DIV} \\
&\leq \frac{1}{2}\left[\rho||u_h^{n+1}-u_h^n||_f^2 + \rho||\nabla\cdot(u_h^{n+1}-u_h^n)||_f^2\right] \\
&+ \frac{1}{2}\triangle t^2\frac{g}{k_{\min}}\left(1+C_{PF}^2(\Omega_p)\right)a_p(\phi_h^{n+1},\phi_h^{n+1}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{2}\left[\rho||u_h^{n+1}||_f^2 + \rho||\nabla\cdot u_h^{n+1}||_f^2 + \rho g S_0||\phi_h^{n+1}||_p^2\right] - \frac{1}{2}\left[\rho||u_h^n||_f^2 + \rho||\nabla\cdot u_h^n||_f^2 + \rho g S_0||\phi_h^n||_p^2\right] + \\
&\qquad + \frac{1}{2}\rho g S_0||\phi_h^{n+1}-\phi_h^n||_p^2 + \triangle t[a_f(u_h^{n+1},u_h^{n+1}) + \\
&\qquad + (1-\frac{1}{2}\triangle t g\left(1+C_{PF}^2(\Omega_p)\right)k_{\min}^{-1})a_p(\phi_h^{n+1},\phi_h^{n+1})] \\
&\qquad \leq \triangle t(f_f^{n+1},u_h^{n+1})_f + \triangle t\rho g(f_p^{n+1},\phi_h^{n+1})_p.
\end{aligned}
$$

Stability then follows under the timestep restriction

$$
(1-\frac{1}{2}\triangle t g\left(1+C_{PF}^2(\Omega_p)\right)k_{\min}^{-1}) \geq \alpha > 0
$$

which is equivalent to

$$
\triangle t \leq (1-\alpha)\frac{2}{g\,(1+C_{PF}^2(\Omega_p))}k_{\min} \equiv (1-\alpha)\triangle T_6.
$$

□

4.4. **Stability of CNsplit. CNsplit** computes two partitioned approximations $(\widehat{u}_h^n, \widehat{p}_h^n, \widehat{\phi}_h^n)$ and $(\widetilde{u}_h^n, \widetilde{p}_h^n, \widetilde{\phi}_h^n) \in X_f^h \times Q_f^h \times X_p^h$ for $n \geq 1$ whereupon

$$
\text{(CNsplit)} \qquad (u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) = \frac{1}{2}[(\widehat{u}_h^{n+1}, \widehat{p}_h^{n+1}, \widehat{\phi}_h^{n+1}) + (\widetilde{u}_h^{n+1}, \widetilde{p}_h^{n+1}, \widetilde{\phi}_h^{n+1})],
$$

that is, *the new approximation to each variable is the average of the two computed approximations.* Since the unit ball in a Hilbert space is convex, stability of

$(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1})$ follows from stability of $(\widehat{u}_h^{n+1}, \widehat{p}_h^{n+1}, \widehat{\phi}_h^{n+1})$ and $(\widetilde{u}_h^{n+1}, \widetilde{p}_h^{n+1}, \widetilde{\phi}_h^{n+1})$. We thus prove stability of the two individual sub-problems. Define

$$\triangle T_6 := \frac{\sqrt{2S_0}}{\sqrt{g} C_p^* C_f^* C_{INV}} h$$

We prove long time stability under a time step condition of the form

$$\triangle t < C\sqrt{S_0} h.$$

Theorem 4 (Stability of one step of CNsplit)**.** *Consider (CNsplit-a) one step of the* ***CNsplit*** *method. Suppose there is an $0 < \alpha < 1/2$ such that $\triangle t$ satisfies the time step restriction*

$$\triangle t \leq (1-\alpha)\triangle T_6$$

Then, CNsplit-a is stable uniformly in time over possibly long time intervals. Specifically, for every $N \geq 1$

$$\begin{aligned}
&\alpha\left[\rho||\widehat{u}_h^N||_f^2 + \rho g S_0 ||\widehat{\phi}_h^N||_p^2\right] \\
&+\triangle t \sum_{n=0}^{N-1} \frac{1}{2}\left[a_f(\widehat{u}_h^{n+1} + \widehat{u}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) + a_p(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n)\right] \\
&\leq \rho||\widehat{u}_h^0||_f^2 + \rho g S_0||\widehat{\phi}_h^0||_p^2 - \triangle t c_I(\widehat{\phi}_h^0, \widehat{u}_h^0) \\
&+\triangle t \sum_{n=0}^{N-1}\left[(f_f^{n+1/2}, \widehat{u}_h^{n+1} + \widehat{u}_h^n)_f + \rho g (f_p^{n+1/2}, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n)_p\right].
\end{aligned}$$

