NUMERICAL ANALYSIS OF FILTER BASED STABILIZATION FOR EVOLUTION EQUATIONS

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Abstract. We consider filter based stabilization for evolution equations (in general) and for the Navier-Stokes equations (in particular). The first method we consider is to advance in time one time step by a given method and then to apply an (uncoupled and modular) filter to get the approximation at the new time level. This filter based stabilization, although algorithmically appealing, is viewed in the literature as introducing far too much numerical dissipation to achieve a quality approximate solution. We show that this is indeed the case. We then consider a modification: *Evolve one time step, Filter, Deconvolve then Relax* to get the approximation at the new time step. We give a precise analysis of the numerical diffusion and error in this process and show it has great promise, confirmed in several numerical experiments.

1. Introduction. Simulations in critical settings often struggle with numerical artifacts created by under resolution of the spacial scales and temporal dynamics in the model, e.g., Brown and Minon [BM95]. Often, as soon as there are sufficient computational resources for full resolution, the demands of the application force coupling to other processes, making the target simulation again under resolved. The needs of under resolved simulations have led to a number of stabilizations in Computational Fluid Dynamics. Herein we consider a filter based stabilization extending the idea of "*evolve one time step then filter*". This idea was developed by Boyd [Boyd98], Fischer and Mullen [FM01], [MF98], and used by Dunca [D02]. Mathew et al. [MLF03] made the important connection that "evolve then filter" induces a new implicit time relaxation term into the discretization that acts to damp oscillations in marginally resolved scales. The connection to time relaxation links it to work of Schochet and Tadmor [ST92], Roseneau [R89], Adams, Kleiser, Leonard and Stolz [AL99], [AS01], [AS02], [SA99], [SAK01a], [SAK01b], [SAK02], Dunca [D02], [D04], [DE04] and [LN07], [LMNR06], [ELN07], [L07b], [CL08].

To present this connection, consider the explicit method for a nonlinear evolution equation

$$\frac{\partial u}{\partial t} + F(u) = 0\,.$$

Let overbar denote a local spacial filter with a filter radius (ultimately related to the spacial mesh width) δ. Thus, $\overline{u}$ denotes the "well resolved" part of u and $u' = u - \overline{u}$ denotes the marginally resolved part of u. Add to the explicit method one uncoupled filter step. The method we extend, and then analyze, adds one uncoupled filter step to stabilize an explicit method: given u^n compute u^{n+1} by

$$\text{Step1: Compute } w^{n+1} \text{ via: } \quad \frac{w^{n+1} - u^n}{\triangle t} + F(u^n) = 0, \tag{1.1}$$

$$\text{Step 2: Filter } w^{n+1} \text{ to obtain } u^{n+1}\text{: } u^{n+1} = \overline{w^{n+1}}. \tag{1.2}$$

Both steps can be done by black box modules. The consistency error of (1.1), (1.2) (the error made after one time step, starting with exact values) is $O(\triangle t^2 + \delta^2)$ provided the filter is second order (i.e., $\overline{u} = u + O(\delta^2)$). This suggests a global error of $O(\triangle t + \frac{\delta^2}{\triangle t})$. Following Mathew et al. [MLF03], eliminating Step 2 gives

$$\frac{u^{n+1} - u^n}{\triangle t} + F(u^n) + \frac{1}{\triangle t}(w^{n+1} - \overline{w^{n+1}}) = 0\,, \tag{1.3}$$

which is a time relaxation discretization of the original problem with time relaxation coefficient $1/\triangle t$.

Suppose in addition that the filter is a differential filter, Germano [Ger86], i.e., that $\overline{w}(x,t)$ is the solution of

$$-\delta^2 \triangle \overline{w} + \overline{w} = w.$$

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It then follows from $w - \overline{w} = -\delta^2 \triangle \overline{w}$, and $u^{n+1} = \overline{w^{n+1}}$, that (1.1),(1.2) becomes

$$\frac{u^{n+1} - u^n}{\triangle t} + F(u^n) - \frac{\delta^2}{\triangle t} \triangle u^{n+1} = 0\,,$$

which is an Implicit-Explicit time discretization of the artificial viscosity method [ARW95]. The artificial viscosity coefficient is $\delta^2/\triangle t$ which can result in low accuracy and large amounts of numerical diffusion depending on the relative scalings of $\triangle t$ and δ.

When used with implicit methods, the effects of filter based stabilization are less clear. The goals of this report are (i) to understand their effect when used with implicit time stepping methods, (ii) to increase the accuracy of filter based stabilization, and (iii) to decrease the large amounts of numerical diffusion implicitly induced by the filter step. We characterize the numerical diffusion induced by the various combinations of filter, deconvolution, relaxation stabilization beginning in Theorem 1.2. In Sections 3 through 5 we apply the stabilization to the Navier-Stokes equations and show how the induced numerical diffusion is reduced by relaxation. We give a complete stability and error analysis for these stabilizations for the Navier-Stokes equations.

1.1. Filter based stabilization of implicit methods. The fundamental problem of filter based stabilization is that, if not carefully done, it reduces accuracy by introducing large amounts of numerical diffusion. We consider next the more complex case of implicit methods with filtering (to stabilize), deconvolution (to increase accuracy) and relaxation (to reduce numerical diffusion). Let $V \hookrightarrow L \hookrightarrow V'$ be Hilbert spaces with duality pairing an extension of the L-inner product, denoted by $< \cdot, \cdot >_L$, and $\| \cdot \|_L$ its associated norm. Let $F : V \to V'$ satisfy

$$< F(v), v >_L = 0 \text{ , for all } v \in V.$$

Consider the exactly conservative evolution equation: *Find* $u : [0, T] \to V$ *satisfying* $u(0) = u^0 \in V$ *and*

$$\frac{du}{dt} + F(u) = 0 \text{ for } t > 0.$$

Since this equation is exactly conservative under the above condition on F, any dissipation of energy is a numerical artifact. Let the filter $G : V' \to V$, and deconvolution operator $D : V \to V$, be bounded, self adjoint and positive linear operators, i.e. SPD, which satisfy the minimal compatibility conditions that as operators : $L \to L$

$$DG \text{ and } (I - DG) \text{ are } L \text{ self adjoint and positive.} \tag{1.4}$$

The van Cittert deconvolution operators defined in Section 2.3 and many others satisfy (1.4).

ALGORITHM 1.1. *Pick* $\chi \in [0, 1]$ *and* $\triangle t > 0$.

Step 1: Given u^n *find* w^{n+1} *satisfying*

$$\frac{w^{n+1} - u^n}{\triangle t} + F(\frac{w^{n+1} + u^n}{2}) = 0. \tag{1.5}$$

Step 2: *(a) Filter: Compute* $\overline{w^{n+1}} = G(w^{n+1})$

(b) Deconvolve: Compute $D(\overline{w^{n+1}})$

(c) Relax:

$$u^{n+1} := (1 - \chi) w^{n+1} + \chi D(\overline{w^{n+1}})$$

Relaxation was introduced into filter based stabilization by Fischer and Mullen [FM01], [MF98] to reduce the induced numerical diffusion. We introduce deconvolution in Step 2 to increase accuracy, see Theorem 5.4 in Section 5, and compare Table 6.1 (van Cittert deconvolution, see Definition 2.2) with Table 6.2 (no deconvolution); see also Table 6.4. We quantify the numerical diffusion introduced by Step 2 next.

THEOREM 1.2. *Let* $u = (1 - \chi) w + \chi D(\overline{w})$. *Then*

$$||w||_L^2 - ||u||_L^2 = \chi(2 - \chi) \langle (I - DG) w, w \rangle_L + \chi^2 \langle (I - DG) w, DGw \rangle_L \tag{1.6}$$

An approximate solution given by Algorithm 1.1 satisfies, for any $l > 0$,

$$||u^l||_L^2 + \triangle t \sum_{n=0}^{l} \left\{ \frac{\chi(2-\chi)}{\triangle t} \left\langle (I-DG)w^n, w^n \right\rangle_L + \frac{\chi^2}{\triangle t} \left\langle (I-DG)w^n, DGw^n \right\rangle_L \right\} = ||u^0||_L^2.$$

The numerical dissipation introduced by Step 2 at each time step is

$$\frac{\chi(2-\chi)}{\triangle t} \left\langle (I-DG)w^n, w^n \right\rangle_L + \frac{\chi^2}{\triangle t} \left\langle (I-DG)w^n, DGw^n \right\rangle_L \geq 0. \tag{1.7}$$

Proof. Take the L inner product of Step 1 with $(w^{n+1} + u^n)$, use $< F(v), v >_L = 0$, and rearrange the result. This gives

$$\frac{1}{\triangle t} \left(||u^{n+1}||_L^2 - ||u^n||_L^2 \right) + \frac{1}{\triangle t} \left[||w^{n+1}||_L^2 - ||u^{n+1}||_L^2 \right] = 0.$$

Step 2c can be rearranged to read

$$u + \chi(I - DG)w = w\,.$$

Take the L inner product with w. This gives

$$\langle u, w \rangle_L + \chi \left\langle (I-DG)w, w \right\rangle_L = ||w||_L^2\,.$$

The second term is nonnegative $\chi \left\langle (I-DG)w, w \right\rangle_L \geq 0$. Apply the polarization identity to the first term. This gives, after simplification

$$||u||_L^2 - ||u-w||_L^2 + 2\chi \left\langle (I-DG)w, w \right\rangle_L = ||w||_L^2.$$

Step 2c can also be rearranged to read

$$u - w = -\chi(I-DG)w\,, \quad \text{so} \quad ||u-w||_L^2 = \chi^2 \left\langle (I-DG)w, (I-DG)w \right\rangle_L.$$

Thus,

$$||w||_L^2 = ||u||_L^2 + \chi(2-\chi) \left\langle (I-DG)w, w \right\rangle_L + \chi^2 \left\langle (I-DG)w, DGw \right\rangle_L,$$

which is the first claim. Inserting this with $u = u^{n+1}, w = w^{n+1}$ gives

$$\frac{1}{\triangle t} \left(||u^{n+1}||_L^2 - ||u^n||_L^2 \right) + \frac{1}{\triangle t} \left[\chi(2-\chi) \left\langle (I-DG)w^{n+1}, w^{n+1} \right\rangle_L + \chi^2 \left\langle (I-DG)w^{n+1}, DGw^{n+1} \right\rangle_L \right] = 0.$$

Summing gives the energy estimate. That the term in brackets is nonnegative and thus dissipated energy follows since $(I - DG)$ is SPD for the first term. The second term is nonnegative since DG and $I - DG$ commute and are both SPD.
□

The form of the induced numerical diffusion suggests the natural scaling (used in our tests in Section 5) of the relaxation parameter

$$\chi \simeq O(\triangle t).$$

With this scaling the second term in the expression for the numerical dissipation is a higher order term; the numerical dissipation is dominated by the first term. On the other hand, fluid flow problems are complex and the above suggestive scaling is not likely to be optimal in general. The determination of parameter choice better than the above (and which can possibly be applied locally) is an open problem. For a properly chosen deconvolution operator, $(I - DG)w$ can be very small for smooth w, so deconvolution reduces the numerical diffusion added by filtering to the large scales.

