NUMERICAL ANALYSIS OF THE SEMIDISCRETE FINITE ELEMENT METHOD FOR COMPUTING THE NOISE GENERATED BY TURBULENT FLOWS

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ABSTRACT. This paper studies the convergence of the Finite Element Method for predicting the noise generated by turbulent flows through the Lighthill model. The model's derivation is given in an intuitive and physical manner. The model gives a non-homogeneous wave equation for the acoustic pressure where the right-hand side depends on the divergence of the nonlinear term of the NSE and external force. The stability, accuracy, convergence and implementation of the semidiscrete FEM scheme for a problem based on this model is studied. The rate of convergence depends on the FEM discretization for the wave equation and the $L^2(0,T;L^2(\Omega))$-error of the flow variables acting as the acoustic source. We also present numerical results that confirm our theoretical predictions.

1. INTRODUCTION

In this paper we study the semidiscrete Finite Element Method for computing the acoustic pressure of the sound generated by turbulent flows of a fluid. The physical model used is the Lighthill analogy [16], reviewed in Section 3. A rigorous analysis of the method's error is given in Theorem 2.

Prediction of the acoustic noise generated by a turbulent flow is a problem of fundamental importance in different fields of acoustics. Noise pollution in transport technologies such as jet airplanes and trains increases every year, e.g. [23]. The next generation fighter jets that are being designed are expected to produce 147 decibels of noise while 150 start to damage internal organs. Other important applications lie in submarine detection and medicine. Measuring characteristics of the sound coming from a blood flow in a valve of a heart would help diagnose heart murmurs.

The fundamental model of noise generated by turbulence is due to Lighthill [16]. Given the turbulent flow's velocity $\mathbf{u}$ and density ρ, the Lighthill's model for the small acoustic pressure fluctuations q is a wave equation driven by nonlinear term :

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta q = \nabla\cdot(\nabla\cdot(\rho_0\mathbf{u}\otimes\mathbf{u}) - \nabla\cdot\mathbb{S} - \rho_0\mathbf{f}), \tag{1.1}$$

with the *viscous* stress tensor $\mathbb{S}$ (the non-pressure part of the stress term), the sound speed $a_0 = \sqrt{\frac{\partial p}{\partial \rho}}|_{\rho=\rho_0}$, the external force $\mathbf{f}$ and the density ρ_0. Interestingly, to the order of accuracy of the approximation leading to (1.1), for small Mach

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numbers the noise can often be predicted by solving the incompressible Navier-Stokes equations (NSE) for $\mathbf{u}$ and inserting the incompressible velocity and density ρ_0 into the right-hand side (RHS) of (1.1) and then solving (1.1) for the acoustic pressure. Further, the last two terms on the RHS of (1.1) are negligible if $\nabla \cdot \mathbf{f} = 0$ and the Reynolds number is moderate. Thus, to this order of accuracy, at high Reynolds number, the RHS of (1.1) is often further simplified to $\nabla \cdot \nabla \cdot (\rho_0 \mathbf{u} \otimes \mathbf{u})$.

In this paper we consider the error in numerical simulation of acoustic noise based on this, so called, hybrid approach. It is assumed that the Mach number is small. In Section 3 for completeness we review the derivation of (1.1) which is called the Lighthill analogy. The whole acoustic domain of our (1.1) may be divided in two parts. These are the turbulent region Ω_1 with the flow where the generation of sound occurs and the far field Ω_2 where the acoustic waves propagate. In this paper, Ω_1 is surrounded by Ω_2. The whole domain is $\Omega = \Omega_1 \cup \Omega_2$. This is shown on figure 1.

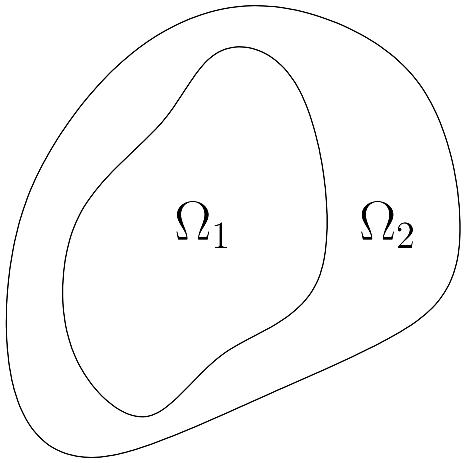

FIGURE 1. One domain inside the other

The semidiscrete finite element scheme will be presented for the following Initial Boundary Value Problem (IBVP):

$$
\begin{aligned}
&\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta q = f(t,x) + G(t,x)\ \forall (t,x) \in (0,T)\times\Omega, \\
&q(0,x) = q_1(x),\ \frac{\partial q}{\partial t}(0,x) = q_2(x)\ \forall x \in \Omega, \\
&\nabla q \cdot \mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g(t,x)\ \forall (t,x) \in (0,T)\times\partial\Omega,
\end{aligned}
\tag{1.2}
$$

where $f(t,x) = \nabla\cdot(\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \nabla\cdot\mathbb{S} - \rho\mathbf{f})$ inside Ω_1 and 0 around it in Ω_2, assumed velocity $\mathbf{u}$ is the solution of the incompressible NSE in Ω_1. The function $G(t,x)$ and $g(t,x)$ are control functions that we add according to the problem's physics and goals. The case $g \equiv 0$ in (1.2) gives the non-reflecting boundary conditions of the first order.

The basic FEM scheme for the wave equation with RHS known exactly and the same boundary conditions as in (1.2) and homogeneous Dirichlet boundary conditions was analyzed by Dupont [9]. With weaker assumptions on the regularity of the solution, Baker [1] presented an analysis for the wave equation with the homogeneous Dirichlet boundary conditions. Our analysis differs by the presence of the computational error in the RHS of the wave equation in (1.2). We show in Section 4 that the FEM formulation of the problem (1.2) has a stable solution for bounded time periods. In the same Section, we state and prove the main convergence theorem. The right hand side of the error estimate has one bounding term that involves the error in the turbulent flow that generates the acoustic noise. The rate of decrease for that error is obtained in Theorem 3 and requires additional regularity on $\mathbf{u}$. The two-dimensional numerical experiments in Section 6 provide the experimental rates of convergence for both the solution of the given IBVP and the divergence of the nonlinear term of the NSE and verify the theoretical predictions.

2. Notation and preliminaries

In this paper we assume that both Ω and Ω_1 are open bounded connected domains in $\mathbb{R}^n$, $n = 2, 3$, having smooth enough boundaries $\partial\Omega$ and $\partial\Omega_1$ respectively. $(\cdot,\cdot)$ and $\|\cdot\|$ without a subscript denote the $L^2(\Omega)$ or $L^2(\Omega_1)$ inner product and norm depending on which domain is considered at the moment. The norms $\|\cdot\|_{L^p(\Omega)}$ may be used for vector functions $\mathbf{u}$ with two or three components in a Banach space H. If $1 \leqslant p < \infty$, they should be understood as

$$\|\mathbf{u}\|_{L^p(\Omega)} = \left(\sum_{i=1}^{n} \|u_i\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}},$$

where u_i denotes i-th component of $\mathbf{u}$ and n is the number of components. The inner product should be understood as

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} (u_i, v_i).$$

$L^2(\partial\Omega)$ denotes the space of the real-valued square-integrable functions on the boundary $\partial\Omega$ of the domain Ω. The inner product in this space is denoted as $< \cdot,\cdot >$:

$$< u, v >= \int_{\partial\Omega} u \cdot v dS \text{ for } u, v \in L^2(\partial\Omega).$$

The norm induced by this inner product is denoted as $|\cdot|$:

$$|v| = \sqrt{< v, v >} \text{ for } v \in L^2(\partial\Omega).$$

For any integer $s \geqslant 0$ let $H^s(\Omega)$ denote a Sobolev space $W^{s,2}(\Omega)$ of real-valued functions on a domain Ω. The inner product and norm in the space $H^s(\Omega)$ are defined by

$$(u, v)_{H^s(\Omega)} = \sum_{|\alpha|=0}^{s} (\partial^\alpha u, \partial^\alpha v), \;\; \|u\|_{H^s(\Omega)} = \sqrt{(u, u)_{H^s(\Omega)}},$$

where α is a multiindex and $\partial^\alpha u$ denotes a weak partial derivative of the order $|\alpha|$ of the function u. Next, if B denotes a Banach space with norm $\|\cdot\|_B$ and

$u : [0,T] \to B$ is Lebesgue measurable, then we define

$$\|u\|_{L^p(0,T;B)} = \left(\int_0^T \|u\|_B^p dt \right)^{\frac{1}{p}}, \|u\|_{L^\infty(0,T;B)} = esssup_{0 \leqslant t \leqslant T} \|u(t,\cdot)\|_B,$$

and the space

$$L^p(0,T;B) = L^p(B) = \{u : [0,T] \to B | \|u\|_{L^p(0,T;B)} < \infty\} \text{ for } 1 \leqslant p \leqslant \infty$$

2.1. **Finite Element Space.** Let us build non-degenerate, edge-to-edge, shape regular mesh by introducing the partition $\Pi = \{T_1, T_2, ..., T_M\}$ of Ω into triangles. The characteristic size of the mesh $h < 1$ is defined by

$$h = max_{1 \leqslant i \leqslant M} diam(T_i).$$

Following [5], define

$$M^m(\Omega) = \{u \in L^2(\Omega) \mid u|_T \in P_{m-1} \ \forall T \in \Pi\} \text{ and } M_0^m(\Omega) = M^m(\Omega) \cap C^0(\Omega),$$

where P_m is the space of polynomials of degree no more than m and $C^0(\Omega)$ is the space of continuous on Ω functions. Therefore, by M_0^m we mean the space of continuous piecewise polynomials of degree no more than $m-1$.