Proof. In (CNsplit-a) set $v_h = \widehat{u}_h^{n+1} + \widehat{u}_h^n, q_h = \widehat{p}_h^{n+1}$, average the incompressibility condition at successive time levels and add. This gives:

$$\begin{aligned}
&\rho||\widehat{u}_h^{n+1}||_f^2 - \rho||\widehat{u}_h^n||_f^2 + \frac{\triangle t}{2} a_f(\widehat{u}_h^{n+1} + \widehat{u}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) + \\
&+\triangle t c_I(\widehat{\phi}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) = \triangle t (f_f^{n+1/2}, \widehat{u}_h^{n+1} + \widehat{u}_h^n)_f.
\end{aligned}$$

Similarly, in the porous media equation, set $\psi_h = \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n$. This gives

$$\begin{aligned}
&\rho g S_0||\widehat{\phi}_h^{n+1}||_p^2 - \rho g S_0||\widehat{\phi}_h^n||_p^2 + \frac{\triangle t}{2} a_p(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n) \\
&-\triangle t c_I(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{u}_h^{n+1}) = \triangle t \rho g (f_p^{n+1/2}, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n).
\end{aligned}$$

Add and consider the sum of the two coupling terms

$$\begin{aligned}
Coupling &= \triangle t\left[c_I(\widehat{\phi}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) - c_I(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{u}_h^{n+1})\right] \\
&= \triangle t\left[c_I(\widehat{\phi}_h^n, \widehat{u}_h^n) - c_I(\widehat{\phi}_h^{n+1}, \widehat{u}_h^{n+1})\right]
\end{aligned}$$

Let us denote $C^n = c_I(\widehat{\phi}_h^n, \widehat{u}_h^n)$ and

$$\begin{aligned}
E^n &= \rho||\widehat{u}_h^n||_f^2 + \rho g S_0||\widehat{\phi}_h^n||_p^2, \\
D^n &= \frac{1}{2} a_f(\widehat{u}_h^{n+1} + \widehat{u}_h^n, \widehat{u}_h^{n+1} + \widehat{u}_h^n) + \frac{1}{2} a_p(\widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n, \widehat{\phi}_h^{n+1} + \widehat{\phi}_h^n).
\end{aligned}$$

Adding the two energy estimates and using the above reduction of the coupling term reduces the total energy estimate to

$$\begin{aligned}&\left[E^{n+1}-\triangle tC^{n+1}\right]-\left[E^{n}-\triangle tC^{n}\right]+\\+\triangle tD^{n}=\ &\triangle t\Big(\ (f_f^{n+1/2},\widehat{u}_h^{n+1}+\widehat{u}_h^{n})_f+\rho g(f_p^{n+1/2},\widehat{\phi}_h^{n+1}+\widehat{\phi}_h^{n})_p\Big)\end{aligned}$$

Sum this inequality from $n=0$ to $N-1$. The energy inequality thus follows provided

$$E^N-\triangle tC^N\geq\alpha E^N\ \text{ for every } N.$$

Consider $\triangle tC^N$. Dropping super and subscripts and applying Lemma 2 gives

$$\begin{aligned}\triangle t|C|\ &\leq\ \triangle t\rho gC_p^*C_f^*C_{INV}h^{-1}||u||_f|||\phi||_p\\&\leq\ \frac{\rho gS_0}{2}||\phi||_p^2+\frac{\triangle t^2}{2\rho gS_0}\left[\rho gC_p^*C_f^*C_{INV}h^{-1}\right]^2||u||_f^2.\end{aligned}$$

We thus have stability provided

$$\frac{\triangle t^2}{2\rho gS_0}\left[\rho gC_p^*C_f^*C_{INV}h^{-1}\right]^2<\rho\text{ or }\triangle t<\triangle T_6.$$

Under the timestep restriction $\triangle t\leq\sqrt{1-\alpha}\triangle T_6$ which is implied by $\triangle t\leq(1-\alpha)\triangle T_6$ we have

$$\rho||\widehat{u}_h^{n+1}||_f^2+\rho gS_0||\widehat{\phi}_h^{n+1}||_p^2-\triangle tc_I(\widehat{\phi}_h^{n+1},\widehat{u}_h^{n+1})\geq\alpha\left[\rho||\widehat{u}_h^{n+1}||_f^2+\rho gS_0||\widehat{\phi}_h^{n+1}||_p^2\right].$$

This proves stability of the first half step. □

Now we consider the second half step.