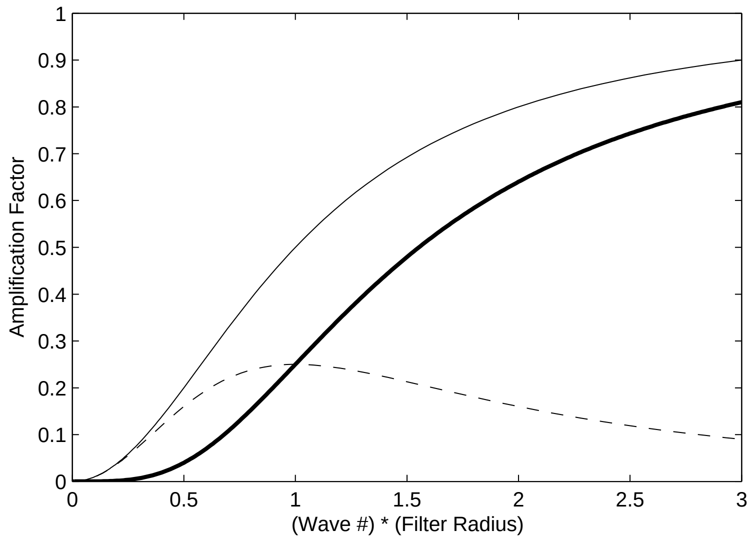

FIG. 1.1. $(I - DG)$ *for* $N = 0$*: thin line,* $N = 1$*: thick line, and DG(I-DG): broken line.*

REMARK 1.3 (Interpreting the numerical dissipation term in Theorem 1.2). *For the periodic problem* $L^2(0, 2\pi)$*,* G *a differential filter (* $G = (-\delta^2 \triangle + 1)^{-1}$ *under periodic boundary conditions),* D *a van Cittert deconvolution operator, Definition 2.2, the action of the numerical dissipation in Theorem 1.2 can be calculated wavenumber by wavenumber. Let* $w(x) = \sum_k \widehat{w}(k)e^{-ikx}$*. The first two van Cittert deconvolution operators are* $D_0 = I$*, and* $D_1 = 2I - G$*. For the first term in the numerical dissipation (1.7) we then have*

$$((I - DG)w, w) = \sum_k \widehat{(I - DG)}(k)|\widehat{w}(k)|^2.$$

We calculate

$$\widehat{(I - D_0 G)}(k) = \frac{(\delta k)^2}{(\delta k)^2 + 1},$$
$$\widehat{(I - D_1 G)}(k) = \left(\frac{(\delta k)^2}{(\delta k)^2 + 1}\right)^2.$$

We plot these in Figure 1.1 (along with the analogous graph for the second term in the numerical diffusion expression (1.7) for $N = 0$*,* $D_0 G \widehat{(I - D_0 G)}(k) = (\delta k)^2 / \left((\delta k)^2 + 1\right)^2$*). Note that the numerical diffusion increases as the wavenumber (length scale) increases (deceases). Increasing the order of deconvolution (here only from* $N = 0$ *(thin line) to* $N = 1$ *(thick line)) enhances this effect. Note also that the dotted line (representing the second term) is considerably smaller than the other terms, before even considering its smaller coefficient.*

2. Preliminaries. We consider the above algorithms for the Navier-Stokes equations (NSE) in a polyhedral domain Ω :

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = f \ \ , \text{ in } \Omega \subset \mathbb{R}^d, (d = 2, 3), \ t > 0, \tag{2.1}$$
$$\nabla \cdot u = 0 \ \ , \text{ in } \Omega, \ t > 0,$$
$$u(x, 0) = u^0(x), \text{ in } \Omega. \tag{2.2}$$

Two boundary conditions are considered: either no slip

$$u = 0 \text{ on } \partial\Omega \text{ for } t > 0,$$

or under periodic with zero-mean boundary conditions. In this later case $\Omega = (0, L)^d$ and

$$u(x + Le_j, t) = u(x, t) \quad j = 1, \cdots, d \quad \text{and}, \tag{2.3}$$
$$\int_\Omega \phi \, dx = 0 \text{ for } \phi = u,\ u^0,\ f, \text{ and } p.$$

There are differences between filtering under no slip or periodic boundary conditions. Under periodic conditions, incompressibility is preserved by the simple differential filter:

$$-\delta^2 \triangle \overline{w} + \overline{w} = w. \tag{2.4}$$

Under no slip boundary conditions incompressibility is not preserved by the above simple differential filter. It must be replaced by the Stokes differential filter. Given $w(x, t)$, $\overline{w}(x, t)$ is the solution of

$$-\delta^2 \triangle \overline{w} + \overline{w} + \nabla \lambda = w\,, \text{ and } \nabla \cdot \overline{w} = 0 \text{ in } \Omega\,, \tag{2.5}$$
$$\overline{w} = 0 \text{ on } \partial\Omega.$$

2.1. Notation. Until this point we have suppressed the (secondary) spacial discretization. To go further it must be specified with accompanying technical points. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot, \cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space $W_2^k(\Omega)$, and $\|\cdot\|_k$ denotes the norm in H^k. The space H^{-k} denotes the dual space of H_0^k. For functions $v(x, t)$ defined on the entire time interval $(0, T)$, we define $(1 \leq m < \infty)$

$$\|v\|_{\infty,k} \ := \ EssSup_{[0,T]} \|v(t, \cdot)\|_k \,, \text{ and } \ \|v\|_{m,k} \ := \ \left(\int_0^T \|v(t, \cdot)\|_k^m \, dt \right)^{1/m} .$$

We base our analysis on the finite element method (FEM) for the spacial discretization (and believe that the results extend to many other variational methods). To begin, if we are in the case of periodic with zero mean boundary conditions, let the velocity space X be the periodic H^1 vector functions with zero mean and the pressure space Q the L^2 functions with zero mean:

$$X := (H^1_\#(\Omega) \cap L^2_0(\Omega))^d,\ Q := L^2_0(\Omega).$$

In the case of no slip boundary conditions the analogous choice is

$$X := (H^1_0(\Omega))^d,\ Q := L^2_0(\Omega).$$

We use as the norm on X the H^1 semi-norm which, because of the boundary condition, is a norm, i.e. for $v \in X$, $\|v\|_X := \|\nabla v\|_{L^2}$. We denote the dual space of X by $X^\star$, with the norm $\|\cdot\|_\star$. The space of divergence free functions is given by

$$V := \{v \in X :\ (\nabla \cdot v, q) = 0 \ \ \forall q \in Q\}\,.$$

We shall denote conforming velocity, pressure finite element spaces based on an edge to edge triangulations of Ω (with maximum triangle diameter h) by

$$X_h \subset X,\ Q_h \subset Q.$$

We shall assume that $X_h,\ Q_h$ satisfy the usual inf-sup condition necessary for the stability of the pressure, e.g. [G89]. The discretely divergence free subspace of X_h is

$$V_h = \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0 \ \ \forall q_h \in Q_h\}\,.$$

Taylor-Hood elements (see e.g. [BS94, G89]) are one common example of such a choice for (X_h, Q_h), and are also the elements we use in our numerical experiments. Define the usual explicitly skew symmetrized trilinear form

$$b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v).$$

2.2. Discrete differential filters. In the periodic case, for $\phi \in L^2(\Omega)$ and $\delta > 0$ fixed, denote the result of filtering operation on ϕ by $\overline{\phi}$, where $\overline{\phi}$ is the unique solution (in X) of

$$-\delta^2 \Delta \overline{\phi} + \overline{\phi} = \phi. \tag{2.6}$$

We let $G : L^2(\Omega) \to X$ denote the filtering operation, i.e. $\overline{\phi} := G(\phi)$. Following Manica and Kaya-Merdan [MKM06] the discrete differential filter is given as follows. For $\phi \in L^2(\Omega)$, for a given filtering radius $\delta > 0$, $G_h : L^2(\Omega) \to X_h$, $\overline{\phi}_h := G_h(\phi)$ where $\overline{\phi}_h \in X_h$ is the unique solution of

$$\delta^2(\nabla\overline{\phi}_h, \nabla v_h) + (\overline{\phi}_h, v_h) = (\phi, v_h) \quad \forall v_h \in X_h. \tag{2.7}$$

In the case of internal flows under no slip boundary conditions we must use the discrete Stokes differential filter to preserve discrete incompressibility. For $\phi \in X^*$, $\delta > 0$ given, $G_h : X^* \to X_h$, $\overline{\phi}_h = G_h(\phi)$ where $(\overline{\phi}_h, \rho) \in X_h \times Q_h$ is the unique solution of

$$\delta^2(\nabla\overline{\phi}_h, \nabla v_h) + (\overline{\phi}_h, v_h) - (\rho, \nabla \cdot v_h) = (\phi, v_h) \quad \forall v_h \in X_h\,, \tag{2.8}$$

$$(\nabla \cdot \overline{\phi}_h, q) = 0 \quad \forall q \in Q_h. \tag{2.9}$$

The Stokes filter results in an exactly divergence free filtered velocity; the discrete Stokes filtered velocity is discretely divergence free.

We begin by recalling from [BIL04, MKM06] some basic facts about discrete differential filters.

LEMMA 2.1. *For $\phi \in X$, we have the following bounds for the discrete filter*

$$\|\overline{\phi}_h\| \le \|\phi\|\,, \quad \|\nabla\overline{\phi}_h\| \le \|\nabla\phi\| \text{ and } \|\nabla \times \overline{\phi}_h\| \le \|\nabla\phi\|\,.$$

For $\phi \in X$ and $\Delta\phi \in L^2(\Omega)$

$$\delta^2||\nabla(\phi - \overline{\phi}_h)||^2 + ||\phi - \overline{\phi}_h||^2 \le C \inf_{v_h \in X_h} \left(\delta^2||\nabla(\phi - v_h)||^2 + ||\phi - v_h||^2\right) + C\delta^4||\Delta\phi||^2.$$

We shall assume that the solution to the NSE that is approximated is a strong solution and in particular satisfies

$$u \in L^2(0,T;X) \cap L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;X), \tag{2.10}$$

$$p \in L^2(0,T;Q), \quad u_t \in L^2(0,T;X^*)\,, \tag{2.11}$$

and

$$(u_t, v) + (u \cdot \nabla u, v) - (p, \nabla \cdot v) + \nu(\nabla u, \nabla v) = (f, v) \quad \forall v \in X, \tag{2.12}$$

$$(\nabla \cdot u, q) = 0 \quad \forall q \in Q. \tag{2.13}$$

For notational clarity let $v(t^{n+1/2}) = v((t^{n+1} + t^n)/2)$ for the continuous variable and $v^{n+1/2} = (v^{n+1} + v^n)/2$ for both, continuous and discrete variables.

2.3. Deconvolution. There are many known procedures for deconvolution, e.g., [BB98]. The minimal conditions we assume throughout are that the (discrete) filter and (discrete) deconvolution used satisfy the consistency conditions of Stanculescu [S07].

Assumption A1: D_h, and G_h are symmetric, positive definite (SPD) *operators and preserve discrete incompressibility,*

Assumption A2: $||D_hG_h||_{\mathcal{L}(L^2 \to L^2)} \le 1$ *and* $||I - D_hG_h||_{\mathcal{L}(L^2 \to L^2)} \le 1$,

Assumption A3: $(I - D_hG_h)$ *and* D_hG_h *are SPD.*

These have been proven to hold for van Cittert deconvolution (next) in Stanculescu [S07]. Our error analysis, while particular to a differential filter and the van Cittert deconvolution operators, can readily be extended to more general ones satisfying the above. The N^{th} van Cittert deconvolution operator is computed by repeated filtering. It can be compactly given as follows. (We will suppress the dependence of D and D_h on the parameter N.)

DEFINITION 2.2. *The N^{th} order van Cittert continuous and discrete deconvolution operators as D and D_h are, respectively,*

$$D\phi := \sum_{n=0}^{N}(I - G)^n\phi\,, \quad \text{and} \quad D_h\phi := \sum_{n=0}^{N}(I - G_h)^n\phi\,. \tag{2.14}$$

3. Evolve then Filter for the Navier-Stokes equations. As a first step, we consider discretization by the FEM in space, Crank-Nicolson (CN) method in time with an added filtering step and no deconvolution or relaxation (i.e., Step 1 and Step 2a of Algorithm 1.1). If no slip boundary conditions are imposed, let the filter be the discrete Stokes filter (2.8),(2.9), and if periodic boundary conditions, the simple discrete differential filter (2.7), denoted by G and G_h.