From now on, C will denote a generic constant, not necessarily the same in two places. As in [9], we suppose there exist a positive constant C and integer $k \geqslant 2$ such that the spaces $M_0^m(\Omega)$ have the property that for $0 \leqslant s \leqslant 1$ and $2 \leqslant m \leqslant k$, and $V \in H^m(\Omega)$

$$inf_{\chi \in M_0^m(\Omega)} \|V - \chi\|_{H^s(\Omega)} \leqslant Ch^{m-s} \|V\|_{H^m(\Omega)}.$$

Following [9], we define the H^1-projection $\tilde{u} \in M_0^m(\Omega)$ for $u \in H^1(\Omega)$ by the formula :

$$a_0^2(\nabla u, \nabla u_h) + (u, u_h) = a_0^2(\nabla \tilde{u}, \nabla u_h) + (\tilde{u}, u_h) \ \forall u_h \in M_0^m(\Omega).$$

Below is the lemma that will be used in the proof of the main theorem about the error estimate.

Lemma 1. *(Dupont [9], Lemma 5) Let $u, \frac{\partial u}{\partial t} \in L^\infty(H^k(\Omega))$ and $\frac{\partial^2 u}{\partial t^2} \in L^2(H^k(\Omega))$ for some positive integer k, $m \geqslant k \geqslant 2$. Then for some positive constant C independent of h the error in the H^1-projection $\tilde{u}$ satisfies*

$$\left\| \frac{\partial^r (u - \tilde{u})}{\partial t^r} \right\|_{L^s(L^2(\Omega))} + \left\| \frac{\partial^r (u - \tilde{u})}{\partial t^r} \right\|_{L^s(H^{-\frac{1}{2}}(\partial\Omega))} \leqslant Ch^k,$$

where $s = \infty, \infty, 2$ for $r = 0, 1, 2$ respectively.

A mesh with above properties is called quasi-uniform, if there exist constants C_1 and C_2 independent of h, such that

$$C_1 \cdot diam(T_i) \leqslant diam(T_j) \leqslant C_2 \cdot diam(T_i)$$

for any distinct triangular elements T_i and T_j of the mesh.

If a mesh is quasi-uniform and functions v_h from the space $M_0^m(\Omega)$ built on this mesh satisfy the following regularity condition for non-negative integers l_1, l_2 and real numbers $p, q > 1$

$$v_h \in W^{l_1,p}(\Omega) \cap W^{l_2,q}(\Omega),$$

then the following inverse estimate holds (see [6]):

$$\|v_h\|_{W^{l_1,p}(\Omega)} \leqslant Ch^{l_2 - l_1 + min(0, \frac{n}{p} - \frac{n}{q})} \|v_h\|_{W^{l_2,q}(\Omega)}$$

for any $v_h \in M_0^m(\Omega)$ and some positive constant C independent on h.

For a given FEM space $M_0^m(\Omega)$, $m \geqslant 2$, consider the nodal basis consisting of functions ϕ_j. An arbitrary function $u \in H^m(\Omega)$ has a unique continuous representation on Ω and therefore we define a piecewise polynomial interpolant $I_h(u)$ for this function. If N_j denote the nodal points then

$$I_h(u) = \sum_j u(N_j)\phi_j.$$

In simulations of the incompressible NSE the FEM spaces for velocity X_h and pressure Q_h must satisfy the LBB-condition. It guarantees the stability of the approximate pressure. It is as follows :

$$inf_{q_h \in Q_h} sup_{v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|\nabla v_h\| \cdot \|q_h\|} \geqslant \beta_h > 0, \tag{2.1}$$

where β_h is bounded away from zero uniformly in h. More on the LBB-condition may be found in [15].

3. Lighthill analogy

To understand Lighthill's contribution, we consider first the derivation of the far-field acoustic equation. We start with the compressible NSE for density ρ, velocity $\mathbf{u}$ and pressure p :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{3.1}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nabla \cdot \mathbb{S} + \rho \mathbf{f}. \tag{3.2}$$

In the far field the external forces $\mathbf{f}$ and the viscous stress tensor $\mathbb{S}$ are typically negligible. Additionally we have a relation $p = P(\rho, s)$ where s denotes the entropy. The wave equation is the result of linearization of the equations with respect to the rest state which is characterized by constants $\mathbf{u}_0 = 0$, p_0, ρ_0, $\mathbf{f} = 0$:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \rho = \rho_0 + r, p = p_0 + q.$$

Next differentiate the linearized continuity equation with respect to time and take the divergence of the linearized momentum equation. Subtraction of the results leads to the equation

$$\frac{\partial^2 r}{\partial t^2} - \Delta q = 0.$$

Using the relation between pressure and density gives the homogeneous wave equation in the form

$$\frac{1}{a_0^2} \frac{\partial^2 q}{\partial t^2} - \Delta q = 0. \tag{3.3}$$

The above wave equation only holds in the far field in which the sound propagates. Coupling equations for the turbulent region and the fluctuations requires some efficient physical model. Lighthill's approach has erased the gap between the turbulent region and the far field in (1.1).

The derivation of the Lighthill analogy is presented below. See, e.g., [16] for extensions, alternate derivation and complementary work. Rewrite (3.2) in the divergence form assuming (3.1):

$$\frac{\partial(\rho\mathbf{u})}{\partial t}+\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})+\nabla p=\nabla\cdot\mathbb{S}+\rho\mathbf{f}. \tag{3.4}$$

Differentiate (3.1) with respect to time and apply divergence operator to (3.4) :

$$\begin{aligned}&\frac{\partial^2\rho}{\partial t^2}+\frac{\partial}{\partial t}\nabla\cdot(\rho\mathbf{u})=0,\\&\frac{\partial}{\partial t}\nabla\cdot(\rho\mathbf{u})+\nabla\cdot\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})+\Delta p=\nabla\cdot\nabla\cdot\mathbb{S}+\nabla\cdot\rho\mathbf{f}.\end{aligned}$$

Subtraction of these two equations gives the following holding in Ω:

$$-\Delta p=\nabla\cdot(\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})-\nabla\cdot\mathbb{S}-\rho\mathbf{f})-\frac{\partial^2\rho}{\partial t^2}. \tag{3.5}$$

Consider the far field where the perturbations of the pressure and density are defined with respect to the rest state. Then (3.5) is mathematically equivalent to

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2}-\Delta p=\nabla\cdot(\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})-\nabla\cdot\mathbb{S}-\rho\mathbf{f})+\frac{\partial^2}{\partial t^2}(\frac{q}{a_0^2}-\rho). \tag{3.6}$$

We choose a_0 to be the speed of sound in the medium at rest state. Equation (3.6) may already be called Lighthill's analogy. Now some considerations must be made. First, in the far field the last term $\frac{\partial^2}{\partial t^2}(\frac{q}{a_0^2}-\rho)=\frac{\partial^2}{\partial t^2}(\frac{q}{a_0^2}-r)$ is negligible because it is simply the LHS of the classical wave equation in the quiescent state (see [14] for details). Moreover, in this medium the first term on the RHS is also negligible because it consists of the nonlinear term and two linear terms that make no significant influence on the sound propagation in the far field. Therefore, in the far field equation (3.6) reduces to the wave equation (3.3) for the acoustic pressure. Lighthill's model extends equation (3.6) to the whole fluid including the turbulent region. Suppose that perturbations of the pressure and density are defined on the whole Ω and the last term on the RHS of (3.6) is negligible on Ω. These two suppositions together give a picture of the whole aerodynamical system as a field of wave propagation with the divergence term playing a role of a sound source.