Theorem 5 (Stability of one step of CNsplit)**.** *Consider (CNsplit-b). Suppose there is an $\alpha, 0<\alpha<1$, such that $\triangle t$ satisfies the time step restriction*

$$\triangle t\leq(1-\alpha)\triangle T_6$$

Then, it is stable over long time intervals. Specifically, for every $N\geq1$

$$\begin{aligned}&\alpha\left[\rho||\widetilde{u}_h^{N}||_f^2+\rho gS_0||\widetilde{\phi}_h^{N}||_p^2\right]\\+\triangle t\sum_{n=0}^{N-1}\frac{1}{2}&\left[a_f(\widetilde{u}_h^{n+1}+\widetilde{u}_h^{n},\widetilde{u}_h^{n+1}+\widetilde{u}_h^{n})+a_p(\widetilde{\phi}_h^{n+1}+\widetilde{\phi}_h^{n},\widetilde{\phi}_h^{n+1}+\widetilde{\phi}_h^{n})\right]\\&\leq\left[\rho||\widetilde{u}_h^{0}||_f^2+\rho gS_0||\widetilde{\phi}_h^{0}||_p^2+\triangle tc_I(\widetilde{\phi}_h^{0},\widetilde{u}_h^{0})\right]\\&+\triangle t\sum_{n=0}^{N-1}\left[(f_f^{n+1/2},\widetilde{u}_h^{n+1}+\widetilde{u}_h^{n})_f+\rho g(f_p^{n+1/2},\widetilde{\phi}_h^{n+1}+\widetilde{\phi}_h^{n})_p\right].\end{aligned}$$

The proof is essentially the same as for the first half-step and is thus omitted.

5. Numerical Experiments

We present numerical experiments to test the algorithms presented herein. First, using the exact solution introduced in [MZ10], we test accuracy. One new aspect is that we also test mass conservation errors across the interface I, the last columns of Tables 1 through 4. While mixed methods are expected to have better conservation properties than the non-mixed formulation we use and we anticipate some penalties for uncoupling the problem across I, we find the mass conservation errors are quite acceptable in this limited test. Second, we test stability over longer time intervals and small values of $k_{\min}$ and S_0. In these tests the splitting based partitioned methods appear to be stable for larger timestep sizes than the IMEX based partitioned methods we have tested previously in [LTT11] and that good partitioned methods are available when one parameter is small. When both are small, a very small timestep is required for stability for the four methods. The code was implemented using the software package *FreeFEM++*.

5.1. **Test 1.** For the first test we select the velocity and pressure field given in [MZ10]. Let the domain Ω be composed of $\Omega_f = (0,1)\times(1,2)$ and $\Omega_p = (0,1)\times(0,1)$ with the interface $\Gamma = (0,1) \times \{1\}$. The exact velocity field is given by

$$\begin{aligned}
u_1(x,y,t) &= (x^2(y-1)^2 + y)\cos t\,,\\
u_2(x,y,t) &= \left(-\frac{2}{3}x(y-1)^3 + 2 - \pi\sin(\pi x)\right)\cos t\,,\\
p(x,y,t) &= (2-\pi\sin(\pi x))\sin\left(\frac{\pi}{2}y\right)\cos t\,,\\
\phi(x,y,t) &= (2-\pi\sin(\pi x))(1-y-\cos(\pi y))\cos t.
\end{aligned}$$

To check the rates of convergence, take the time interval $0 \le t \le 1$ and in this first test the physical parameters ρ, g, μ, K, S_0 and α are simply set to 1. We utilize Taylor-Hood $P2-P1$ finite elements for the Stokes subdomain and continuous piecewise quadratic finite element for the Darcy subdomain. The boundary conditions on the exterior boundaries (not including the interface I) are inhomogeneous Dirichlet: $u_h = u_{exact}, \phi_h = \phi_{exact}$ on the exterior boundaries. The initial data and source terms are chosen to correspond the exact solution.

For convenience, we denote $\|\cdot\|_I = \|\cdot\|_{L^2(0,T;L^2(I))}$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(0,T;L^2(\Omega_{f|p}))}$ and $\|\cdot\|_2 = \|\cdot\|_{L^2(0,T;L^2(\Omega_{f|p}))}$. We show below in Table 1–4 the errors of approximated velocity and Darcy pressure in several different norms. In the last columns of the tables are the errors in mass conservation on I.

h	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\phi-\phi_h\|_I$	$\|(u_h^f - u_h^p)\cdot \mathbf{n}\|_I$
1/5	2.921e-3	7.194e-2	4.030e-3	4.626e-3	2.280e-1
1/10	8.954e-4	2.181e-2	1.183e-2	1.661e-3	4.070e-2
1/20	4.198e-4	5.751e-3	6.367e-4	9.080e-4	9.566e-3
1/40	2.105e-4	1.959e-3	3.399e-4	4.977e-4	2.376e-3
1/80	1.057e-4	8.328e-4	1.771e-4	2.668e-4	5.047e-4

TABLE 1. The convergence performance for SDsplit method. The time step Δt is set to be equal to mesh size h.