ALGORITHM 3.1. *[Evolve then Filter for NSE]*
Step 1: Given u_h^n find $w_h^{n+1} \in X_h$, $p_h^{n+1} \in Q_h$ satisfying

$$
\begin{aligned}
(\frac{w_h^{n+1} - u_h^n}{\triangle t}, v_h) + b^*(\frac{w_h^{n+1} + u_h^n}{2}, \frac{w_h^{n+1} + u_h^n}{2}, v_h) + \nu(\nabla \frac{w_h^{n+1} + u_h^n}{2}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h) \\
= (f^{n+1/2}, v_h), \text{ for all } v_h \in X_h, \\
(\nabla \cdot w_h^{n+1}, q_h) = 0, \text{ for all } q_h \in Q_h.
\end{aligned} \tag{3.1}
$$

Step 2: Filter w_h^{n+1} to give u_h^{n+1} satisfying

$$
\begin{aligned}
\delta^2(\nabla u_h^{n+1}, \nabla v_h) + (u_h^{n+1}, v_h) = (w_h^{n+1}, v_h) \quad \forall v_h \in X_h \,, \\
\textit{imposing, in the non-periodic case, } \; (\nabla \cdot u_h^{n+1}, q_h) = 0, \; \forall q_h \in Q_h \,.
\end{aligned}
$$

The temporal consistency error and associated *forecasted* global error in the basic algorithm (3.1) is:

$$
\text{Temporal Consistency Error} = O(\triangle t^3 + \delta^2), \; \text{Global Error} = O(\triangle t^2 + \frac{\delta^2}{\triangle t} + \text{Spacial Error}),
$$

We turn to stability.

LEMMA 3.2. *Let $u_h^n = \overline{w_h}_h$. We have*

$$
2\delta^2 ||\nabla u_h^n||^2 + ||u_h^n||^2 + ||u_h^n - w_h^n||^2 = ||w_h^n||^2 \,.
$$

Proof. Consider the discrete Stokes filter (the proof is the same in the periodic case). Recall that

$$
\delta^2(\nabla u_h^n, \nabla v_h) + (u_h^n, v_h) = (w_h^n, v_h) \quad \forall v_h \in V_h.
$$

Set $v_h = u_h^n$. This gives

$$
\delta^2 ||\nabla u_h^n||^2 + ||u_h^n||^2 = (w_h^n, u_h^n).
$$

The polarization identity $(w_h^n, u_h^n) = \frac{1}{2}||w_h^n||^2 + \frac{1}{2}||u_h^n||^2 - \frac{1}{2}||u_h^n - w_h^n||^2$ gives, after rearrangement, the result:

$$
||w_h^n||^2 = 2\delta^2 ||\nabla u_h^n||^2 + ||u_h^n||^2 + ||u_h^n - w_h^n||^2.
$$

□

Next we prove a strong energy equality and associated strong, unconditional stability property.

PROPOSITION 3.3. *The approximate velocity u_h^{n+1} given by the Algorithm 3.1 satisfies the energy equality*

$$
\begin{aligned}
\frac{1}{2}||u_h^{l+1}||^2 + \triangle t \sum_{n=0}^{l} \left(\frac{\delta^2}{\triangle t} ||\nabla u_h^{n+1}||^2 + \frac{1}{2\triangle t} ||u_h^{n+1} - w_h^{n+1}||^2 + \nu ||\nabla (\frac{w_h^{n+1} + u_h^n}{2})||^2 \right) \\
= \frac{1}{2}||u^0||^2 + \triangle t \sum_{n=0}^{l} (f^{n+1/2}, \frac{w_h^{n+1} + u_h^n}{2}) \,,
\end{aligned}
$$

and the stability bound

$$
\begin{aligned}
\frac{1}{2}||u_h^{l+1}||^2 + \triangle t \sum_{n=0}^{l} \left(\frac{\delta^2}{\triangle t} ||\nabla u_h^{n+1}||^2 + \frac{1}{2\triangle t} ||u_h^{n+1} - w_h^{n+1}||^2 + \frac{\nu}{2} ||\nabla (\frac{w_h^{n+1} + u_h^n}{2})||^2 \right) \\
\leq \frac{1}{2}||u^0||^2 + \frac{1}{2\nu} \triangle t \sum_{n=0}^{l} ||f^{n+1/2}||_*^2 \;.
\end{aligned}
$$

Proof. In Step 1 set $v_h = (w_h^{n+1} + u_h^n)/2$. This gives

$$\frac{1}{2\triangle t}(||w_h^{n+1}||^2 - ||u_h^n||^2) + \nu||\nabla(\frac{w_h^{n+1} + u_h^n}{2})||^2 = (f^{n+1/2}, \frac{w_h^{n+1} + u_h^n}{2}). \tag{3.2}$$

We use the stability equality of Lemma 3.2

$$||w_h^{n+1}||^2 = 2\delta^2||\nabla u_h^{n+1}||^2 + ||u_h^{n+1}||^2 + ||u_h^{n+1} - w_h^{n+1}||^2$$

in the first term in the LHS of (3.2). Rearranging gives

$$\frac{1}{2\triangle t}(||u_h^{n+1}||^2 - ||u_h^n||^2) + \left(\frac{\delta^2}{\triangle t}||\nabla u_h^{n+1}||^2 + \frac{1}{2\triangle t}||u_h^{n+1} - w_h^{n+1}||^2 + \nu||\nabla(\frac{w_h^{n+1} + u_h^n}{2})||^2\right)$$
$$= (f^{n+1/2}, \frac{w_h^{n+1} + u_h^n}{2}).$$

Summing this establishes the energy equality. Using the Cauchy-Schwarz-Young inequality on the RHS, subsuming one term into the LHS gives

$$\frac{1}{2\triangle t}(||u_h^{n+1}||^2 - ||u_h^n||^2) + \left(\frac{\delta^2}{\triangle t}||\nabla u_h^{n+1}||^2 + \frac{1}{2\triangle t}||u_h^{n+1} - w_h^{n+1}||^2 + \frac{\nu}{2}||\nabla(\frac{w_h^{n+1} + u_h^n}{2})||^2\right)$$
$$\leq \frac{1}{2\nu}||f^{n+1/2}||_*^2$$

Summing over the index n, the global stability estimate follows.
□

The method is thus stable. The viscous and numerical dissipation in the method are respectively

$$\text{Viscous dissipation} := \nu||\nabla(\frac{w_h^{n+1} + u_h^n}{2})||^2$$
$$\text{Numerical dissipation} := \frac{\delta^2}{\triangle t}||\nabla u_h^n||^2 + \frac{1}{2\triangle t}||u_h^{n+1} - w_h^{n+1}||^2.$$

The first term in the numerical dissipation is the same as induced by the explicit method plus simple filtering. The second term resembles the numerical dissipation in the backward Euler method.

4. Evolve, Filter, Deconvolve and Relax. Consider discretization of the Navier-Stokes equations by the FEM in space, CN method in time with an added filtering step including deconvolution and time relaxation. To give concrete estimates specific choices of deconvolution operators and filters must be made. In this section we choose the (discrete) van Cittert deconvolution operator and (discrete) Stokes filter.

ALGORITHM 4.1. *[Evolve, Filter, Deconvolve then Relax for NSE] Let the filter be the discrete Stokes filter (2.8),(2.9). Choose χ with $0 \leq \chi \leq 1$.*
Step 1: Given u_h^n find $w_h^{n+1} \in X_h$, $p_h^{n+1} \in Q_h$ satisfying

$$(\frac{w_h^{n+1} - u_h^n}{\triangle t}, v_h) + b^*(\frac{w_h^{n+1} + u_h^n}{2}, \frac{w_h^{n+1} + u_h^n}{2}, v_h) + \nu(\nabla\frac{w_h^{n+1} + u_h^n}{2}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h) \tag{4.1}$$
$$= (f^{n+1/2}, v_h), \text{ for all } v_h \in X_h, \tag{4.2}$$
$$(\nabla \cdot w_h^{n+1}, q_h) = 0, \text{ for all } q_h \in Q_h.$$

Step 2: Compute $D_h\left(\overline{w_h^{n+1}}^h\right)$, then relax:

$$u_h^{n+1} = (1-\chi)w_h^{n+1} + \chi D_h\left(\overline{w_h^{n+1}}^h\right)$$

We next analyze the methods enhanced stability.

LEMMA 4.2. *Let* $u_h = (1-\chi)w_h + \chi D_h(\overline{w_h}_h)$. *Then*

$$||w_h||^2 - ||u_h||^2 = \chi(2-\chi)((I - D_hG_h)w_h, w_h) + \chi^2((I - D_hG_h)w_h, D_hG_hw_h), \tag{4.3}$$

$$||w_h||^2 - ||u_h||^2 = -||u_h - w_h||^2 + 2\chi((I - D_hG_h)w_h, w_h)\,. \tag{4.4}$$

If Assumptions A1, A2 and A3 hold then

$$||u_h|| \le ||w_h||\,. \tag{4.5}$$

Proof. The first equality is proven in Theorem 1.2. For the second, take the inner product of $u_h = (1-\chi)w_h + \chi D_hG_hw_h$ with w_h. This gives

$$(u_h, w_h) = (1-\chi)(w_h, w_h) + \chi(D_hG_hw_h, w_h) = ||w_h||^2 - \chi((I - D_hG_h)w_h, w_h).$$

Applying the polarization identity to the LHS we obtain

$$\frac{1}{2}||w_h||^2 + \frac{1}{2}||u_h||^2 - \frac{1}{2}||u_h - w_h||^2 = (u_h, w_h) = ||w_h||^2 - \chi((I - D_hG_h)w_h, w_h)\,,$$

from which (4.4) follows. For (4.5), note that since $||D_hG_h|| \le 1$ and $0 \le \chi \le 1$,

$$||u_h|| \le (1-\chi)||w_h|| + \chi||D_hG_hw_h|| \le ((1-\chi) + \chi||D_hG_h||)\,||w_h|| \le ||w_h||.$$

□

Next we prove an energy equality, unconditional stability and give the precise formula for the numerical dissipation in the algorithm.

PROPOSITION 4.3. *[Stability with Deconvolution and Relaxation]Suppose Assumptions A1, A2 and A3 above hold. Algorithm 4.1 satisfies the energy equality*

$$\frac{1}{2}||u_h^{l+1}||^2 + \nu\triangle t\sum_{n=0}^{l}||\nabla(\frac{w_h^{n+1}+u_h^n}{2})||^2 +$$

$$+\triangle t\sum_{n=0}^{l}\left[\frac{\chi(2-\chi)}{2\triangle t}((I - D_hG_h)w_h^{n+1}, w_h^{n+1}) + \frac{\chi^2}{2\triangle t}((I - D_hG_h)w_h^{n+1}, D_hG_hw_h^{n+1})\right]$$

$$= \frac{1}{2}||u^0||^2 + \triangle t\sum_{n=0}^{l}(f^{n+1/2}, \frac{w_h^{n+1}+u_h^n}{2})\,,$$

and the stability bound

$$||u_h^{l+1}||^2 + \nu\triangle t\sum_{n=0}^{l}||\nabla(\frac{w_h^{n+1}+u_h^n}{2})||^2 +$$

$$+\triangle t\sum_{n=0}^{l}\left[\frac{\chi(2-\chi)}{\triangle t}((I - D_hG_h)w_h^{n+1}, w_h^{n+1}) + \frac{\chi^2}{\triangle t}((I - D_hG_h)w_h^{n+1}, D_hG_hw_h^{n+1})\right]$$

$$\le ||u^0||^2 + \frac{\triangle t}{\nu}\sum_{n=0}^{l}||f^{n+1/2}||_*^2\,.$$

Proof. In Step 1 in Algorithm 4.1 set $v_h = (w_h^{n+1} + u_h^n)/2$. This gives

$$\frac{1}{2\triangle t}(||u_h^{n+1}||^2 - ||u_h^n||^2) + \nu||\nabla(\frac{w_h^{n+1}+u_h^n}{2})||^2 + \frac{1}{2\triangle t}\left[||w_h^{n+1}||^2 - ||u_h^{n+1}||^2\right] = (f^{n+1/2}, \frac{w_h^{n+1}+u_h^n}{2}). \tag{4.6}$$

We use Lemma 4.2 in the bracketed term in the LHS of (4.6). Rearranging gives

$$\frac{1}{2\triangle t}(||u_h^{n+1}||^2 - ||u_h^n||^2) + \nu||\nabla(\frac{w_h^{n+1}+u_h^n}{2})||^2 +$$

$$\left[\frac{\chi(2-\chi)}{2\triangle t}((I - D_hG_h)w_h^{n+1}, w_h^{n+1}) + \frac{\chi^2}{2\triangle t}((I - D_hG_h)w_h^{n+1}, D_hG_hw_h^{n+1})\right]$$

$$= (f^{n+1/2}, \frac{w_h^{n+1}+u_h^n}{2}).$$

Summing this proves the energy equality. Using the Cauchy-Schwarz-Young inequality on the RHS, subsuming one term into the LHS and summing over the index n, proves the global stability estimate.
□

The numerical dissipation in Algorithm 4.1 is exactly described by the bracketed term in the energy equality:

$$\text{Numerical dissipation} := \left[\frac{\chi(2-\chi)}{2\triangle t}((I-D_hG_h)w_h^{n+1},w_h^{n+1})+\frac{\chi^2}{2\triangle t}((I-D_hG_h)w_h^{n+1},D_hG_hw_h^{n+1})\right].$$

5. Error Analysis of the Time Relaxation Approximation. In this section we present a detailed error analysis for the approximation scheme, Algorithm 4.1 incorporating both (differential) filtering and (van Cittert) deconvolution. We focus our attention on the simplest differential filter, G defined via (2.6) and its discrete counterpart G_h defined by (2.7). Lemma 2.1 presents the error estimate for the error between ϕ and $D_hG_h(\phi) := D_0\overline{\phi}_h$ for the 0^{th} order deconvolution operator. For van Cittert deconvolution operators (Definition 2.2) we have the following result.