Definition 1. $\mathbb{T}=\rho\mathbf{u}\otimes\mathbf{u}-\mathbb{S}$ *is called the Lighthill tensor.*

The Lighthill tensor is not negligble in the turbulent region and is negligible in laminar regions including the far field. The whole system is described by the following equation :

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2}-\Delta q=\nabla\cdot(\nabla\cdot\mathbb{T}-\rho\mathbf{f}). \tag{3.7}$$

This model of sound generated by turbulence allows breaking this problem in two subproblems. In the turbulent region we can use methods applicable for solving incompressible NSE and this will provide us with tensor $\mathbb{T}$. Knowing the RHS of the equation (3.7) we can solve the non-homogeneous hyperbolic problem for the whole domain. In the far field we set the RHS to zero.

In fact, for relatively small Mach numbers the compressibility of the flow has negligible impact on the sound generation (see, e.g., [23]). The fluctuations of the density $r=\rho-\rho_0$ in the RHS of (1.1) are the terms of high order and may be neglected. Thus we consider the coupled problem of (3.7) holding in Ω and

$$\begin{aligned} &\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p = \nabla \cdot \mathbb{S} + \rho \mathbf{f}, \\ &\nabla \cdot \mathbf{u} = 0, \end{aligned} \tag{3.8}$$

holding in Ω_1. The boundary conditions for (3.8) depend on a certain application.

Lemma 2. *If $\rho \equiv \rho_0$ and $\nabla \cdot \mathbf{u} = 0$, then $\nabla \cdot \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho_0 \nabla \mathbf{u} : \nabla \mathbf{u}^t$.*

Proof. Since ρ is constant,

$$\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho_0 \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \rho_0 (u_i u_j)_{,i} = \rho_0 (u_{i,i} u_j + u_i u_{j,i}),$$

where u_i denotes the i-th component of the vector $\mathbf{u}$ and repeating index means summation. Due to the incompressibility condition, the last expression equals

$$\rho_0 (u_{i,i} u_j + u_i u_{j,i}) = \rho_0 u_i u_{j,i} = \rho_0 \mathbf{u} \cdot \nabla \mathbf{u}.$$

Finally,

$$\nabla \cdot (\rho_0 \mathbf{u} \cdot \nabla \mathbf{u}) = \rho_0 \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = \rho_0 (u_i u_{j,i})_{,j} = \rho_0 (u_{i,j} u_{j,i} + u_i u_{j,i,j}) = \rho_0 u_{i,j} u_{j,i}.$$

The last term is precisely $\rho_0 \nabla \mathbf{u} : \nabla \mathbf{u}^t$. □

Lemma 3. *If $\nabla \cdot \mathbf{u} = 0$, then $\nabla \cdot \nabla \cdot \mathbb{S}(\mathbf{u}) = 0$.*

Proof. Let $\mu > 0$ be the shear viscosity coefficient of the fluid. Since in incompressible flows

$$\nabla \cdot \mathbb{S}(\mathbf{u}) = \mu \Delta \mathbf{u},$$

then

$$\nabla \cdot \nabla \cdot \mathbb{S}(\mathbf{u}) = \mu \nabla \cdot \Delta \mathbf{u}.$$

Consider

$$\nabla \cdot \Delta \mathbf{u} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\Delta u_i) = \sum_{i=0}^{3} \frac{\partial^2}{\partial x_i^2} (\nabla \cdot \mathbf{u}) = 0.$$

□

The last two lemmas allow us to rewrite the RHS of the Lighthill analogy in the form

$$\rho_0 \cdot (\nabla \mathbf{u} : \nabla \mathbf{u}^t - \nabla \cdot \mathbf{f}).$$

4. Finite element scheme

Definition 2. *Define*

$$Q(\mathbf{u}, \mathbf{v}) = \rho_0 \nabla \mathbf{u} : \nabla \mathbf{v}^t.$$

In fluid mechanics the term $\rho_0 \nabla \mathbf{u} : \nabla \mathbf{u}^t$ is called Q. The $Q > 0$ is used for eduction of persistent coherent vortices. It is interesting that this same quantity occurs in the RHS of (1.1) as the dominant sound source in its generation by turbulent flows.

Consider the following initial boundary-value problem

$$
\begin{aligned}
&\frac{\partial^2 q}{\partial t^2} - a_0^2 \Delta q = a_0^2 (Q(\mathbf{u},\mathbf{u}) - \rho_0 \nabla \cdot \mathbf{f}) + G(t,x)\ \forall (t,x) \in (0,T) \times \Omega_1, \\
&\frac{\partial^2 q}{\partial t^2} - a_0^2 \Delta q = 0\ \forall (t,x) \in (0,T) \times \Omega/\Omega_1, \\
&q(0,x) = q_1(x),\ \frac{\partial q}{\partial t}(0,x) = q_2(x)\ \forall x \in \Omega, \\
&\nabla q \cdot \mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g(t,x)\ \forall (t,x) \in (0,T) \times \partial\Omega,
\end{aligned} \tag{4.1}
$$

where all functions on the RHS are known and $\mathbf{n}$ being the outward normal on the boundary $\partial\Omega$. The case $G \equiv 0$ refers to the turbulent flow being the only source of the sound. The question of proper boundary conditions depends on the physical problem. $g(t,x) \equiv 0$ gives the case of the first-order non-reflecting boundary conditions. Although more accurate non-reflecting boundary conditions are known, those in (4.1) with $g(t,x) \equiv 0$ are the first step in applications where the interest lies in the sound waves that propagate in infinite space without reflection. This allows a simulation to measure acoustic power of the waves generated solely by the turbulent flow. The non-zero boundary control function $g(t,x)$ may be used if we want to consider additional sources of sound on the boundary. Adding $g(t,x)$ to the right-hand side of the boundary condition has no effect on the error estimates.

In computations, $Q(\mathbf{u},\mathbf{u})$ is given approximately due to two reasons. First, Q consists of the solution of the incompressible NSE. Second, the solution of the incompressible NSE is found via computations and thus contains error which follows from inaccuracy of the scheme used. Let h_1 denote the mesh size of this scheme. The modeling error due to incompressibity is analyzed in [18]. The second one is of computational importance and is analyzed here. Thus we have

$$
\begin{aligned}
&\frac{\partial^2 q}{\partial t^2} - a_0^2 \Delta q = a_0^2 (Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) - \rho_0 \nabla \cdot \mathbf{f}) + G(t,x)\ \forall (t,x) \in (0,T) \times \Omega_1, \\
&\frac{\partial^2 q}{\partial t^2} - a_0^2 \Delta q = 0\ \forall (t,x) \in (0,T) \times \Omega/\Omega_1, \\
&q(0,x) = q_1(x),\ \frac{\partial q}{\partial t}(0,x) = q_2(x)\ \forall x \in \Omega, \\
&\nabla q \cdot \mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g(t,x)\ \forall (t,x) \in (0,T) \times \partial\Omega,
\end{aligned} \tag{4.2}
$$

The total error between the exact solution q of (4.1) and the approximate q_h will consist of the FEM error caused by computations and the perturbation of the RHS caused by replacing $Q(\mathbf{u},\mathbf{u}) - \rho_0\nabla \cdot \mathbf{f}$ with $Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) - \rho_0\nabla \cdot \mathbf{f}$.

The variational formulation is as follows. Assume that

$$
Q(\mathbf{u},\mathbf{u}) - \rho_0 \nabla \cdot \mathbf{f} + \frac{1}{a_0^2} G \in L^2(0,T;L^2(\Omega_1)), q(0,\cdot) \in H^1(\Omega),
$$

$$
\frac{\partial q}{\partial t}(0,\cdot) \in L^2(\Omega), g \in L^2(0,T;L^2(\partial\Omega)).
$$

Find $q \in L^2(0,T;H^1(\Omega))$ such that $\frac{\partial q}{\partial t} \in L^2(0,T;H^1(\Omega))$, $\frac{\partial^2 q}{\partial t^2} \in L^2(0,T;L^2(\Omega))$ and

(4.3)
$$\left(\frac{\partial^2 q}{\partial t^2}, v\right) + a_0^2\,(\nabla q, \nabla v) + a_0 \left\langle \frac{\partial q}{\partial t}, v \right\rangle =$$
$$= a_0^2 \left(Q(\mathbf{u},\mathbf{u}) - \rho_0 \nabla\cdot\mathbf{f} + \frac{1}{a_0^2} G, v\right)_{\Omega_1} + a_0^2 < g, v >$$
$$\forall v \in H^1(\Omega), 0 < t < T,$$

$$(q(0,\cdot), v) = (q_1(\cdot), v)\ \forall v \in H^1(\Omega), \tag{4.4}$$

$$\left(\frac{\partial q}{\partial t}(0,\cdot), v\right) = (q_2(\cdot), v)\ \forall v \in H^1(\Omega). \tag{4.5}$$

The condition that $Q(\mathbf{u},\mathbf{u}) \in L^2(0,T;L^2(\Omega_1))$ is satisfied if we impose the following regularity condition for $\mathbf{u}$:

$$\mathbf{u} \in L^4(0,T;W^{1,4}(\Omega_1)).$$

This fact easily follows from Holder's inequality.