From the tables, we see that SDsplit, BEsplit1 and BEsplit2 are first order methods while CNsplit is second order accuracy, as predicted. Further, the error levels of the first order methods seem quite acceptable as are the mass conservation errors across I.

5.2. **Test 2.** Stokes-Darcy flows with small hydraulic conductivity tensor and storativity coefficient are of special interest in some applications. We test herein and compare the performance of our proposed methods for uncoupling Stokes-Darcy flows for three cases: small $k_{\min}$ and $O(1)$ S_0, $O(1)$ $k_{\min}$ and small S_0 and small

h	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\phi-\phi_h\|_I$	$\|(u_h^f-u_h^p)\cdot\mathbf{n}\|_I$
1/5	3.448e-3	7.371e-2	4.289e-3	4.766e-3	2.278e-1
1/10	1.657e-3	2.343e-2	1.163e-3	1.665e-3	4.694e-2
1/20	8.405e-4	7.200e-3	5.409e-4	8.126e-3	9.531e-3
1/40	4.239e-4	2.923e-3	2.705e-4	4.081e-4	2.369e-3
1/80	2.128e-4	1.367e-3	1.356e-4	2.046e-4	5.035e-4

TABLE 2. The convergence performance for BEsplit1 method. The time step Δt is set to be equal to mesh size h.

h	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\phi-\phi_h\|_I$	$\|(u_h^f-u_h^p)\cdot\mathbf{n}\|_I$
1/5	2.768e-3	7.130e-2	9.738e-3	1.649e-2	2.547e-1
1/10	9.282e-4	2.164e-2	4.833e-3	8.441e-3	7.087e-2
1/20	4.390e-4	5.610e-3	2.447e-3	4.231e-3	2.722e-2
1/40	2.196e-4	1.860e-3	1.233e-3	2.119e-3	1.212e-2
1/80	1.100e-4	7.739e-4	6.188e-4	1.060e-3	6.258e-3

TABLE 3. The convergence performance for BEsplit2 method. The time step Δt is set to be equal to mesh size h.

h	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\phi-\phi_h\|_I$	$\|(u_h^f-u_h^p)\cdot\mathbf{n}\|_I$
1/5	3.044e-3	7.789e-2	7.647e-3	1.112e-2	2.284e-1
1/10	4.323e-4	2.259e-2	1.520e-3	2.085e-3	4.795e-2
1/20	5.466e-5	5.193e-3	3.654e-4	4.961e-4	9.849e-3
1/40	7.829e-6	1.270e-3	9.081e-5	1.227e-4	2.487e-3
1/80	1.573e-6	3.187e-4	2.265e-5	3.056e-5	5.273e-4

TABLE 4. The convergence performance for CNsplit method. The time step Δt is set to be equal to mesh size h.

$k_{\min}$ and small S_0. The last case is separated into several sub-cases to distinguish 'extremely small' and 'moderately small' S_0 and $k_{\min}$. Our test here is to check the largest timestep for which the four methods are stable over long time intervals. Since the problem is linear we can take the body force terms to be zero. The true solution decays as $t \to \infty$, so any growth in the approximate solution is an instability. We take the initial condition

$$\begin{aligned}
u_1(x,y,0) &= (x^2(y-1)^2+y)\,,\\
u_2(x,y,0) &= \left(-\frac{2}{3}x(y-1)^3+2-\pi\sin(\pi x)\right)\,,\\
p(x,y,0) &= (2-\pi\sin(\pi x))\sin\left(\frac{\pi}{2}y\right)\,,\\
\phi(x,y,0) &= (2-\pi\sin(\pi x))(1-y-\cos(\pi y)).
\end{aligned}$$

Define the kinetic energy $E^n = \|u_h^n\|_f^2 + \|\phi_h^n\|_p^2$. The final time T_f in our experiment is 10.0 and the system parameters are simply set to be 1.0, except hydraulic conductivity $k_{\min}$ and storativity coefficient S_0. We take the mesh size $h = 1/10$ and run the experiment with different time-step sizes. With each value of Δt, we compute the kinetic energy at final time, i.e., E^N where $N = T_f/\Delta t$. However, we