LEMMA 5.1. *[LMNR06] For smooth ϕ the discrete N^{th} order approximate deconvolution operator satisfies for $0 \le s \le N$*

$$\|\phi\,-\,D_hG_h\phi\| \;\le\; C_1\,\delta^{2s+2}\,\|\overline{\phi}\|_{H^{2s+2}} \;+\; C_2\left(\delta h^k\,+\,h^{k+1}\right)\left(\sum_{n=1}^{N+1}|G^n(\phi)|_{k+1}\right). \qquad (5.1)$$

The dependence of the $|G^n(\phi)|_{k+1}$ terms in (5.1) upon the *filter radius* δ, for a general smooth function ϕ, is not fully understood. In the case of ϕ periodic the $|G^n(\phi)|_{k+1}$ are independent of δ. Also, for ϕ satisfying homogeneous Dirichlet boundary conditions, with the additional property that $\Delta^j\phi = 0$ on $\partial\Omega$ for $0 \le j \le \left[\frac{k+1}{2}\right]-1$, the $|G^n(\phi)|_{k+1}$ are independent of δ, see [LMNR06], [L07].

As mentioned above, Taylor-Hood approximating elements are a common choice for (X_h, Q_h) and correspond to $k = 2$ in (5.1). For these approximating elements we have from Lemma 5.1 the following corollary.

COROLLARY 5.2. *Suppose $\phi \in H_0^1(\Omega)\cap H^4(\Omega)$. Suppose the order of deconvolution is $N = 1$ and (X_h, Q_h) are chosen to be the Taylor-Hood elements. We have*

$$\|\phi\,-\,D_hG_h\phi_h\| \;\le\; C_1\,\delta^3\|\phi\|_3 \;+\; C_2\left(\delta h^2\,+\,h^3\right)\|\phi\|_3\,. \qquad (5.2)$$

Proof. This follows from the previous lemma by taking $s = 1/2$, $k = 2, N = 1$ and thus $\left[\frac{k+1}{2}\right]-1 = 0$. We have then $\|\overline{\phi}\|_3 \le C\|\phi\|_3$ with constant independent of δ and h.
□

Extending Corollary 5.2, we will make the following assumption.
Assumption DG1: *The $|G^n(\phi)|_{k+1}$ terms in (5.1) are independent of δ* and

$$\|\phi\,-\,D_hG_h\phi\| \;\le\; C_1\,\delta^{2N+2}\,\|\phi\|_{H^{2N+2}} \;+\; C_2\left(\delta h^k\,+\,h^{k+1}\right)\|\phi\|_{k+1}\,. \qquad (5.3)$$

With $t^n = n\Delta t$, $n = 0, 1, 2, \ldots, N_T$, $T := N_T\Delta t$, and $d_tf^n := (f(t^n) - f(t^{n-1})/\Delta t$, we introduce the following discrete norms:

$$|||v|||_{\infty,k} := \max_{0\le n\le N_T}\|v^n\|_k\,, \qquad |||v_{1/2}|||_{\infty,k} \;:=\; \max_{1\le n\le N_T}\|v^{n-1/2}\|_k\,,$$

$$|||v|||_{m,k} := \left(\sum_{n=0}^{N_T}\|v^n\|_k^m\Delta t\right)^{1/m}, \qquad |||v_{1/2}|||_{m,k} \;:=\; \left(\sum_{n=1}^{N_T}\|v^{n-1/2}\|_k^m\Delta t\right)^{1/m}.$$

We denote time level averages by

$$\breve{w}^n := \frac{w^n+u^{n-1}}{2} \quad\text{and}\quad \tilde{w}^n := \frac{w^n+w^{n-1}}{2}\,. \qquad (5.4)$$

To begin the analysis we rewrite Algorithm 4.1 in the following form.

Algorithm 4.1 restated. *Assume that the filtering, deconvolution, and relaxation parameters δ, N, and χ are given. Then, for $n = 1, 2, \ldots, N_T$, find w_h^n, $D_h G_h(w_h^n)$, $u_h^n \in X_h$, $p_h^n \in Q_h$, such that*

$$(w_h^n, v_h) + \Delta t\, b^*(\breve{w}_h^n, \breve{w}_h^n, v_h) - \Delta t\,(p_h^n, \nabla \cdot v_h) + \Delta t\,\nu(\nabla \breve{w}_h^n, \nabla v_h) = (u_h^{n-1}, v_h) + \Delta t\,(f^{n-1/2}, v_h)\,,\ \forall v_h \in X_h\,, \tag{5.5}$$

$$(\nabla \cdot w_h^n, q_h) = 0\,,\ \forall q_h \in Q_h\,, \tag{5.6}$$

$$u_h^n = (1-\chi) w_h^n + \chi\, D_h G_h(w_h^n)\,. \tag{5.7}$$

As the spaces X_h and Q_h satisfy the usual inf-sup condition, we can equivalent consider the problem: *For $n = 1, 2, \ldots, N_T$ find w_h^n, $\overline{w_h^n}_h$, $u_h^n \in V_h$ such that*

$$(w_h^n, v_h) + \Delta t\, b^*(\breve{w}_h^n, \breve{w}_h^n, v_h) + \Delta t\,\nu(\nabla \breve{w}_h^n, \nabla v_h) = (u_h^{n-1}, v_h) + \Delta t\,(f^{n-1/2}, v_h),\ \forall v_h \in V_h\,, \tag{5.8}$$

$$\delta^2(\nabla \cdot \overline{w_h^n}_h, \nabla \cdot v_h) + (\overline{w_h^n}_h, v_h) = (w_h^n, v_h)\ ,\ \forall v_h \in V_h\,, \tag{5.9}$$

$$u_h^n = (1-\chi) w_h^n + \chi\, D_h \overline{w_h^n}_h\,. \tag{5.10}$$

We first establish computability of the approximation at each time step.

Lemma 5.3. *For the approximation scheme (5.8)-(5.10) we have that w_h^n, $\overline{w_h^n}_h$, u_h^n, exist at each time step.*

Proof. The existence of a solution w_h^n to (5.8) follows from the Leray-Schauder Principle [Z95]. Specifically, with $A : V_h \to V_h$, defined by $y = A(w)$

$$(y, v) := -\Delta t\, b^*((w + u_h^{n-1})/2, (w + u_h^{n-1})/2, v) - \Delta t\,\nu(\nabla(w + u_h^{n-1})/2, \nabla v) + (u_h^{n-1}, v) + \Delta t (f^{n-1/2}, v)\ ,$$

the operator A is compact and any solution of $w = s\,A(w)\,,\quad$ for $0 \le s < 1$, satisfied the bound $\|w\| \le \gamma$, where γ is independent of s. Existence and uniqueness of $\overline{w_h^n}_h$ follows directly from the Riesz representation theorem. Existence and uniqueness of u_h^n follows for w_h^n and $\overline{w_h^n}_h$ from the definition of D_h.
□

To establish the optimal asymptotic error estimates for the approximation we need to assume that the true solution is more regular than that given by (2.10),(2.11). Specifically, assuming $2N + 2 \ge k + 1$,

$$u \in L^\infty(0, T; W_4^{k+1}(\Omega)) \cap L^\infty(0, T; H^{2N+2}(\Omega)) \cap \tag{5.11}$$

$$H^1(0, T; H^{k+1}(\Omega)) \cap H^3(0, T; L^2(\Omega)) \cap W_4^2(0, T; H^1(\Omega)),$$

$$p \in L^\infty(0, T; H^{s+1}(\Omega))\,, \text{ and } f \in H^2(0, T; L^2(\Omega))\,. \tag{5.12}$$

For the errors $e^n := u(t^n) - u_h^n$, and $\varepsilon^n := u(t^n) - w_h^n$, we have the following.

Theorem 5.4. *For u, p, and f as described by (5.11),(5.12), satisfying (2.12),(2.13), and u_h^n, w_h^n given by (5.5)-(5.7) we have that for Δt sufficiently small*

$$|||u - u_h|||_{\infty,0} + |||u - w_h|||_{\infty,0} \le F(\Delta t, h, \delta, \chi) + C h^{k+1} |||u|||_{\infty,k+1}\,, \tag{5.13}$$

$$\left(\nu \Delta t \sum_{n=1}^{l} \|\nabla(u^{n-1/2} - (w_h^n + u_h^{n-1})/2)\|^2 \right)^{1/2} \le F(\Delta t, h, \delta, \chi) + C\nu^{1/2} (\Delta t)^2 \|\nabla u_{tt}\|_{2,0} + C \nu^{1/2} h^k |||u|||_{2,k+1}\ ,\ \textit{for } 1 \le l \le N_T\,. \tag{5.14}$$

where

$$F(\Delta t, h, \delta, \chi) := C\nu^{-1/2}\left(h^{k+1/2}|||u|||_{4,k+1}^2 \;+\; h^{k+1/2}|||\nabla u|||_{4,0}^2 \;+\; h^{s+1}|||p_{1/2}|||_{2,s+1}\right)$$
$$+ C\,\nu^{1/2}\,h^k|||u|||_{2,k+1} \;+\; C\,h^{k+1}\,\|u_t\|_{2,k+1} + C\,\nu^{-1}\,h^k|||u|||_{\infty,k+1}$$
$$+ C\,\chi\,\left(\nu^{-3/2}\,|||\nabla u|||_{\infty,0}^2 + (\Delta t)^{-1/2}\right)\,(\Delta t)^{-1/2}\,h^{k+1}\,|||u|||_{2,k+1}$$
$$+ C\,\chi\,\left(\nu^{-3/2}\,|||\nabla u|||_{\infty,0}^2 + (\Delta t)^{-1/2}\right)\,(\Delta t)^{-1/2}\,(\delta^{2N+2} \;+\; \delta\,h^k \;+\; h^{k+1})\,(|||u|||_{2,2N+2} \;+\; |||u|||_{2,k+1})$$
$$+\; C\,(\Delta t)^2\left(\|u_{ttt}\|_{2,0} \;+\; \|f_{tt}\|_{2,0} \;+\; \nu^{1/2}\;|\nabla u_{tt}\|_{2,0}\right.$$
$$\left.+\,\nu^{-1/2}\|\nabla u_{tt}\|_{4,0}^2 \;+\; \nu^{-1/2}|||\nabla u|||_{4,0}^2 \;+\; \nu^{-1/2}|||\nabla u_{1/2}|||_{4,0}^2\right)\,.$$