The finite element approximation will be based on finite-dimensional spaces $\{M_0^m(\Omega)\} \subset H^1(\Omega)$ of continuous piecewise polynomials of degree no more than $m-1$, section 2. It is as follows. Assume that

$$Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - \rho_0 \nabla\cdot\mathbf{f} + \frac{1}{a_0^2} G \in L^2(0,T;L^2(\Omega_1)),\ g \in L^2(0,T;L^2(\partial\Omega)).$$

Find such twice differentiable map $q_h : [0,T] \to M_0^m(\Omega)$ that

(4.6)
$$\left(\frac{\partial^2 q_h}{\partial t^2}, v_h\right) + a_0^2\,(\nabla q_h, \nabla v_h) + a_0 \left\langle \frac{\partial q_h}{\partial t}, v_h \right\rangle =$$
$$= a_0^2 \left(Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) - \rho_0 \nabla\cdot\mathbf{f} + \frac{1}{a_0^2} G, v_h\right)_{\Omega_1} + a_0^2 < g, v_h >$$
$$\forall v_h \in M_0^m(\Omega), 0 < t < T,$$
$$q_h(0,\cdot) \text{ approximates } q_1 \text{ well},$$
$$\frac{\partial q_h}{\partial t}(0,\cdot) \text{ approximates } q_2 \text{ well}.$$

The regularity condition $Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) \in L^2(0,T;L^2(\Omega_1))$ is handled by the following lemma.

Lemma 4. *Suppose the exact velocity* $\mathbf{u}$ *satisfies condition*

$$\mathbf{u} \in L^4(0,T;H^2(\Omega_1)) \cap L^4(0,T;W^{1,4}(\Omega_1))$$

and the mesh used for computing $\mathbf{u}_{h_1}$ *in* Ω_1 *is quasi-uniform. Finally, let* $\|\mathbf{u} - \mathbf{u}_{h_1}\|_{L^4(L^2(\Omega_1))}$ *converge to zero no slower than* $O(h_1^{1+\frac{n}{4}})$*, where* $n = 2$ *or* 3 *is the dimension of the physical space. Then*

$$\mathbf{u}_{h_1} \in L^4(0,T;W^{1,4}(\Omega_1)),$$

and thus $Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) \in L^2(0,T;L^2(\Omega_1))$.

Proof. Due to the triangle inequality, it is obvious that

$$\|\mathbf{u}_{h_1}\|_{L^4(W^{1,4}(\Omega_1))} \leqslant \|\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} + \|\mathbf{u}-I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} + \|\mathbf{u}_{h_1}-I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))}.$$

Here I_{h_1} is a piecewise polynomial interpolant, section 2. The first term on the RHS is bounded due to the assumption of the lemma. The second term may be bounded as shown (see, for example, [6]):

$$\|\mathbf{u} - I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} \leqslant C\|\nabla\mathbf{u}\|_{L^4(L^4(\Omega_1))}.$$

To bound the third term, we need to use the inverse estimate in the following manner :

$$\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} \leqslant Ch_1^{-1-\frac{n}{4}}\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\|_{L^4(L^2(\Omega_1))}$$

The final step is to use the triangle inequality

$$\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\|_{L^4(W^{1,4}(\Omega_1))} \leqslant Ch_1^{-1-\frac{n}{4}}\left(\|\mathbf{u} - I_{h_1}\mathbf{u}\|_{L^4(L^2(\Omega_1))} + \|\mathbf{u} - \mathbf{u}_{h_1}\|_{L^4(L^2(\Omega_1))}\right)$$

The first term on the RHS may be bounded by $Ch^{1-\frac{n}{4}}\|\nabla\nabla\mathbf{u}\|_{L^4(L^2(\Omega_1))}$. The assumption on the speed of convergence of $\|\mathbf{u}-\mathbf{u}_{h_1}\|_{L^4(L^2(\Omega_1))}$ finishes the proof. □

Theorem 1. *The solution q_h of* (4.6) *is stable and the following inequality holds :*

$$\left\|\frac{\partial q_h}{\partial t}\right\| + a_0\|\nabla q_h\| \leqslant C\left(\left\|Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) - \rho_0\nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G\right\|_{L^2(L^2(\Omega_1))} + \|g\|_{L^2(L^2(\partial\Omega))} + \right.$$
$$\left. + \left\|\frac{\partial q_h}{\partial t}(0,\cdot)\right\| + \|\nabla q_h(0,\cdot)\|\right)$$

with positive constant $C = C(t)$.

Proof. Set $v_h = \frac{\partial q_h}{\partial t}$. Then

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{\partial q_h}{\partial t}\right\|^2 + a_0^2\,\|\nabla q_h\|^2\right) + a_0\left|\frac{\partial q_h}{\partial t}\right|^2 =$$
$$= a_0^2\left(Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) - \rho_0\nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G, \frac{\partial q_h}{\partial t}\right)_{\Omega_1} + a_0^2\left\langle g, \frac{\partial q_h}{\partial t}\right\rangle,$$

$$\frac{d}{dt}\left(\left\|\frac{\partial q_h}{\partial t}\right\|^2 + a_0^2\|\nabla q_h\|^2\right) \leqslant a_0^4\left\|Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) - \rho_0\nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G\right\|^2 + \left\|\frac{\partial q_h}{\partial t}\right\|^2 + \frac{a_0^3}{2}|g|^2,$$

$$\left\|\frac{\partial q_h}{\partial t}\right\|^2 + a_0^2\|\nabla q_h\|^2 \leqslant \int_0^t\left(a_0^4\left\|Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) - \rho_0\nabla\cdot\mathbf{f} + \frac{1}{a_0^2}G\right\|^2 + \right.$$
$$\left. + \left\|\frac{\partial q_h}{\partial t}\right\|^2 + \frac{a_0^3}{2}|g|^2\right)d\tau + \left\|\frac{\partial q_h}{\partial t}(0,\cdot)\right\|^2 + a_0^2\|\nabla q_h(0,\cdot)\|^2.$$

Applying Gronwall's lemma to the inequality above finishes the proof. □

Remark 1. *The function $C(t)$ from the theorem may grow exponentially fast. This fact may be related to the phenomena of resonance which is common for hyperbolic problems.*

Consider the H^1-projection $\tilde{q} \in L^2(0,T;M_0^m(\Omega))$ of the solution of (4.3)-(4.5) given by the formula :

$$a_0^2(\nabla q, \nabla v_h) + (q, v_h) = a_0^2(\nabla\tilde{q}, \nabla v_h) + (\tilde{q}, v_h)\ \forall v_h \in M_0^m(\Omega). \tag{4.7}$$

Theorem 2. *Let the solution q of* (4.3) *satisfy the conditions : q, $\frac{\partial q}{\partial t} \in L^\infty(H^k(\Omega))$ and $\frac{\partial^2 q}{\partial t^2} \in L^2(H^k(\Omega))$ for some positive integer k, $m \geqslant k \geqslant 2$. If the initial conditions are taken so that*

$$\|(q_h - \tilde{q})(0,\cdot)\|_{H^1(\Omega)} + \left\| \frac{\partial}{\partial t}(q_h - \tilde{q})(0,\cdot) \right\| \leqslant C_1 h^k$$

with some posititve constant C_1 independent of h, then the solution of (4.6) *satisfies:*

$$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \left\| \frac{\partial}{\partial t}(q - q_h) \right\|_{L^\infty(L^2(\Omega))} \leqslant$$
$$\leqslant C\left(h^k + \|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))}\right)$$

with some constant $C > 0$ independent of h.