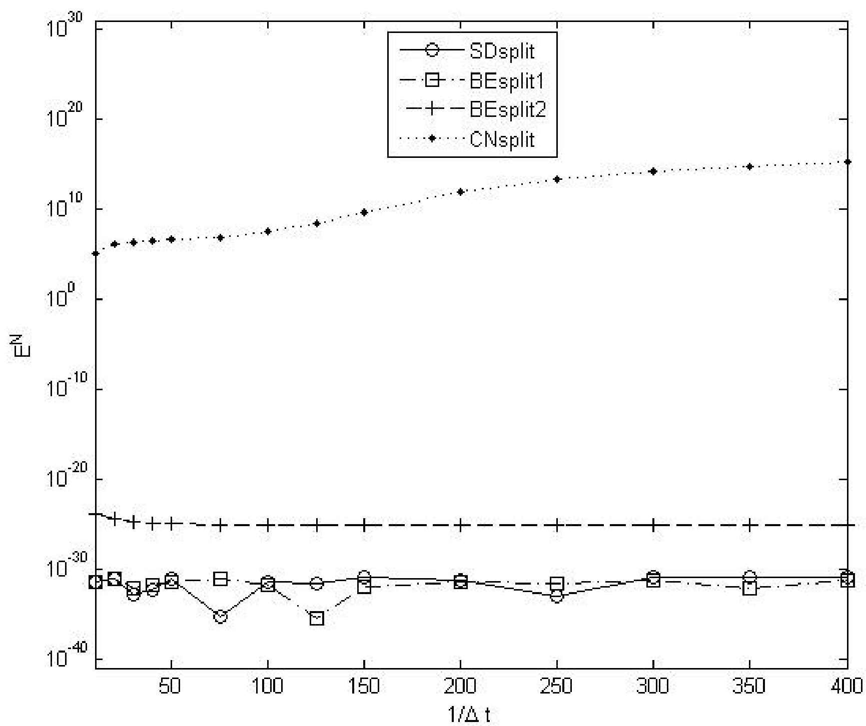

FIGURE 1. E^N for time-step sizes and different splitting methods with $k_{\min} = 1$ and $S_0 = 10^{-12}$

use 10^{250} as a 'cut-off' value for E^n. If E^n exceeds 10^{250} at some n, we stop and output E^n, the kinetic energy at that point. By looking at these figures, we can estimate the largest Δt for which numerical methods is stable.

Since Stokes flows and porous media flows are not typically high velocity flows, and since the domains are large with associated significant costs for subdomain solves, the ability to take large timesteps is desirable. In the stability tests for small parameter $k_{\min}$ or S_0 the three first order methods are superior. They are stable for larger timesteps, as predicted by the theory. The CNsplit method generally requires a much smaller timestep to attain stability. Thus, in some of the figures, the largest timesteps needed for the stability of CNsplit are not shown in some cases. To present the CNsplit case, Figure 7 gives a graph showing stability of CNsplit alone with numerous small values of S_0 and $k_{\min}$.

6. Conclusions and open problems

In both our analysis and tests on problems k_{min} and S_0 are small it seems that stability over long time intervals (and the associated time step restriction) is a key issue in uncoupling the Stokes-Darcy problem. With one small parameter, the first order splitting methods had significant advantages in stability and are a good option when $k_{\min}$ or S_0 is small.

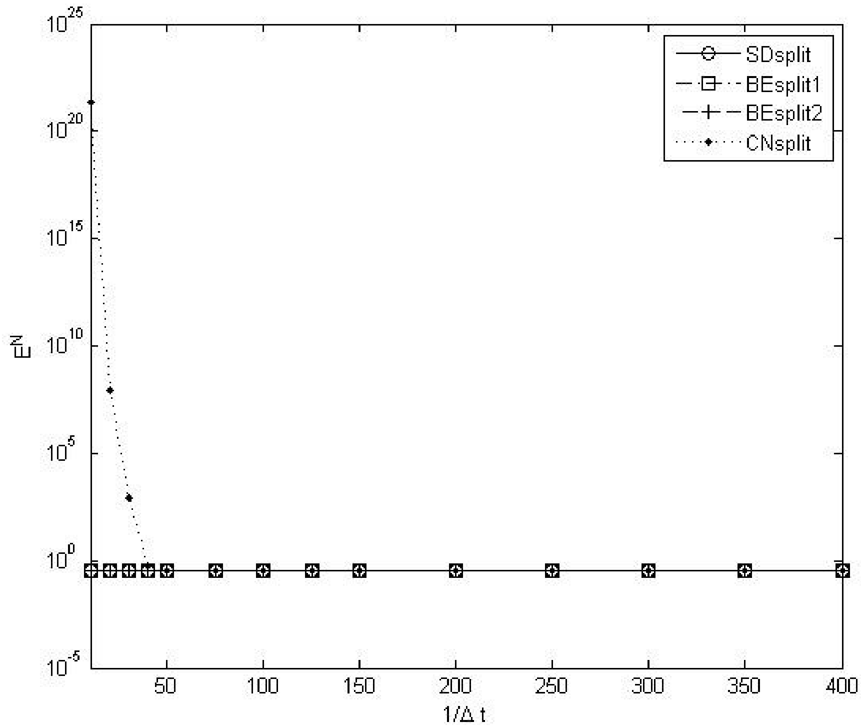

FIGURE 2. E^N using different $\triangle t$ sizes for different splitting methods with $k_{\min} = 1$ and $S_0 = 10^{-12}$.