Proof. At time $t = (n-1/2)\Delta t$, u given by (2.12)-(2.13) satisfies

$$(u^n - u^{n-1}, v_h) \;+\; \frac{\Delta t}{2}\,\nu(\nabla\tilde{u}^n\,,\,\nabla v_h) + \Delta t\,b^*(\tilde{u}^n\,,\tilde{u}^n\,,\,v_h) \;-\; \Delta t\,(p^{n-1/2}, \nabla\cdot v_h)$$
$$= \Delta t\,(\tilde{f}^n, v_h) \;+\; \Delta t\,Intp(u^n; v_h)\;, \qquad (5.15)$$

for all $v_h \in V_h$, where $Intp(u^n; v_h)$, representing the interpolating error, denotes

$$Intp(u^n; v_h) = \left((u^n - u^{n-1})/\Delta t - u_t^{n-1/2}, v_h\right) \;+\; \nu(\nabla\tilde{u}^n - \nabla u^{n-1/2}\;,\;\nabla v_h)$$
$$+\; b^*(\tilde{u}^n\,,\tilde{u}^n\,,\,v_h) \;-\; b^*(u^{n-1/2}, u^{n-1/2}, v_h)$$
$$+\;(f^{n-1/2} - \tilde{f}^n, v_h)\,. \qquad (5.16)$$

Subtracting (5.8) from (5.15), we have for $\varepsilon^n \;=\; u^n \;-\; w_h^n$, $e^n \;=\; u^n \;-\; u_h^n$,

$$(\varepsilon^n - e^{n-1}, v_h) \;+\; \Delta t\,\nu(\nabla(\varepsilon^n + e^{n-1})/2\,,\nabla v_h) = -\,\Delta t\,b^*(\tilde{u}^n\,,\tilde{u}^n\,,\,v_h) \;+\; \Delta t\,b^*(\breve{w}_h^n, \breve{w}_h^n, v_h)$$
$$+\;\Delta t(p^{n-1/2}, \nabla\cdot v_h) \;+\; \Delta t\,Intp(u^n; v_h)\;, \qquad (5.17)$$

for all $v_h \in V_h$. Let $U^n \in V_h$, $\varepsilon^n \;=\; u^n \;-\; w_h^n \;=\; (u^n - U^n) \;+\; (U^n - w_h^n) := \Lambda^n \;+\; F^n$, and $e^n \;=\; u^n \;-\; u_h^n \;=\; (u^n - U^n) + (U^n - u_h^n) := \Lambda^n \;+\; E^n$.

Similar to the notation defined in (5.4), we use $\breve{F}^n := (F^n + E^{n-1})/2$. With the choice $v_h = \breve{F}^n$, and using $(\nabla\cdot\breve{F}^n, q_h) = 0, \;\; \forall q_h \in Q_h$, equation (5.17) becomes

$$\frac{1}{2}\left(\|F^n\|^2 \;-\; \|E^{n-1}\|^2\right) \;+\; \Delta t\,\nu\|\nabla\breve{F}^n\|^2 = -(\Lambda^n - \Lambda^{n-1}, \breve{F}^n) \;-\; \Delta t\,\nu(\nabla\tilde{\Lambda}^n, \nabla\breve{F}^n)$$
$$-\;\Delta t\,b^*(\tilde{u}^n\,,\tilde{u}^n\,,\,\breve{F}^n) \;+\; \Delta t\,b^*(\breve{w}_h^n, \breve{w}_h^n, \breve{F}^n)$$
$$+\;\Delta t(p^{n-1/2} - q_h, \nabla\cdot\breve{F}^n) \;+\; \Delta t\,Intp(u^n; \breve{F}^n)\;. \qquad (5.18)$$

Next we estimate the terms on the RHS of (5.18).

$$(\Lambda^n - \Lambda^{n-1}, \breve{F}^n) \le \frac{1}{2}\Delta t\|\frac{\Lambda^n - \Lambda^{n-1}}{\Delta t}\|^2 + \frac{1}{2}\Delta t\|\breve{F}^n\|^2$$
$$= \;\frac{1}{2}\Delta t\int_\Omega\left(\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\Lambda_t\,dt\right)^2\,d\Omega \;+\;\frac{1}{2}\Delta t\|\breve{F}^n\|^2$$
$$\le \;\frac{1}{2}\Delta t\int_\Omega\left(\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}|\Lambda_t|^2\,dt\right)\,d\Omega \;+\;\frac{1}{2}\Delta t\|\breve{F}^n\|^2$$
$$= \;\frac{1}{2}\int_{t^{n-1}}^{t^n}\|\Lambda_t\|^2\,dt \;+\;\frac{1}{2}\Delta t\left(\|F^n\|^2 \;+\; \|E^{n-1}\|^2\right)\;. \qquad (5.19)$$

$$\nu(\nabla\tilde{\Lambda}^n, \nabla\breve{F}^n) \;\le\; \frac{\nu}{10}\|\nabla\breve{F}^n\|^2 \,+\, C\nu\,\|\nabla\tilde{\Lambda}^n\|^2\,. \tag{5.20}$$

We rewrite $b^*(\tilde{u}^n\,, \tilde{u}^n\,,\, \tilde{F}^n) \,-\, b^*(\breve{w}_h^n, \breve{w}_h^n, \tilde{F}^n)$ as

$$\begin{aligned}
b^*(\tilde{u}^n\,, \tilde{u}^n\,,\, \breve{F}^n) \,-\, b^*(\breve{w}_h^n, \breve{w}_h^n, \breve{F}^n) &= b^*(\tilde{u}^n\,, \tilde{u}^n\,,\, \breve{F}^n) \,-\, b^*(\breve{w}_h^n, \tilde{u}^n, \breve{F}^n) \\
&\quad + b^*(\breve{w}_h^n, \tilde{u}^n, \breve{F}^n) \,-\, b^*(\breve{w}_h^n, \breve{w}_h^n, \breve{F}^n) \\
&= b^*((\varepsilon^n + e^{n-1})/2\,, \tilde{u}^n\,,\, \breve{F}^n) \,+\, b^*(\breve{w}_h^n, (\varepsilon^n + e^{n-1})/2, \breve{F}^n) \\
&= b^*(\tilde{\Lambda}^n + \breve{F}^n\,, \tilde{u}^n\,,\, \breve{F}^n) \,+ b^*(\breve{w}_h^n, \tilde{\Lambda}^n + \breve{F}^n\,, \breve{F}^n) \\
&= b^*(\tilde{\Lambda}^n, \tilde{u}^n\,,\, \breve{F}^n) \,+ b^*(\breve{F}^n\,, \tilde{u}^n\,,\, \breve{F}^n) \\
&\quad + b^*(\breve{w}_h^n, \tilde{\Lambda}^n\,, \breve{F}^n) + b^*(\breve{w}_h^n, \breve{F}^n\,, \breve{F}^n)\,.
\end{aligned} \tag{5.21}$$

Using $b^*(u,v,w) \le C(\Omega)\sqrt{\|u\|\,\|\nabla u\|}\,\|\nabla v\|\,\|\nabla w\|$, for $u,\ v,\ w\ \in X$, and Young's inequality, we bound the terms on the RHS of (5.21) as follows.

$$\begin{aligned}
b^*(\tilde{\Lambda}^n\,, \tilde{u}^n\,,\, \breve{F}^n) &\le C\,\sqrt{\|\tilde{\Lambda}^n\|\,\|\nabla\tilde{\Lambda}^n\|}\,\|\nabla\tilde{u}^n\|\,\|\nabla\breve{F}^n\| \\
&\le \frac{\nu}{10}\,\|\nabla\breve{F}^n\|^2 \;+\; C\,\nu^{-1}\,\|\tilde{\Lambda}^n\|\,\|\nabla\tilde{\Lambda}^n\|\,\|\nabla\tilde{u}^n\|^2
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
b^*(\breve{F}^n\,, \tilde{u}^n\,,\, \breve{F}^n) &\le C\,\|\breve{F}^n\|^{1/2}\,\|\nabla\breve{F}^n\|^{3/2}\,\|\nabla\tilde{u}^n\| \\
&\le \frac{\nu}{10}\,\|\nabla\breve{F}^n\|^2 \;+\; C\,\nu^{-3}\,\|\nabla\tilde{u}^n\|^4\,\|\breve{F}^n\|^2 \\
&\le \frac{\nu}{10}\,\|\nabla\breve{F}^n\|^2 \;+\; C\,\nu^{-3}\,\|\nabla\tilde{u}^n\|^4\,\left(\|F^n\|^2 \,+\, \|E^{n-1}\|^2\right)
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
b^*(\breve{w}_h^n\,, \tilde{\Lambda}^n\,,\, \breve{F}^n) &\le C\,\|\nabla\breve{w}_h^n\|\,\|\nabla\tilde{\Lambda}^n\|\,\|\nabla\breve{F}^n\| \\
&\le \frac{\nu}{10}\,\|\nabla\breve{F}^n\|^2 \;+\; C\,\nu^{-1}\,\|\nabla\breve{w}_h^n\|^2\,\|\nabla\tilde{\Lambda}^n\|^2
\end{aligned} \tag{5.24}$$

$$b^*(\breve{w}_h^n\,, \breve{F}^n, \breve{F}^n) \;=\; 0 \tag{5.25}$$

$$\begin{aligned}
(p^{n-1/2} - q_h, \nabla\cdot\breve{F}^n) &\le \|p^{n-1/2} - q_h\|\,\|\nabla\cdot\breve{F}^n\| \\
&\le \frac{\nu}{10}\|\nabla\breve{F}^n\|^2 \,+\, C\,\nu^{-1}\,\|p^{n-1/2} - q_h\|^2\,.
\end{aligned} \tag{5.26}$$

With the bounds (5.19)–(5.26), (5.18)becomes

$$\begin{aligned}
&\frac{1}{2}\left(\|F^n\|^2 \,-\, \|E^{n-1}\|^2\right) + \Delta t\,\frac{\nu}{2}\|\nabla\breve{F}^n\|^2 \\
&\qquad \le\; C\Delta t\,(1 + \nu^{-3}\|\nabla\tilde{u}^n\|^4)\left(\|F^n\|^2 \,+\, \|E^{n-1}\|^2\right) \,+\, C\nu\Delta t\|\nabla\tilde{\Lambda}^n\|^2 \\
&\qquad + C\nu^{-1}\Delta t\,\|\nabla\breve{w}_h^n\|^2\,\|\nabla\tilde{\Lambda}^n\|^2 \,+\, C\nu^{-1}\Delta t\,\|\nabla\tilde{u}^n\|^2\|\tilde{\Lambda}^n\|\,\|\nabla\tilde{\Lambda}^n\| \\
&\qquad + C\int_{t^{n-1}}^{t^n}\|\Lambda_t\|^2\,dt \,+\, C\,\nu^{-1}\Delta t\,\|p^{n-1/2} - q_h\|^2 \,+\, \Delta t\,|Intp(u^n;\breve{F}^n)|\,.
\end{aligned} \tag{5.27}$$

As u_h^n and w_h^n are connected through the *filter-deconvolve-relax* equation, we next use that equation to obtain a relationship between $\|F^n\|$ and $\|E^n\|$. The true solution $u(\cdot,t^n) = u^n$ satisfies

$$u^n \;=\; (1-\chi)u^n \,+\, \chi D_h G_h u^n \,+\, \chi(I - D_h G_h)u^n\,. \tag{5.28}$$

Subtracting (5.7) from (5.28) yields

$$
\begin{aligned}
e^n =&\ (1-\chi)\varepsilon^n + \chi D_h G_h \varepsilon^n + \chi(I - D_h G_h) u^n \qquad (5.29)\\
\text{i.e., } E^n =&\ (1-\chi)F^n + \chi D_h G_h F^n - \chi(I - D_h G_h)\Lambda^n + \chi(I - D_h G_h) u^n\\
\Rightarrow\ \|E^n\| \le&\ (1-\chi)\|F^n\| + \chi\|D_h G_h\|\,\|F^n\| + \chi\|(I - D_h G_h)\Lambda^n\| + \chi\|(I - D_h G_h)u^n\|\\
\|E^n\| \le&\ \|F^n\| + \chi\|(I - D_h G_h)\Lambda^n\| + \chi\|(I - D_h G_h)u^n\| \qquad (5.30)
\end{aligned}
$$