Proof. Denote $Q(\mathbf{u},\mathbf{u}) = Q$ and $Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1}) = Q_{h_1}$ for simplicity. Consider $\eta = \tilde{q} - q$, $\psi = q_h - \tilde{q}$ and $e = q - q_h$. The proof of the error estimates in the beginning is similar to that in [9] except there appeares the term with $Q_{h_1} - Q$. Start with the variational formulation (4.3) using the definition of the H^1-projection:

$$\left(\frac{\partial^2 \tilde{q}}{\partial t^2}, v_h\right) + a_0^2(\nabla\tilde{q},\nabla v_h) + a_0\left\langle \frac{\partial \tilde{q}}{\partial t}, v_h \right\rangle =$$
$$= \left(a_0^2 Q + G - \eta + \frac{\partial^2 \eta}{\partial t^2}, v_h\right)_{\Omega_1} + a_0\left\langle \frac{\partial \eta}{\partial t}, v_h\right\rangle + a_0^2 < g, v_h > .$$

Using (4.6) and the previous result, we obtain :

$$\left(\frac{\partial^2 \psi}{\partial t^2}, v_h\right) + a_0^2(\nabla\psi, \nabla v_h) + a_0\left\langle \frac{\partial \psi}{\partial t}, v_h \right\rangle =$$
$$= a_0^2(Q_{h_1} - Q, v_h)_{\Omega_1} + \left(\eta - \frac{\partial^2 \eta}{\partial t^2}, v_h\right) - a_0\left\langle \frac{\partial \eta}{\partial t}, v_h\right\rangle.$$

Choose $v_h = \frac{\partial \psi}{\partial t}$. This yields :

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{\partial \psi}{\partial t}\right\|^2 + a_0^2\|\nabla\psi\|^2\right) + a_0\left\langle \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial t}\right\rangle =$$
$$= a_0^2(Q_{h_1} - Q, \frac{\partial \psi}{\partial t})_{\Omega_1} + \left(\eta - \frac{\partial^2 \eta}{\partial t^2}, \frac{\partial \psi}{\partial t}\right) - a_0\left\langle \frac{\partial \eta}{\partial t}, \frac{\partial \psi}{\partial t}\right\rangle.$$

Adding

$$\frac{1}{2}\frac{d}{dt}\|\psi\|^2 \leqslant \frac{1}{2}\left(\left\|\frac{\partial \psi}{\partial t}\right\|^2 + \|\psi\|^2\right)$$

to the previous equation results in

$$\frac{d}{dt}\left(\left\|\frac{\partial \psi}{\partial t}\right\|^2 + \|\psi\|^2 + a_0^2\|\nabla\psi\|^2\right) + 2\left|\sqrt{a_0}\cdot\frac{\partial \psi}{\partial t}\right|^2 \leqslant$$
$$C\left(\left\|\frac{\partial \psi}{\partial t}\right\|^2 + \|\psi\|^2 + \|\eta\|^2 + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2\right) - 2a_0\left\langle \frac{\partial \eta}{\partial t}, \frac{\partial \psi}{\partial t}\right\rangle + a_0^2\|Q_{h_1} - Q\|^2$$

with some positive constant C. Note that

$$\int_0^t \left\langle \frac{\partial \eta}{\partial t}, \frac{\partial \psi}{\partial t}\right\rangle d\tau = \left\langle \frac{\partial \eta}{\partial t}, \psi\right\rangle(t) - \left\langle \frac{\partial \eta}{\partial t}, \psi\right\rangle(0) - \int_0^t \left\langle \frac{\partial^2 \eta}{\partial t^2}, \psi\right\rangle d\tau.$$

Following [9], we shall estimate the boundary term on the time level t as shown below

$$\left|-2a_0\int_0^t\left\langle\frac{\partial\eta}{\partial t},\frac{\partial\psi}{\partial t}\right\rangle d\tau\right|\leqslant C\left(\left\|\frac{\partial\eta}{\partial t}\right\|_{H^{-\frac{1}{2}}(\partial\Omega)}\cdot\|\psi\|_{H^1(\Omega)}+\right.$$
$$\left.+\left\|\frac{\partial\eta}{\partial t}(0,\cdot)\right\|_{H^{-\frac{1}{2}}(\partial\Omega)}\cdot\|\psi(0,\cdot)\|_{H^1(\Omega)}+\int_0^t\left\|\frac{\partial^2\eta}{\partial t^2}\right\|_{H^{-\frac{1}{2}}(\partial\Omega)}\cdot\|\psi\|_{H^1(\Omega)}\right).$$

The last expression may be bounded by

$$\epsilon\|\psi\|^2_{H^1(\Omega)}+$$
$$+C\left(\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))}+\|\psi(0,\cdot)\|^2_{H^1(\Omega)}+\int_0^t\|\psi\|^2_{H^1(\Omega)}d\tau+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}\right)$$

with $\epsilon>0$ of our choice. Here $C=C(\epsilon)$. Integration gives

$$\left\|\frac{\partial\psi}{\partial t}\right\|^2+\|\psi\|^2+a_0^2\|\nabla\psi\|^2+2\int_0^t\left|\sqrt{a}\cdot\frac{\partial\psi}{\partial t}\right|^2\leqslant$$
$$C\int_0^t\left(\left\|\frac{\partial\psi}{\partial t}\right\|^2+\|\psi\|^2+\|\eta\|^2+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2\right)d\tau+\epsilon\|\psi\|^2_{H^1(\Omega)}+$$
$$C\left(\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))}+\|\psi(0,\cdot)\|^2_{H^1(\Omega)}+\int_0^t\|\psi\|^2_{H^1(\Omega)}d\tau+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}\right)$$
$$+a_0^2\int_0^t\|Q_{h_1}-Q\|^2d\tau+\left\|\frac{\partial\psi}{\partial t}(0,\cdot)\right\|^2+\|\psi(0,\cdot)\|^2+a_0^2\|\nabla\psi(0,\cdot)\|^2,$$

or

$$\left\|\frac{\partial\psi}{\partial t}\right\|^2+\|\psi\|^2_{H^1(\Omega)}+\int_0^t\left|\sqrt{a}\cdot\frac{\partial\psi}{\partial t}\right|^2\leqslant$$
$$C\left[\int_0^t\left(\left\|\frac{\partial\psi}{\partial t}\right\|^2+\|\psi\|^2_{H^1(\Omega)}\right)d\tau+\|\eta\|^2_{L^2(L^2(\Omega))}+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}\right.$$
$$\left.+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))}+\left\|\frac{\partial\psi}{\partial t}(0,\cdot)\right\|^2+\|\psi(0,\cdot)\|^2_{H^1(\Omega)}+\int_0^t\|Q_{h_1}-Q\|^2d\tau\right]$$

with some positive constant C. Apply Gronwall's lemma to yield

$$\left\|\frac{\partial\psi}{\partial t}\right\|^2_{L^\infty(L^2(\Omega))}+\|\psi\|^2_{L^\infty(H^1(\Omega))}+\left\|\sqrt{a}\cdot\frac{\partial\psi}{\partial t}\right\|^2_{L^2(L^2(\partial\Omega))}\leqslant$$
$$C\left[\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))}+\|\eta\|^2_{L^2(L^2(\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))}\right.$$
$$\left.+\left\|\frac{\partial\psi}{\partial t}(0,\cdot)\right\|^2+\|\psi(0,\cdot)\|^2_{H^1(\Omega)}+\int_0^t\|Q_{h_1}-Q\|^2d\tau\right],$$

where $C = C(T)$ grows exponentially fast. Next we can use Lemma 1, i.e. for some constant C independent of h

$$\left\| \frac{\partial^r \eta}{\partial t^r} \right\|_{L^s(L^2(\Omega))} + \left\| \frac{\partial^r \eta}{\partial t^r} \right\|_{L^s(H^{-\frac{1}{2}}(\partial\Omega))} \leqslant Ch^k,$$

where $s = \infty, \infty, 2$ for $r = 0, 1, 2$ respectively. If $q_h(0, \cdot)$, $\frac{\partial q_h}{\partial t}(0, \cdot)$ are taken so that $\|(q_h - \tilde{q})(0, \cdot)\|_{H^1(\Omega)} + \left\|\frac{\partial}{\partial t}(q_h - \tilde{q})(0, \cdot)\right\| \leqslant C_1 h^k$, where C_1 is independent of h, then there is a constant C independent of h such that

$$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \left\| \frac{\partial}{\partial t}(q - q_h) \right\|_{L^\infty(L^2(\Omega))} \leqslant C \left(h^k + \|Q_{h_1} - Q\|_{L^2(L^2(\Omega_1))} \right).$$

□

Now the estimate for $Q(\mathbf{u}, \mathbf{u}) - Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1})$ must be found. Here we deal with another finite element scheme of the mesh size h_1 used for computing velocity field $\mathbf{u}_{h_1}$ of the turbulent flow in the inner domain Ω_1. Let X_{h_1} and Q_{h_1} denote the finite element spaces satisfying the LBB-condition (2.1).