Many other open problems remain. Finding partitioned methods stable for large timesteps when both $k_{\min}$, S_0 are small is an open problem. Further, while the first order methods gave acceptable error levels, more accuracy is always desirable. The stability of higher order partitioned methods for large timesteps and small parameters also is also largely an open problem. We have not tried to optimize the dependence of the timestep barriers upon the domain size. This is an important and open problem, especially for domains with large aspect ratios. At this point we do not know if a partitioned method exists with timestep restriction independent of $S_0, k_{\min}, \mu$ and h. If $k_{\min}, \mu \to 0$ the problem reduces to $u_t + C\phi = 0$ and $\phi_t - Cu = 0$ and any such algorithm would be an explicit method for an abstract wave-like equation written as a first order system. The behavior of numerical methods (both partitioned time stepping methods and iterative decoupling methods for use with monolithic time discretizations) in the quasi-static limit (as $S_0 \to 0$) is an open question critical in applications to aquifers since quasi static models are common, e.g., [CR08] for an example and [M11] for a first step to its resolution. In many problems k_{min} and S_0 are both small and the double asymptotics of both parameters is important and open. Since fluid flow acts on different time scales in free flow and in porous media, developing algorithms with good properties that allow different

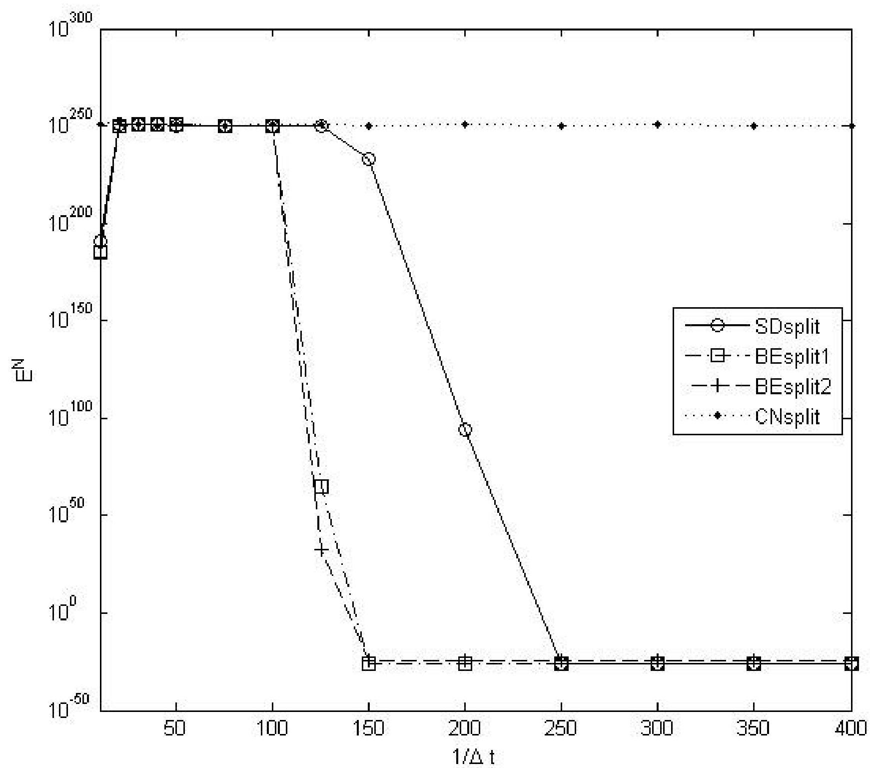

FIGURE 3. E^N using different $\triangle t$ sizes for different splitting methods with $k_{\min} = 10^{-3}$ and $S_0 = 10^{-3}$

time step sizes in the two domains (multi-rate or asynchronous methods) is an important and largely open challenge.