Squaring both sides of (5.30) and simplifying we obtain

$$
\|E^n\|^2 \ \le\ \|F^n\|^2 + \Delta t\,\|F^n\|^2 + 2\,\chi^2\left(1 + (\Delta t)^{-1}\right)\|(I - D_h G_h)\Lambda^n\|^2 + 2\,\chi^2\left(1 + (\Delta t)^{-1}\right)\|(I - D_h G_h)u^n\|^2 . \qquad (5.31)
$$

Substituting (5.31) into (5.27) and summing from $n = 1$ to l, (assuming that $\|F^0\| = 0$, i.e. $u^0 \in X_h$) we obtain

$$
\begin{aligned}
&\|F^l\|^2 + \nu\,\Delta t \sum_{n=1}^{l} \|\nabla \breve{F}^n\|^2\\
&\le C\,\Delta t \sum_{n=0}^{l} (1 + \nu^{-3}\,\|\nabla \tilde{u}^n\|^4)\,\|F^n\|^2\\
&+\ C\,\nu\,\Delta t \sum_{n=1}^{l} \|\nabla \tilde{\Lambda}^n\|^2 +\ C\,\nu^{-1}\,\Delta t \sum_{n=1}^{l} \|\nabla \tilde{u}^n\|^2\,\|\tilde{\Lambda}^n\|\,\|\nabla \tilde{\Lambda}^n\|\\
&+\ C\,\nu^{-1}\,\Delta t \sum_{n=1}^{l} \|\nabla \breve{w}_h^n\|^2\,\|\nabla \tilde{\Lambda}^n\|^2 \ +\ C \sum_{n=1}^{l} \int_{t^{n-1}}^{t^n} \|\Lambda_t\|^2\,dt\\
&+\ C\,\Delta t \sum_{n=0}^{l-1} \left(1 + \nu^{-3}\,\|\nabla \tilde{u}^n\|^4 + (\Delta t)^{-1}\right)\left(1 + (\Delta t)^{-1}\right)\chi^2\|(I - D_h G_h)\Lambda^n\|^2\\
&+\ C\,\Delta t \sum_{n=0}^{l-1} \left(1 + \nu^{-3}\,\|\nabla \tilde{u}^n\|^4 + (\Delta t)^{-1}\right)\left(1 + (\Delta t)^{-1}\right)\chi^2\|(I - D_h G_h)u^n\|^2\\
&+\ \Delta t \sum_{n=1}^{l} C\,\nu^{-1}\|p^{n-1/2} - q_h\|^2\\
&+\ 2\Delta t \sum_{n=1}^{l} |Intp(u^n; \breve{F}^n)| . \qquad (5.32)
\end{aligned}
$$

The terms on the RHS of (5.32) can be further simplified as follows.

$$
C\,\nu\,\Delta t \sum_{n=1}^{l} \|\nabla \tilde{\Lambda}^n\|^2 \ \le\ C\,\nu\,\Delta t \sum_{n=0}^{l} \|\nabla \Lambda^n\|^2 \ \le\ C\,\nu\,\Delta t \sum_{n=0}^{l} h^{2k} |u^n|_{k+1}^2 \ \le C\,\nu\,h^{2k}\||u|\|_{2,k+1}^2 . \qquad (5.33)
$$

For the next term

$$
\begin{aligned}
& C\,\nu^{-1}\,\Delta t \sum_{n=1}^{l} \|\nabla \tilde{u}^n\|^2\,\|\tilde{\Lambda}^n\|\,\|\nabla \tilde{\Lambda}^n\| \\
& \quad \le\; C\,\nu^{-1}\,\Delta t \sum_{n=1}^{l} \left(\|\Lambda^n\|\,\|\nabla \Lambda^n\| \;+\; \|\Lambda^{n-1}\|\,\|\nabla \Lambda^{n-1}\| \;+\; \|\Lambda^{n-1}\|\,\|\nabla \Lambda^n\| + \|\Lambda^n\|\,\|\nabla \Lambda^{n-1}\| \right) \|\nabla \tilde{u}^n\|^2 \\
& \quad \le\; C\,\nu^{-1}\,h^{2k+1} \Bigg(\Delta t \sum_{n=1}^{l} |u^n|_{k+1}^2\,\|\nabla \tilde{u}^n\|^2 \;+ \Delta t \sum_{n=1}^{l} |u^n|_{k+1} |u^{n-1}|_{k+1}\,\|\nabla \tilde{u}^n\|^2 \\
& \qquad\qquad +\; \Delta t \sum_{n=1}^{l} |u^{n-1}|_{k+1}^2\,\|\nabla \tilde{u}^n\|^2 \Bigg) \\
& \quad \le C\,\nu^{-1}\,h^{2k+1} \left(\Delta t \sum_{n=0}^{l} |u^n|_{k+1}^4 \;+\; \Delta t \sum_{n=0}^{l} \;\|\nabla u^n\|^4 \right) \\
& \quad \le C\,\nu^{-1}\,h^{2k+1} \left(|||u|||_{4,k+1}^4 \;+\; |||\nabla u|||_{4,0}^4 \right)\,. \qquad (5.34)
\end{aligned}
$$

Using the boundedness of $\nu\,\Delta t\,\sum_{n=1}^{l} \|\nabla \breve{w}_h^n\|$ (Proposition 4.3)

$$
\begin{aligned}
C\,\nu^{-1}\,\Delta t \sum_{n=1}^{l} \|\nabla \breve{w}_h^n\|^2 \|\nabla \tilde{\Lambda}^n\|^2 &\le C\,\nu^{-1}\,\Delta t \sum_{n=1}^{l} \|\nabla \breve{w}_h^n\|^2 \frac{1}{2} h^{2k} (\|u^n\|_{k+1}^2 \;+\; \|u^{n-1}\|_{k+1}^2) \\
&\le C\nu^{-2}\,h^{2k}\,|||u|||_{\infty,k+1}^2\,. \qquad (5.35)
\end{aligned}
$$

Next

$$
C \sum_{n=1}^{l} \int_{t^{n-1}}^{t^n} \|\Lambda_t\|^2\,dt \;\le\; C \sum_{n=1}^{l} \int_{t^{n-1}}^{t^n} h^{2k+2} \|u_t\|^2\,dt \;\le\; C\,h^{2k+2}\,\|u_t\|_{2,k+1}^2\,. \qquad (5.36)
$$

$$
\Delta t \sum_{n=1}^{l} C\,\nu^{-1} \|p^{n-1/2} - q_h\|^2 \;\le\; C\,\nu^{-1}\,\Delta t \sum_{n=1}^{l} h^{2s+2}\,\|p^{n-1/2}\|_{s+1}^2 \quad \le C\,\nu^{-1}\,h^{2s+2}\,|||p_{1/2}|||_{2,s+1}^2\,. \qquad (5.37)
$$

From Lemma 5.1 and Assumption A2, and assuming $\Delta t < 1$

$$
\begin{aligned}
& C\,\Delta t \sum_{n=0}^{l-1} (1 + \nu^{-3}\,\|\nabla \tilde{u}^n\|^4 + (\Delta t)^{-1}) \left(1 \;+\; (\Delta t)^{-1}\right) \chi^2 \|(I - D_h G_h)\Lambda^n\|^2 \\
& \qquad \le\; C\,\Delta t \sum_{n=0}^{l-1} \left(\nu^{-3}\,\|\nabla \tilde{u}^n\|^4 + (\Delta t)^{-1} \right)\,(\Delta t)^{-1}\,\chi^2 \|I - D_h G_h\|^2\,\|\Lambda^n\|^2 \\
& \qquad \le\; C\,\Delta t \sum_{n=0}^{l-1} \left(\nu^{-3}\,\|\nabla \tilde{u}^n\|^4 + (\Delta t)^{-1} \right)\,(\Delta t)^{-1}\,\chi^2\,h^{2k+2}\,\|u^n\|_{k+1}^2 \\
& \qquad \le\; C\,\chi^2\,\left(\nu^{-3} |||\nabla u|||_{\infty,0}^4 + (\Delta t)^{-1} \right)\,(\Delta t)^{-1}\,h^{2k+2}\,|||u|||_{2,k+1}^2\,. \qquad (5.38)
\end{aligned}
$$

$$
\begin{aligned}
& C\,\Delta t \sum_{n=0}^{l-1} \left(1 + \nu^{-3}\,\|\nabla \tilde{u}^n\|^4 + (\Delta t)^{-1}\right)\,\left(1 \;+\; (\Delta t)^{-1}\right)\,\chi^2 \|(I - D_h G_h) u^n\|^2 \\
& \quad \le\; C\,\Delta t \sum_{n=0}^{l-1} \left(\nu^{-3}\,\|\nabla \tilde{u}^n\|^4 + (\Delta t)^{-1} \right)\,(\Delta t)^{-1}\,(\delta^{4N+4} \;+\; \delta^2\,h^{2k} \;+\; h^{2k+2})\,\chi^2\,(\|u^n\|_{2N+2}^2 \;+\; \|u^n\|_{k+1}^2) \\
& \le\; C\,\chi^2\,\left(\nu^{-3}\,|||\nabla u|||_{\infty,0}^4 + (\Delta t)^{-1} \right)\,(\Delta t)^{-1}\,(\delta^{4N+4} \;+\; \delta^2\,h^{2k} \;+\; h^{2k+2})\,(|||u|||_{2,2N+2}^2 \;+\; |||u|||_{2,k+1}^2)\,. \qquad (5.39)
\end{aligned}
$$

As given in [ELN07] the *interpolation error term* $2\Delta t\,\sum_{n=1}^{l}\,|Intp(u^n;\breve{F}^n)|$ in (5.32) can be bounded as

$$
\begin{aligned}
2\Delta t\sum_{n=1}^{l}|Intp(u^n;\breve{F}^n)| \le \Delta t\, C\sum_{n=1}^{l}\left(\|F^n\|^2 + \|E^{n-1}\|^2\right) + (\epsilon_1 + \epsilon_2 + \epsilon_3)\Delta t\,\nu\sum_{n=1}^{l}\|\nabla\breve{F}^n\|^2 \\
+ C\,(\Delta t)^4\left(\|u_{ttt}\|_{2,0}^2 + \|f_{tt}\|_{2,0}^2 + \nu\|\nabla u_{tt}\|_{2,0}^2\right. \\
\left.+\nu^{-1}\|\nabla u_{tt}\|_{4,0}^4 + \nu^{-1}|||\nabla u|||_{4,0}^4 + \nu^{-1}|||\nabla u_{1/2}|||_{4,0}^4\right)\,. \qquad (5.40)
\end{aligned}
$$

Combining (5.33)–(5.40), equation (5.32) simplifies to

$$
\begin{aligned}
&\|F^l\|^2 + \Delta t\sum_{n=1}^{l}\nu\|\nabla\breve{F}^n\|^2 \\
&\le\; C\,\Delta t\sum_{n=0}^{l}\left(1 + \nu^{-3}\|\nabla\tilde{u}^n\|^4\right)\|F^n\|^2 \\
&+ C\nu^{-1}\left(h^{2k+1}|||u|||_{4,k+1}^4 \;+\; h^{2k+1}|||\nabla u|||_{4,0}^4 \;+\; h^{2s+2}|||p_{1/2}|||_{2,s+1}^2\right) \;+\; C\,\nu\, h^{2k}|||u|||_{2,k+1}^2 \\
&+ C\,h^{2k+2}\,\|u_t\|_{2,k+1}^2 + C\,\nu^{-2}\,h^{2k}|||u|||_{\infty,k+1}^2 \\
&+ C\,\chi^2\,\left(\nu^{-3}\,|||\nabla u|||_{\infty,0}^4 + (\Delta t)^{-1}\right)\,(\Delta t)^{-1}\,h^{2k+2}\,|||u|||_{2,k+1}^2 \\
&+ C\,\chi^2\,\left(\nu^{-3}\,|||\nabla u|||_{\infty,0}^4 + (\Delta t)^{-1}\right)\,(\Delta t)^{-1}\,(\delta^{4N+4} + \delta^2\,h^{2k} + h^{2k+2})\,(|||u|||_{2,2N+2}^2 + |||u|||_{2,k+1}^2) \\
&+\; C\,(\Delta t)^4\left(\|u_{ttt}\|_{2,0}^2 + \|f_{tt}\|_{2,0}^2 + \nu\|\nabla u_{tt}\|_{2,0}^2\right. \\
&\qquad\left.+\nu^{-1}\|\nabla u_{tt}\|_{4,0}^4 + \nu^{-1}|||\nabla u|||_{4,0}^4 + \nu^{-1}|||\nabla u_{1/2}|||_{4,0}^4\right)\,. \qquad (5.41)
\end{aligned}
$$