Theorem 3. *Suppose the solution* $\mathbf{u}$ *of the incompressible NSE in* Ω_1 *satisfies the following regularity condition :*

$$\mathbf{u} \in L^\infty(0, T; H^2(\Omega_1)) \cap L^\infty(0, T; W^{1,4}(\Omega_1)) \cap L^2(0, T; W^{m,4}(\Omega_1))$$

for some integer $m \geqslant 2$. *In addition, assume that the mesh which is used for computing* $\mathbf{u}_{h_1}$, *is quasi-uniform. If the approximating space* $M_0^k(\Omega_1)$ *is used for computing velocity* $\mathbf{u}$ *with* $k \geqslant m$ *and the error* $\|\mathbf{u} - \mathbf{u}_{h_1}\|_{L^\infty(L^2(\Omega_1))}$ *converges to zero no slower than* $O(h_1^{1+\frac{n}{4}})$, *where* $n = 2$ *or* 3 *is the dimension of the physical space, then there exists such positive constant* C *independent of* h_1 *that*

$$\|Q(\mathbf{u}, \mathbf{u}) - Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))} \leqslant$$
$$Ch_1^{-\frac{n}{4}} \cdot \left(h_1^{m-1} \|\partial^m \mathbf{u}\|_{L^2(L^4(\Omega_1))} + \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))} \right).$$

Proof. It's easy to see that

$$Q(\mathbf{u}, \mathbf{u}) - Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) = \rho_0 \cdot (\nabla \mathbf{u} : \nabla \mathbf{u}^t - \nabla \mathbf{u}_{h_1} : \nabla \mathbf{u}_{h_1}^t) =$$
$$= \rho_0 \cdot (\nabla \mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t) + \rho_0 \cdot (\nabla(\mathbf{u} - \mathbf{u}_{h_1}) : \nabla \mathbf{u}_{h_1}^t).$$

Bound both terms separetely. For the L^2-norm of the first one we obtain

$$\|\rho_0 \cdot (\nabla \mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t)\|^2 \leqslant C \int_\Omega |\nabla \mathbf{u}|^2 |\nabla(\mathbf{u} - \mathbf{u}_{h_1})|^2$$

for some suitable positive constant C. By Holder's inequality

$$C \int_{\Omega_1} |\nabla \mathbf{u}|^2 |\nabla(\mathbf{u} - \mathbf{u}_{h_1})|^2 \leqslant C \|\nabla \mathbf{u}\|^2_{L^4(\Omega_1)} \cdot \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|^2_{L^4(\Omega_1)}.$$

Consider the continuous piecewise polynomial interpolant $I_{h_1}(\mathbf{u})$ for $\mathbf{u}$ of order $s \geqslant m - 1$. Obviously,

$$\nabla(\mathbf{u} - \mathbf{u}_{h_1}) = \nabla(\mathbf{u} - I_{h_1}(\mathbf{u})) + \nabla(I_{h_1}(\mathbf{u}) - \mathbf{u}_{h_1}).$$

Hence,

$$\|\rho_0 \cdot (\nabla \mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t)\| \leqslant$$
$$C \|\nabla \mathbf{u}\|_{L^4(\Omega_1)} \left(\|\nabla(\mathbf{u} - I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla(I_{h_1}(\mathbf{u}) - \mathbf{u}_{h_1})\|_{L^4(\Omega_1)} \right).$$

In the same manner,

$$\|\rho_0 \cdot (\nabla(\mathbf{u}-\mathbf{u}_{h_1}) : \nabla \mathbf{u}^t_{h_1})\| \leqslant$$
$$C\|\nabla \mathbf{u}_{h_1}\|_{L^4(\Omega_1)} \left(\|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla(I_{h_1}(\mathbf{u})-\mathbf{u}_{h_1})\|_{L^4(\Omega_1)}\right).$$

To bound the term $\|\nabla \mathbf{u}_{h_1}\|_{L^4(\Omega_1)}$ use the triangle inequality :

$$\|\nabla \mathbf{u}_{h_1}\|_{L^4(\Omega_1)} \leqslant \|\nabla(\mathbf{u}_{h_1}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla \mathbf{u}\|_{L^4(\Omega_1)}.$$

Using the inverse estimate, we obtain

$$\|\nabla \mathbf{u}_{h_1}\|_{L^4(\Omega_1)} \leqslant C h_1^{-1-\frac{n}{4}} \|\mathbf{u}_{h_1}-I_{h_1}(\mathbf{u})\| + \|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + \|\nabla \mathbf{u}\|_{L^4(\Omega_1)}.$$

The last two terms are bounded uniformly in time due to the regularity assumption of the theorem. For the first term on the RHS apply the triangle inequality as shown below :

$$C h_1^{-1-\frac{n}{4}} \|\mathbf{u}_{h_1}-I_{h_1}(\mathbf{u})\| \leqslant C h_1^{-1-\frac{n}{4}} \left(\|\mathbf{u}-I_{h_1}(\mathbf{u})\| + \|\mathbf{u}_{h_1}-\mathbf{u}\|\right).$$

Both terms on the RHS are bounded uniformly in time, which follows from the assumption on the regularity and the speed of convergence. We obtain

$$\|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\| \leqslant C(\|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} + C_1 h_1^{-\frac{n}{4}} \|\nabla(I_{h_1}(\mathbf{u})-\mathbf{u}_{h_1})\|),$$

where $C = C(\mathbf{u})$ is a function of $\mathbf{u}$ independent of h_1. Further,

$$\|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\|_{L^4(\Omega_1)} \leqslant C_1 h_1^{m-1} \|\partial^m \mathbf{u}\|_{L^4(\Omega_1)}.$$

Next

$$\|\nabla(I_{h_1}(\mathbf{u})-\mathbf{u}_{h_1})\| \leqslant \|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\| + \|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|,$$
$$\|\nabla(\mathbf{u}-I_{h_1}(\mathbf{u}))\| \leqslant C h_1^{m-1} \|\partial^m \mathbf{u}\| \leqslant C_1 h_1^{m-1} \|\partial^m \mathbf{u}\|_{L^4(\Omega_1)}.$$

So finally

$$\|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\| \leqslant$$
$$C(h_1^{m-1-\frac{n}{4}} \|\partial^m \mathbf{u}\|_{L^4(\Omega_1)} + h_1^{-\frac{n}{4}} \|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|).$$

Since we're interested in $L^2(L^2)$-norm of $Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})$, we square and integrate the last inequality :

$$\int_0^t \|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\|^2 d\tau \leqslant$$
$$C h_1^{-\frac{n}{2}} \cdot \int_0^t (h_1^{2m-2} \|\partial^m \mathbf{u}\|^2_{L^4(\Omega_1)} + \|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|^2) d\tau.$$

The statement of the theorem follows after extracting the square root of both sides of the last inequality. □

Remark 2. *The term* $\|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))}$ *may be bounded by* Ch_1^p *with some positive integer p for the no-slip boundary condition '*$\mathbf{u} = 0$ *on* $\partial\Omega_1$*', depending on which FEM space is used to solve the incompressible NSE in* Ω_1*. For example, for the space of MINI-element* $p = 1$ *and for Taylor-Hood element* $p = 2$ *(see* [15] *or* [5] *for details).*

5. Open problems

Open problems in the error analysis of the Lighthill model include the analysis of the error in negative Sobolev norms. Improving the estimate of the rate of convergence of $Q(\mathbf{u}, \mathbf{u}) - Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1})$ is an open question since the $L^2(L^2(\Omega_1))$-norm of it requires regularity condition $\|\nabla \mathbf{u}\|_{L^4(\Omega_1)} < \infty$. Negative norms give a hope of decreasing the regularity needed for the error estimate of $\|Q(\mathbf{u}, \mathbf{u}) - Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1})\|_{L^2(H^{-k}(\Omega_1))}$ to hold.

Another promising approach is based on the observation that according to (3.5), the RHS of the Lighthill analogy may be expressed in terms of the fluid's pressure and density :

$$\nabla \cdot (\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \mathbb{S} - \rho \mathbf{f}) = \frac{\partial^2 \rho}{\partial t^2} - \Delta p.$$

If compressibility is neglected, we are left with the term $-\Delta p$ on the right. So instead of analyzing the nonlinear term $Q(\mathbf{u}, \mathbf{u})$, it may be more convenient to work with

$$\frac{1}{a_0^2} \frac{\partial^2 q}{\partial t^2} - \Delta q = -\Delta p.$$

From this equation there follows an interesting physical result. According to Lighthill's model, for small Mach numbers sound is generated if and only if Δp is non-zero. If the pressure in the turbulent flow is constant or even changing linearly in space, there are no sound generating sources.

The analysis of the fully discrete scheme for the (4.2) is also another open question.

6. Numerical experiments

In this section we present the results of some numerical experiments in two-dimensional case. Our main purpose in this section is to check the rate of convergence for some exact smooth solution and compare the theoretical predictions with the experimental results. We focus on the case when the no-slip boundary condition is imposed for the NSE in the inner domain Ω_1. Physically this simulation may represent the turbulent flow in the center of the medium, which decays in space fast enough to vanish in the quiescent media. For example, this could be a large storm eddy that does not affect the air far from its epicentre but generates a noise.