7. Acknowledgements

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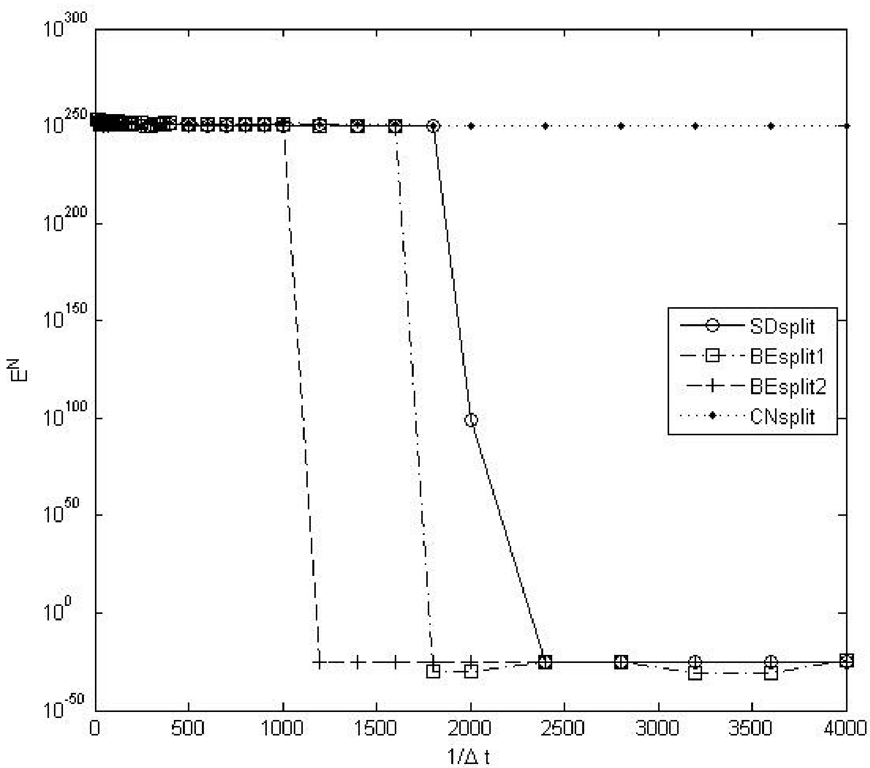

FIGURE 4. E^N using different $\triangle t$ sizes for different splitting methods with $k_{\min} = 10^{-4}$ and $S_0 = 10^{-4}$

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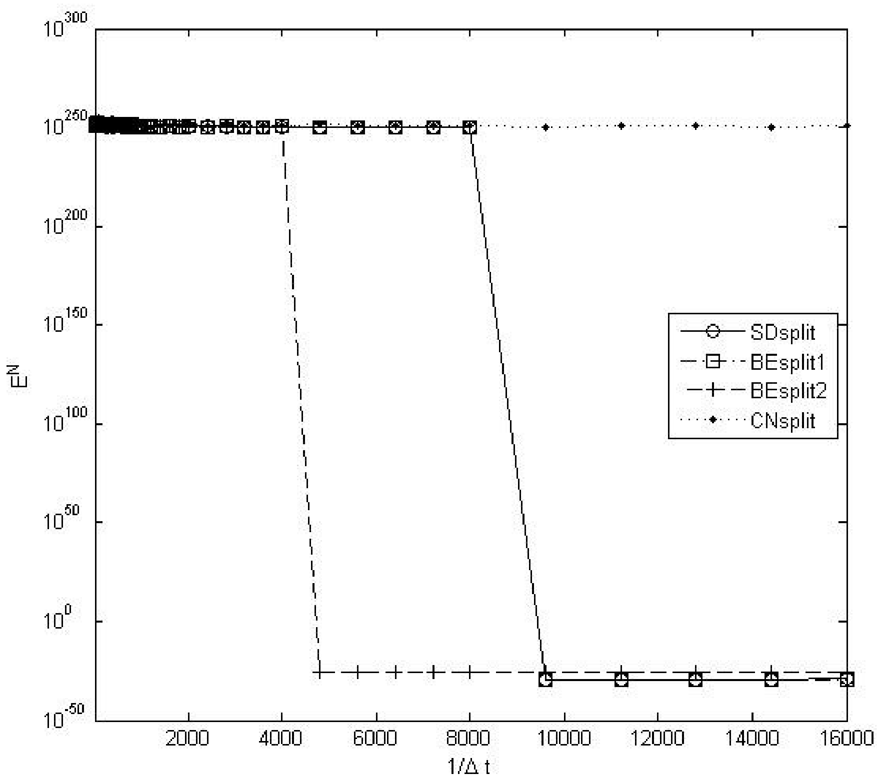

FIGURE 5. E^N using different $\triangle t$ sizes for different splitting methods with $k_{\min} = 10^{-4}$ and $S_0 = 10^{-12}$