Hence, with Δt sufficiently small, i.e. $\Delta t < C(1 + \nu^{-3}|||\nabla u|||_{\infty,0}^4)^{-1}$, from the discrete Gronwall's Lemma [HeRa], we have

$$
\begin{aligned}
&\|F^l\|^2 + \Delta t\sum_{n=1}^{l}\nu\|\nabla\breve{F}^n\|^2 \\
&\le\; C\nu^{-1}\left(h^{2k+1}|||u|||_{4,k+1}^4 \;+\; h^{2k+1}|||\nabla u|||_{4,0}^4 \;+\; h^{2s+2}|||p_{1/2}|||_{2,s+1}^2\right) \;+\; C\,\nu\, h^{2k}|||u|||_{2,k+1}^2 \\
&+ C\,h^{2k+2}\,\|u_t\|_{2,k+1}^2 + C\,\nu^{-2}\,h^{2k}|||u|||_{\infty,k+1}^2 \\
&+ C\,\chi^2\,\left(\nu^{-3}\,|||\nabla u|||_{\infty,0}^4 + (\Delta t)^{-1}\right)\,(\Delta t)^{-1}\,h^{2k+2}\,|||u|||_{2,k+1}^2 \\
&+ C\,\chi^2\,\left(\nu^{-3}\,|||\nabla u|||_{\infty,0}^4 + (\Delta t)^{-1}\right)\,(\Delta t)^{-1}\,(\delta^{4N+4} + \delta^2\,h^{2k} + h^{2k+2})\,(|||u|||_{2,2N+2}^2 + |||u|||_{2,k+1}^2) \\
&+\; C\,(\Delta t)^4\left(\|u_{ttt}\|_{2,0}^2 + \|f_{tt}\|_{2,0}^2 + \nu\|\nabla u_{tt}\|_{2,0}^2\right. \\
&\qquad\left.+\nu^{-1}\|\nabla u_{tt}\|_{4,0}^4 + \nu^{-1}|||\nabla u|||_{4,0}^4 + \nu^{-1}|||\nabla u_{1/2}|||_{4,0}^4\right)\,. \qquad (5.42)
\end{aligned}
$$

The estimate given in (5.13) for $|||u - w_h|||_{\infty,0}$ then follows from the triangle inequality and (5.42). The estimate for $|||u - u_h|||_{\infty,0}$ follows from the estimate (5.30), the triangle inequality, and the estimate for $|||u - w_h|||_{\infty,0}$. To obtain (5.14), we use (5.42) and

$$
\begin{aligned}
\|\nabla\left(u^{n-1/2} - (w_h^n + u_h^{n-1})/2\right)\|^2 &\le \|\nabla(u^{n-1/2} - \tilde{u}^n)\|^2 + \|\nabla\tilde{\Lambda}^n\|^2 + \|\nabla\breve{F}^n\|^2 \\
&\le \frac{(\Delta t)^3}{48}\int_{t_{n-1}}^{t_n}\|\nabla u_{tt}\|^2\,dt + Ch^{2k}|u^n|_{k+1}^2 + Ch^{2k}|u^{n-1}|_{k+1}^2 \\
&\quad + \|\nabla\breve{F}^n\|^2\,.
\end{aligned}
$$

□

For Taylor-Hood approximating elements, i.e. $k = 2$, $s = 1$, we have the following asymptotic estimate.

COROLLARY 5.5. *Under the assumptions of Theorem 5.4, with $\delta = Ch$, $\Delta t = Ch$, $N = 1$, and (X_h, Q_h) given by the Taylor-Hood approximation elements, we have*

$$
|||u - w_h|||_{\infty,0} + |||u - u_h|||_{\infty,0} + \left(\nu\Delta t\sum_{n=1}^{l}\|\nabla(u^{n-1/2} - (w_h^n + u_h^{n-1})/2)\|^2\right)^{1/2} \le C\left((\Delta t)^2 + h^2\right)\,. \qquad (5.43)
$$

The *filter-deconvolve-relax* step is more than a simple perturbation of the usual FEM at each time step. Computationally its positive influence on the stability of the approximation algorithm is very apparent. Mathematically to see this increased stability we need to consider a different norm than the L^2 (in space) norm. Define the (mesh dependent) weighted norm and inner product

$$(v, w)_{IDG} := ((I - D_h G_h)v, w) , \text{ and } \|v\|_{IDG} := \sqrt{(v, v)_{IDG}} . \tag{5.44}$$

From Figure 1.1 we see that this norm has the property of measuring high frequency components of a function. Estimate (5.45) shows that the high frequency components of w_h^n, where spurious oscillations would concentrate, are reduced in u_h^n, i.e. $\|u_h^n\|_{IDG} < \|w_h^n\|_{IDG}$. Estimates (5.46), (5.47), establishes a relationship between the L^2 errors of u_h^n and w_h^n, and the high frequency components of the error in u_h^n and w_h^n, i.e. $\|u^n - u_h^n\|_{IDG}$ and $\|u^n - w_h^n\|_{IDG}$, respectively. We remark that the terms in the brackets on the RHS of (5.46) and (5.47) represent consistency error terms for the *filter-deconvolve* operation.

THEOREM 5.6. *Under the assumptions of Theorem 5.4, for* $n = 1, 2, \ldots, N_T$, $0 \le \delta \le 1$,

$$\|u_h^n\|_{IDG}^2 = \|w_h^n\|_{IDG}^2 - \chi\,(2-\chi)\,\|(I-D_hG_h)w_h^n\|^2 - \chi^2\,\left((I - D_hG_h)w_h^n\,,\,(I - D_hG_h)D_hG_h\,w_h^n\right)\,, \tag{5.45}$$

$$\begin{aligned} ||u^n - u_h^n||^2 \le{} & ||u^n - w_h^n||^2 - \frac{3}{2}\chi(1 - \chi)\|u^n - w_h^n\|_{IDG}^2 \\ & + \left[\chi^2||(I - D_hG_h)u^n||^2 + \frac{1}{2}\chi(1 + 2\chi)\|u^n\|_{IDG}^2 + \chi^2((I - D_hG_h)u^n, D_hG_hu^n)\right] . \end{aligned} \tag{5.46}$$

$$\begin{aligned} \|u^n - u_h^n\|_{IDG}^2 \le{} & \|u^n - w_h^n\|_{IDG}^2 - \chi^2((I - D_hG_h)(u^n - w_h^n), (I - D_hG_h)D_hG_h(u^n - w_h^n)) \\ & - \frac{3}{2}\chi(1 - \chi)\|(I - D_hG_h)(u^n - w_h^n)\|^2 \\ & + \left[2\chi(1 + \chi)\|(I - D_hG_h)u^n\|^2 + \chi^2\|(I - D_hG_h)u^n\|_{IDG}^2\right] . \end{aligned} \tag{5.47}$$

Proof. We note that from **Assumption A3**, $((I - D_hG_h)v\,,\,D_hG_h\,v) \ge 0$, and $((I - D_hG_h)v\,,\,(I - D_hG_h)D_hG_h\,v) \ge 0$. To simplify notation, in the proof we use $e := e^n = u(t^n) - u_h^n$ and $\varepsilon := \varepsilon^n = u(t^n) - w_h^n$, and IDG to denote $I - D_hG_h$. From (5.7), taking the inner-product of both sides with respect to $(I - D_hG_h)u_h^n = IDGu_h^n$,

$$\begin{aligned} (u_h^n\,,\,IDGu_h^n) &= (w_h^n - \chi\,IDGw_h^n\,,\,IDG(w_h^n - \chi\,IDGw_h^n)) \\ &= (w_h^n, IDGw_h^n) + \chi^2\left(IDGw_h^n\,,\,(IDG)^2w_h^n\right) - \chi\left(w_h^n\,,\,(IDG)^2w_h^n\right) \\ &\quad - \chi\,(IDGw_h^n\,,\,IDGw_h^n) \\ &= (w_h^n\,,\,IDGw_h^n) + \chi^2\,(IDGw_h^n\,,\,IDGw_h^n) - \chi^2\,(IDGw_h^n\,,\,(IDG)D_hG_hw_h^n) \\ &\quad - 2\chi\,(IDGw_h^n\,,\,IDGw_h^n) \\ &= (w_h^n\,,\,IDGw_h^n) - \chi(2-\chi)\,(IDGw_h^n\,,\,IDGw_h^n) - \chi^2\,(IDGw_h^n\,,\,(IDG)D_hG_hw_h^n)\,, \end{aligned}$$

which establishes (5.45).

To establish (5.47) we begin with (5.29). Taking the inner-product of both sides with respect to $IDG\varepsilon$,

$$(e, IDG\varepsilon) = (\varepsilon, IDG\varepsilon) - \chi(IDG\varepsilon, IDG\varepsilon) + \chi(IDGu^n, IDG\varepsilon)\,,$$

i.e.

$$\frac{1}{2}\|e\|_{IDG}^2 + \frac{1}{2}\|\varepsilon\|_{IDG}^2 - \frac{1}{2}\|e - \varepsilon\|_{IDG}^2 = \|\varepsilon\|_{IDG}^2 - \chi(IDG\varepsilon, IDG\varepsilon) + \chi(IDGu^n, IDG\varepsilon)\,.$$

Thus,

$$\|\varepsilon\|_{IDG}^2 = \|e\|_{IDG}^2 - \|e - \varepsilon\|_{IDG}^2 + 2\chi(IDG\varepsilon, IDG\varepsilon) - 2\chi(IDGu^n, IDG\varepsilon)\,. \tag{5.48}$$

In addition, rearranging (5.29) we have

$$e - \varepsilon \;=\; -\chi IDG\varepsilon \,+\, \chi IDGu^n$$

and thus,

$$\begin{aligned}\|e-\varepsilon\|_{IDG}^2 &= ((e-\varepsilon), IDG(e-\varepsilon)) &(5.49)\\ &= ((-\chi IDG\varepsilon \,+\, \chi IDGu^n)\,,\, IDG(-\chi IDG\varepsilon \,+\, \chi IDGu^n))\\ &= \chi^2\|IDG\varepsilon\|_{IDG}^2 \,+\, \chi^2\|IDGu^n\|_{IDG}^2 \,-\, 2\chi^2(IDG\varepsilon\,,\,(IDG)^2u^n)\,. &(5.50)\end{aligned}$$

Substituting (5.50) into (5.48) and rearranging

$$\begin{aligned}\|\varepsilon\|_{IDG}^2 = \|e\|_{IDG}^2 \,-\, \chi^2\|IDG\varepsilon\|_{IDG}^2 \,-\, \chi^2\|IDGu^n\|_{IDG}^2 \,+\, 2\chi^2(IDG\varepsilon\,,\,(IDG)^2u^n)\\ +\, 2\chi(IDG\varepsilon, IDG\varepsilon) \,-\, 2\chi(IDGu^n, IDG\varepsilon)\,. \qquad (5.51)\end{aligned}$$

Note that

$$\begin{aligned}2\chi(IDG\varepsilon, IDG\varepsilon) \,-\, \chi^2\|IDG\varepsilon\|_{IDG}^2 &= 2\chi(IDG\varepsilon, IDG\varepsilon) \,-\, \chi^2(IDG\varepsilon, (IDG)(IDG)\varepsilon)\\ &= \,(2\chi \,-\, \chi^2)(IDG\varepsilon, IDG\varepsilon) + \chi^2(IDG\varepsilon, D_hG_h(IDG)\varepsilon)\,, &(5.52)\end{aligned}$$

$$2\chi^2(IDG\varepsilon\,,\,(IDG)^2u^n) \;\le\; \frac{1}{2}\chi^2(IDG\varepsilon\,,\,IDG\varepsilon) \,+\, 2\chi^2((IDG)^2u^n\,,\,(IDG)^2u^n)\,, \qquad (5.53)$$

$$2\chi(IDGu^n, IDG\varepsilon) \;\le\; \frac{1}{2}\chi(IDG\varepsilon, IDG\varepsilon) \,+\, 2\chi(IDGu^n, IDGu^n)\,. \qquad (5.54)$$