Let Ω_1 and Ω be squares such that $\Omega_1 = [0, 1]^2$ and $\Omega = [-0.25, 1.25]^2$, so Ω_1 is embedded into Ω symetrically, as shown on figure 2. The time-dependent incompressible flow is taking place inside Ω_1. The fluid's viscosity $\mu = 0.0172\ kg/m \cdot s$ and density $\rho = 1.2047\ kg/m^3$. This gives a fluid with viscosity being 1000 times as large as that of atmospheric air and the same density. The external forces $\mathbf{f}$ are given explicitly by :

$$\begin{aligned}
&f_1(x, y, t) = -C \cdot (\mu/\rho) \cdot sin(\pi \cdot t) \cdot ((x^2 - 2x^3 + x^4) \cdot (-12 + 24y) + \\
&+ (2 - 12x + 12x^2) \cdot (2y - 6y^2 + 4y^3)) + C^2 \cdot (sin(\pi \cdot t))^2 \cdot (x^4 - 2x^3 + x^2) \cdot \\
&\cdot (4x^3 - 6x^2 + 2x) \cdot ((4y^3 - 6y^2 + 2y)^2 - (y^4 - 2y^3 + y^2) \cdot (12y^2 - 12y + 2)) + \\
&+ C \cdot (x^4 - 2x^3 + x^2) \cdot (4y^3 - 6y^2 + 2y) \cdot \pi \cdot cos(\pi \cdot t) + (\nabla p)_1,
\end{aligned}$$

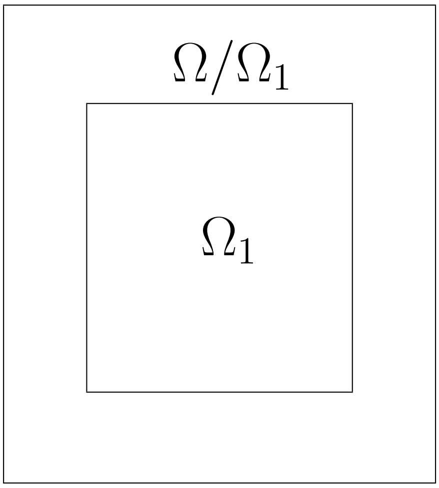

FIGURE 2. One domain inside the other

$$f_2(x,y,t) = -C\cdot(\mu/\rho)\cdot sin(\pi\cdot t)\cdot((-2x+6x^2-4x^3)\cdot(2-12y+12y^2)+ \\ +(12-24x)\cdot(y^2-2y^3+y^4)) + C^2\cdot(sin(\pi\cdot t))^2\cdot(y^4-2y^3+y^2)\cdot \\ \cdot(4y^3-6y^2+2y)\cdot((4x^3-6x^2+2x)^2-(x^4-2x^3+x^2)\cdot(12x^2-12x+2))- \\ -C\cdot(4x^3-6x^2+2x)\cdot(y^4-2y^3+y^2)\cdot\pi\cdot cos(\pi\cdot t)+(\nabla p)_2$$

with positive constant C and the fluid pressure p of our choice. Driven by this force $\mathbf{f}$, the fluid has the following velocity :

$$u_1(x,y,t) = C\cdot(x^4-2x^3+x^2)\cdot(4y^3-6y^2+2y)\cdot sin(\pi\cdot t),$$
$$u_2(x,y,t) = -C\cdot(y^4-2y^3+y^2)\cdot(4x^3-6x^2+2x)\cdot sin(\pi\cdot t).$$

The pressure in this case is constant : $\nabla p = 0$. This incompressible flow gives a vortex with periodically changing direction. The velocity vector field for that flow looks like the one shown on figure 3. The no-slip boundary condition here is satisfied. The exact nonlinear term Q is given by

$$Q(x,y,t) = 2\cdot C^2\cdot(sin(\pi\cdot t))^2\cdot((4x^3-6x^2+2x)^2\cdot(4y^3-6y^2+2y)^2- \\ -(12x^2-12x+2)\cdot(y^4-2y^3+y^2)\cdot(x^4-2x^3+x^2)\cdot(12y^2-12y+2)).$$

Consider the following hyperbolic problem :

$$\frac{\partial^2 q}{\partial t^2} - a_0^2\Delta q = a_0^2 Q(\mathbf{u},\mathbf{u}) + G,\ \forall(x,t)\in\Omega\times(0,T) \tag{6.1}$$

with

$$\nabla q\cdot\mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g,\ \forall(x,t)\in\partial\Omega\times(0,T).$$

We set $a_0 = 2$. As an exact solution we choose q to be the following :

$$q(x,y,t) = cos(\omega t + k(x+y) - k) + cos(\omega t - k(x+y) + k) + q_1(x,y,t),\ \forall(x,y,t)\in\Omega\times(0,T).$$

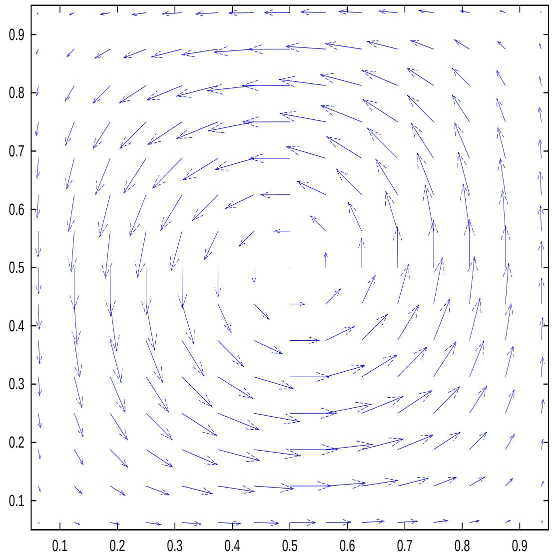

FIGURE 3. The flow for $C = 10$ and time level $T = 0.5$

where $\omega = 2$, $k = \frac{\omega}{a_0\sqrt{2}}$,

$$q_1(x, y, t) = \begin{cases} e^{4-\frac{1}{\frac{1}{4}-(x-\frac{1}{2})^2-(y-\frac{1}{2})^2}} \cdot (cos(\omega_1 t + k_1(x+y) - k_1)+ \\ +cos(\omega_1 t - k_1(x+y) + k_1)), \text{ if } (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 < \frac{1}{4}, \\ 0, \text{ otherwise} \end{cases}$$

with $\omega_1 = 4$ and $k_1 = \frac{\omega_1}{a_0\sqrt{2}}$. The plots of the acoustic pressure as a function of space are given for $t = 0$ and $t = 0.5$ on figures 4 and 5 respectively.

For our tests we take a uniform triangular mesh in Ω_1 of the size $N \times N$ with $N \geqslant 4$ even and $h_1 = 1/N$. Let the finite element space for the velocity field consist of piecewise linear functions, while for the pressure we use piecewise constants on the coarser mesh of size $2h_1$ (see figure 6). These spaces satisfy the LBB-condition (for example, [10]). For the wave equation in Ω consider the trinagluar mesh of the same size $h = h_1$ and the space of the piecewise linears. Both grids for the NSE and the wave equation are the same in Ω_1. The example is shown on figure 7. For the simulation of the incompressible flow we choose Stabilized Extrapolated Backward Euler Method in time with parameter $\delta = 0.005$ (see [15]). This means that the values of the velocity $\mathbf{u}^h_{n+1}$ and pressure p^h_{n+1} at the time step $n+1$ can be found from their values at the previous step n via the relation :

$$\left(\frac{\mathbf{u}^h_{n+1} - \mathbf{u}^h_n}{\Delta t_1}, \mathbf{v}^h\right) + \left(\frac{\mu}{\rho} + \delta\right)(\nabla \mathbf{u}^h_{n+1}, \nabla \mathbf{v}^h) + \frac{1}{2}(\mathbf{u}^h_n \cdot \nabla \mathbf{u}^h_{n+1}, \mathbf{v}^h)$$
$$-\frac{1}{2}(\mathbf{u}^h_n \cdot \nabla \mathbf{v}^h, \mathbf{u}^h_{n+1}) - (p^h_{n+1}, \nabla \cdot \mathbf{v}^h) = (\mathbf{f}(t_{n+1}), \mathbf{v}^h) + \delta(\nabla \mathbf{u}^h_n, \nabla \mathbf{v}^h), \ \forall \mathbf{v}^h \in X^h$$