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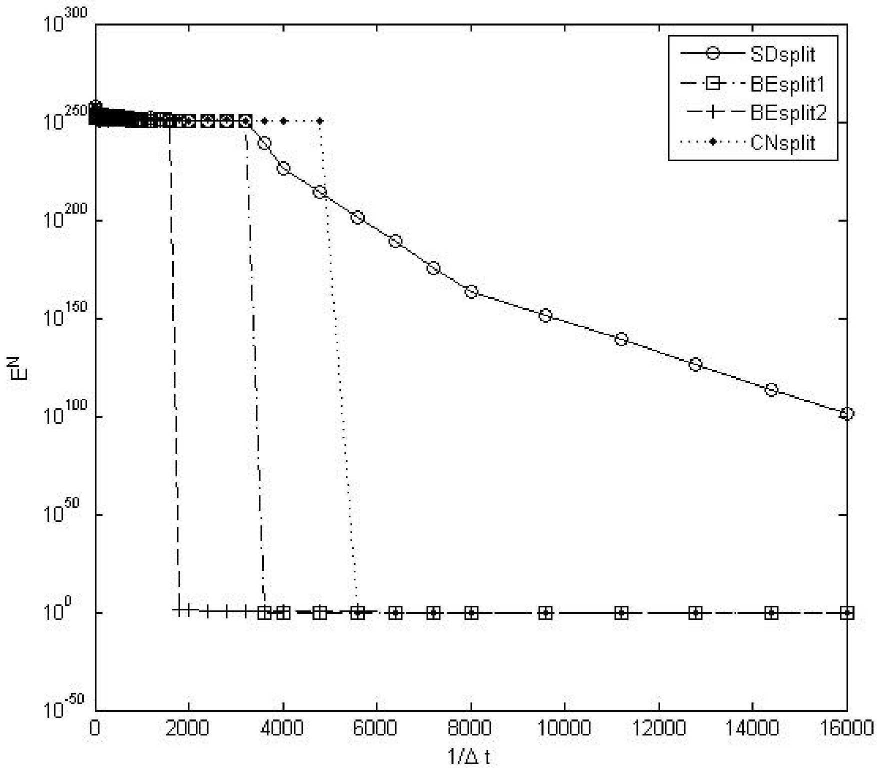

FIGURE 6. E^N using different $\triangle t$ sizes for different splitting methods with $k_{\min} = 10^{-12}$ and $S_0 = 10^{-4}$

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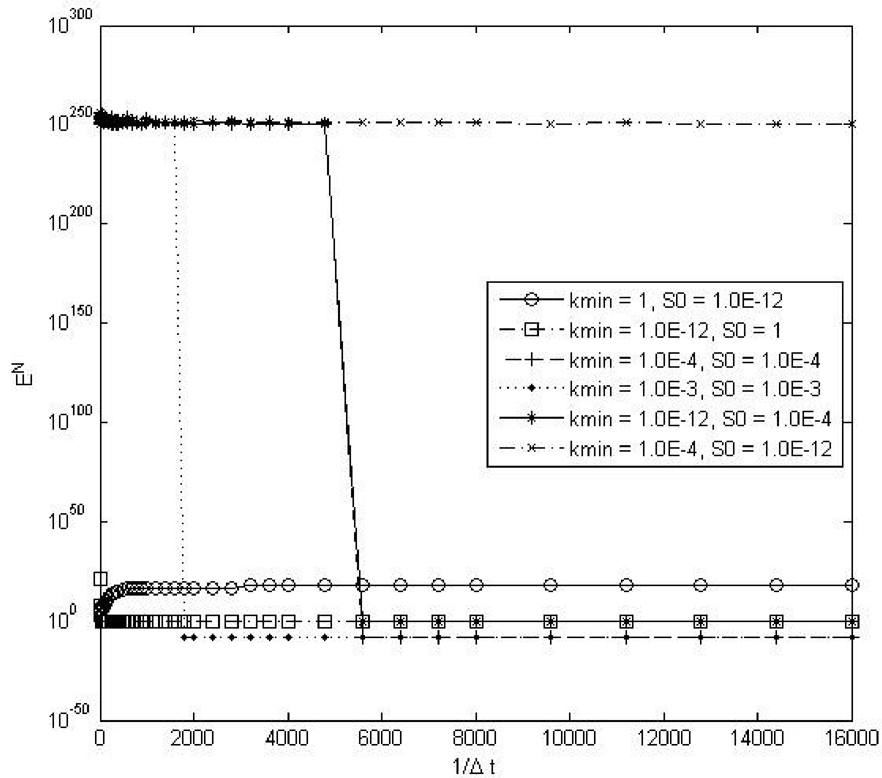

FIGURE 7. Stability of CNsplit at different small values of k_{min} and S_0

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