and

$$(\nabla \cdot \mathbf{u}^h_{n+1}, q^h) = 0, \ \forall q^h \in Q^h,$$

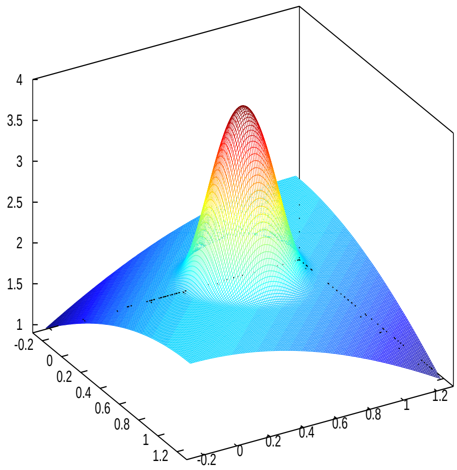

FIGURE 4. The graph of q at $t = 0$

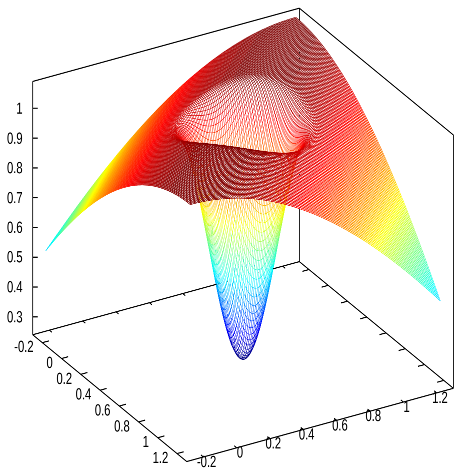

FIGURE 5. The graph of q at $t = 0.5$

where X^h and Q^h denote the finite-dimensional spaces described earlier for velocity and pressure respectively.

The dimension of the space of piecewise linears built on the elements in Ω is equal to $d = (\frac{3}{2}N + 1)^2$. If functions ϕ_i denote the basis functions in that space, then the solution q_h of the wave equation (4.6) can be written as a linear combination

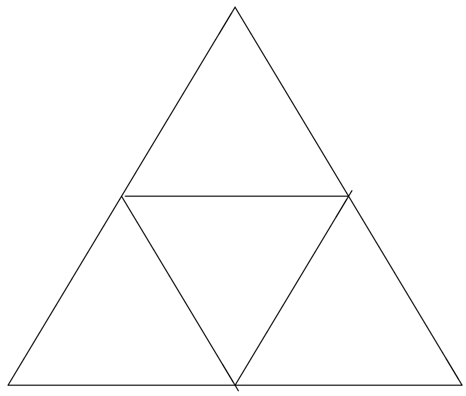

FIGURE 6. Four small triangles inside a big one

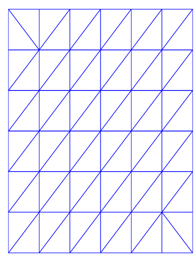

FIGURE 7. The grid for the whole Ω when $N = 4$

$q_h = \sum_i a_i \phi_i$ with coefficients a_i. Let these coefficients organize a vector $\mathbf{q}_h = (a_1, a_2, ..., a_d)^t$. This vector satisfies a linear differential equation in the form

$$M\ddot{\mathbf{q}}_h + a_0 L\dot{\mathbf{q}}_h + a_0^2 S\mathbf{q}_h = \mathbf{f}_{RHS}$$

with the mass matrix M and the stiff matrix S and matrix L related to the boundary term in the LHS of (4.6). This equation may be rewritten as the first-order system of differential equations :

$$\begin{pmatrix} \dot{\mathbf{q}}_h \\ \dot{\mathbf{r}}_h \end{pmatrix} = \begin{pmatrix} 0 & I \\ -a_0^2 M^{-1} S & -a_0 M^{-1} L \end{pmatrix} \begin{pmatrix} \mathbf{q}_h \\ \mathbf{r}_h \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}\mathbf{f}_{RHS} \end{pmatrix}$$

Initial conditions $\mathbf{q}_h(0,\cdot)$ and $\mathbf{r}_h(0,\cdot)$ are found from the H^1-projections of the functions $q(0,\cdot)$ and $\frac{\partial q}{\partial t}(0,\cdot)$ via the formula (4.7).

For time integration we use the Trapezoidal Method with the time step $\Delta t = 0.025$, while for the Backward Euler Method above we use $\Delta t_1 = 0.0125 = \Delta t/2$.

Every time step for the wave equation is done after two time steps for the NSE. We perform 20 steps for the wave equation until we reach the final time $T = 0.5$. This corresponds to the case, when the vortex in Ω_1 reaches its maximum velocity. Among the computed values of the error $\|q - q_h\|_{L^2(\Omega)}$ and $\|\frac{\partial}{\partial t}(q - q_h)\|_{L^2(\Omega)}$ at each time step we choose the greatest ones for both and add them. The result is the total error on the LHS of the inequality in Theorem 2. At the same time, we also compute the error $\|Q(\mathbf{u}, \mathbf{u}) - Q(\mathbf{u}_h, \mathbf{u}_h)\|_{L^2(L^2(\Omega_1))}$.

Since the estimate from Theorem 3 is not sharp due to regularity assumptions, the actual rate of convergence for term Q may only be obtained experimentally. Suppose it is of order α. Then according to Theorem 2 the error for the acoustic pressure satisfies an inequality

$$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial}{\partial t}(q - q_h)\right\|_{L^\infty(L^2(\Omega))} \leqslant K_1 h^2 + K_2 h^\alpha \tag{6.2}$$

with some positive constants K_1 and K_2 independent of h. The actual rate of convergence for the acoustic pressure in this case is $\gamma = min(\alpha, 2)$.

The rate of convergence may be estimated by evaluating the ratios of the error relating to the mesh of size $2h$ to the error relating to the mesh of size h. Indeed, for the first and the second grids we have $error_1$ and $error_2$ respectively :

$$error_1 \sim K \cdot (2h)^\gamma, \; error_2 \sim K \cdot h^\gamma$$

Division gives

$$\frac{error_1}{error_2} \sim 2^\gamma$$

As we refine the mesh by halving h, i.e. doubling N, the above fraction approaches constant 2^γ. The tables below present the results of numerical simulations for different external force vectors $\mathbf{f}$.

Test 1: $C = 10$, $p(x, y, t) = x(1 - x)y(1 - y)$.

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	0.10927		0.9619	
8	0.0932	1.1720	0.3635	2.6463
16	0.0557	1.6736	0.1060	3.4292
32	0.02567	2.1704	0.0271	3.9056

Test 2: $C = 100$, $p(x, y, t) = const$.

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	8.8487		4.9921	
8	3.7567	2.3555	1.4432	3.4590
16	1.7709	2.1214	0.4119	3.5038
32	0.88578	1.9992	0.14264	2.8878

Test 3: $C = 100$, $p(x, y, t) = x(1 - x)y(1 - y)$.

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	9.1412		5.1537	
8	4.0375	2.2641	1.6182	3.1849
16	1.8930	2.1329	0.4889	3.3097
32	0.9338	2.0273	0.1626	3.0075

Test 4: $C = 100$, $p(x, y, t) = 4x(1 - x)y(1 - y)$.

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	10.3300		5.9369	
8	5.6410	1.8313	3.0508	1.9460
16	2.7856	2.0251	1.2187	2.5034
32	1.3353	2.0860	0.3774	3.2292

According to Theorem 2, the rate of convergence for the solution of the wave equation in the absence of the error $Q - Q_{h_1}$ is expected to be quadratic, i.e. $k = 2$. This means that the ratio in this case must be reaching 4 as we refine our mesh. The experimental rate of convergence for the term Q appears to be linear. This fact follows from the third column of all tables where the ratio is approaching 2. So that means that the total rate of decrease of $L^\infty(L^2(\Omega))$-error for fluctuations of pressure must eventually reach 1 as we refine the mesh. This tendency of the rate to decrease may be seen in cases when the $L^2(L^2(\Omega_1))$-error for the term Q is large compared to the $L^\infty(L^2(\Omega))$-error for q and its time derivative. The example is presented below in the 5-th table. We can see that for $N = 32$ the ratio is dropping.

Test 5: $C = 300$, $p(x, y, t) = const.$

N	$\|Q - Q_h\|_{L^2(L^2(\Omega_1))}$	ratio	$\|q - q_h\|_{L^\infty(L^2(\Omega))} + \|\frac{\partial}{\partial t}(q - q_h)\|_{L^\infty(L^2(\Omega))}$	ratio
4	125.90		49.614	
8	42.593	2.9558	15.295	3.2437
16	19.714	2.1605	4.3770	3.4945
32	9.7621	2.0194	1.6236	2.6958

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