Thus, using (5.52)-(5.54) in (5.51), we obtain

$$\begin{aligned}\|\varepsilon\|_{IDG}^2 \ge \|e\|_{IDG}^2 \,+\, \frac{3}{2}\chi(1 \,-\, \chi)\|IDG\varepsilon\|^2 \,+\, \chi^2(IDG\varepsilon, D_hG_h(IDG)\varepsilon)\\ -\, 2\chi(1 \,+\, \chi)\|IDGu^n\|^2 \,-\, \chi^2\|IDGu^n\|_{IDG}^2\,.\end{aligned}$$

For the final result we begin again with $e - \varepsilon \;=\; -\chi IDG\varepsilon \,+\, \chi IDGu^n$. Take the inner product with ε, use the polarization identity on the (e, ε) term, multiply by 2 and simplify. This gives

$$||e||^2 + 2\chi(IDG\varepsilon, \varepsilon) = ||\varepsilon||^2 + ||e-\varepsilon||^2 + 2\chi(IDGu, \varepsilon). \qquad (5.55)$$

Taking norms we also have $||e-\varepsilon||^2 = ||\chi IDG\varepsilon - \chi IDGu^n||^2 = \chi^2(IDG\varepsilon \,-\, IDGu^n, IDG\varepsilon \,-\, IDGu^n)$. Inserting this into the RHS of (5.55), expanding and collecting terms yields

$$\begin{gathered}||e||^2+\\ \chi\left\{(2-\chi)\,(IDG\varepsilon, \varepsilon) + \chi(IDG\varepsilon, DG\varepsilon) - \chi||IDGu||^2 - 2(IDGu, \varepsilon) + 2\chi(IDG\varepsilon, IDGu)\right\}\\ = ||\varepsilon||^2.\end{gathered}$$

For the last two terms inside the braces we have, using operator weighted Cauchy-Schwarz inequalities,

$$\begin{aligned}2(IDGu, \varepsilon) &\le 2(IDGu, u) \,+\, \frac{1}{2}(IDG\varepsilon, \varepsilon),\\ 2\chi(IDG\varepsilon, IDGu) &= 2\chi(IDG\varepsilon, u) - 2\chi(IDG\varepsilon, DGu)\\ &\le 2\chi(IDGu, u) + \frac{1}{2}\chi(IDG\varepsilon, \varepsilon) + \chi(IDG\varepsilon, DG\varepsilon) + \chi(IDGu, DGu).\end{aligned}$$

Inserting these two into the braces and collecting terms, we have, as claimed:

$$\begin{aligned}||e||^2 \le ||\varepsilon||^2 - \frac{3}{2}\chi(1 \,-\, \chi)(IDG\varepsilon, \varepsilon)\\ +\left[\chi^2||IDGu||^2 + 2\chi(1+\chi)](IDGu, u) + \chi^2(IDGu, DGu)\right].\end{aligned}$$

□

6. Numerical Experiments. In this section, we present numerical experiments to test the algorithms presented herein. Using the Green-Taylor vortex problem, we confirm the predicted convergence rates of the previous section, and also use it to compare the accuracy of the algorithms. Further testing is then performed using the flow around a cylinder benchmark problem. We use the software *FreeFEM++* [HePi] to run the numerical tests which provided a multi-frontal Gauss LU factorization for the linear solver. The scheme presented in Algorithm 4.1 is discretized in space using the finite element method with Taylor-Hood elements (continuous piecewise quadratic polynomials for the velocity and continuous linears for the pressure). The nonlinear system was solved by a fixed point iteration. For the Stokes filter, used in all the computations, we applied the same boundary conditions as given for the problem being solved.

m	$\|\|\|u - u_h\|\|\|_{\infty,0}$	rate	$\|\|\|\nabla u - \nabla u_h\|\|\|_{2,0}$	rate
16	$2.55117 \cdot 10^{-2}$		3.36582	
24	$2.05028 \cdot 10^{-2}$	0.54	2.27963	0.46
32	$1.35885 \cdot 10^{-2}$	1.43	1.6152	1.18
40	$8.1897 \cdot 10^{-3}$	2.27	1.18214	1.73
48	$5.01347 \cdot 10^{-3}$	2.69	$8.85739 \cdot 10^{-1}$	2.01
56	$3.20674 \cdot 10^{-3}$	2.90	$6.76137 \cdot 10^{-1}$	2.20
64	$2.14058 \cdot 10^{-3}$	3.03	$5.24409 \cdot 10^{-1}$	2.33
72	$1.48243 \cdot 10^{-3}$	3.12	$4.12513 \cdot 10^{-1}$	2.44

TABLE 6.1
Errors and convergence rates for order of deconvolution $N = 1$.

m	$\|\|\|u - u_h\|\|\|_{\infty,0}$	rate	$\|\|\|\nabla u - \nabla u_h\|\|\|_{2,0}$	rate
16	$2.72906 \cdot 10^{-2}$		$3.09567 \cdot 10^{-1}$	
24	$2.64931 \cdot 10^{-2}$	0.07	$3.01949 \cdot 10^{-1}$	0.06
32	$2.5414 \cdot 10^{-2}$	0.14	$2.91435 \cdot 10^{-1}$	0.12
40	$2.40916 \cdot 10^{-2}$	0.24	$2.78326 \cdot 10^{-1}$	0.21
48	$2.2577 \cdot 10^{-2}$	0.36	$2.63147 \cdot 10^{-1}$	0.31
56	$2.09387 \cdot 10^{-2}$	0.49	$2.46598 \cdot 10^{-1}$	0.42
64	$1.9251 \cdot 10^{-2}$	0.63	$2.29443 \cdot 10^{-1}$	0.54

TABLE 6.2
Errors and convergence rates for order of deconvolution $N = 0$.

m	$\|\|\|u - u_h\|\|\|_{\infty,0}$	rate	$\|\|\|\nabla u - \nabla u_h\|\|\|_{2,0}$	rate
16	$2.54983 \cdot 10^{-2}$		$2.93004 \cdot 10^{-1}$	
24	$2.0471 \cdot 10^{-2}$	0.54	$2.42837 \cdot 10^{-1}$	0.46
32	$1.35493 \cdot 10^{-2}$	1.43	$1.72972 \cdot 10^{-1}$	1.18
40	$2.40916 \cdot 10^{-3}$	2.27	$1.17613 \cdot 10^{-1}$	1.73
48	$4.99331 \cdot 10^{-3}$	2.69	$8.1466 \cdot 10^{-2}$	2.01
56	$3.19319 \cdot 10^{-3}$	2.90	$5.80612 \cdot 10^{-2}$	2.18
64	$2.13123 \cdot 10^{-3}$	3.03	$4.25236 \cdot 10^{-2}$	2.33

TABLE 6.3
Errors and convergence rates for order of deconvolution $N = 1$ with relaxation $\chi = \Delta t$.

6.1. Convergence Rate Verification. Our first test is designed to test (and does confirm) the predicted rates of convergence. The problem of simulating decay of the Green-Taylor vortex, [GT37], [T23], is an interesting test problem in which the true solution is known. It was used as a numerical test in Chorin [Cho68], Tafti [Tafti] and John and Layton [JL02]. For a very insightful and detailed analysis of the problem for LES models see Barbato, Berselli and Grisanti [BBG07] and Berselli [B05]. The prescribed solution in

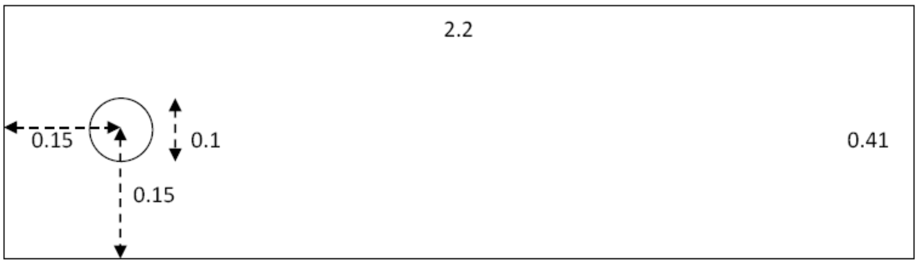

FIG. 6.1. *Domain.*

$\Omega = (0,1) \times (0,1)$ is given by

$$u_1(x,y,t) = -\cos(\omega\pi x)\sin(\omega\pi y)e^{-2\omega^2\pi^2 t/\tau}\,,$$
$$u_2(x,y,t) = \sin(\omega\pi x)\cos(\omega\pi y)e^{-2\omega^2\pi^2 t/\tau}\,,$$
$$p(x,y,t) = -\frac{1}{4}(\cos(2\omega\pi x) + \cos(2\omega\pi y))e^{-2\omega^2\pi^2 t/\tau}\,.$$

When the relaxation time $\tau = Re$, this is a solution of the NSE with $f = 0$, consisting of an $\omega \times \omega$ array of oppositely signed vortices that decay as $t \to \infty$.

In our test, with Dirichlet boundary conditions for the velocity, we choose $\omega = 1$, $\triangle t = 0.005$, $T = 1$, Reynolds number $Re = 100$ and $\delta = h = 1/m$, where m is the number of subdivisions of the interval $(0,1)$. The results for the approximation method described in Algorithm 4.1 are presented in Table 6.1, using order of deconvolution $N = 1$ without relaxation (i.e. $\chi = 0$) and in Table 6.3 for $N = 1$ with relaxation for $\chi = \Delta t$. Results using the simple averaging filter, i.e. deconvolution with order $N = 0$, are presented in Table 6.2. The convergence rate is calculated from the error at two successive values of h in the usual manner by postulating $e(h) = Ch^\beta$ and solving for β via $\beta = \ln(e(h_1)/e(h_2))/\ln(h_1/h_2)$.

From the tables we see the convergence rate approaches the second order predicted for $|||\nabla u - \nabla u_h|||_{2,0}$ and we also see what appears to be an L^2 lift for $|||u-u_h|||_{\infty,0}$ for order of deconvolution $N = 1$. The method with order of deconvolution $N = 0$, has much larger errors and slower rates of convergence, as expected. From this test it is clear that *deconvolution makes an important contribution to improving the accuracy of the approximation.*

6.2. Flow around a cylinder. Our next numerical illustration is for two dimensional under-resolved flow around a cylinder. We compute values for the maximal drag $c_{d,max}$ and lift $c_{l,max}$ coefficient at the cylinder, and for the pressure difference $\Delta p(t)$ between the front and back of the cylinder at the final time $T = 8$. This is a well known benchmark problem taken from Shäfer and Turek [ST96] and John [J04]. It is not turbulent but does have interesting features. The flow patterns are driven by the interaction of a fluid with a wall which is an important scenario for industrial flows. This simple flow can be difficult to simulate successfully by a model with sufficient regularization to handle higher Reynolds number problems. The domain is presented in Figure 6.1.

The time dependent inflow profile is

$$u_1(0,y,t) = u_1(2.2,y,t) = \frac{6}{0.41^2}\sin(\pi\, t/8)y(0.41-y)\,,$$
$$u_2(0,y,t) = u_2(2.2,y,t) = 0.$$

No slip boundary conditions are prescribed along the top and bottom walls, “do-nothing” at the outflow and the initial condition is $u(x,y,0) = 0$. The viscosity is $\nu = 10^{-3}$ and the external force $f = 0$. The Reynolds number of the flow, based on the diameter of the cylinder and on the mean velocity inflow is $0 \le Re \le 100$. A mesh with 62757 number of degrees of freedom is used for all simulation for a clear comparison of the different algorithms presented in this report. The filter radius is chosen as the length of the cylinder divided by the number of mesh points around the cylinder.

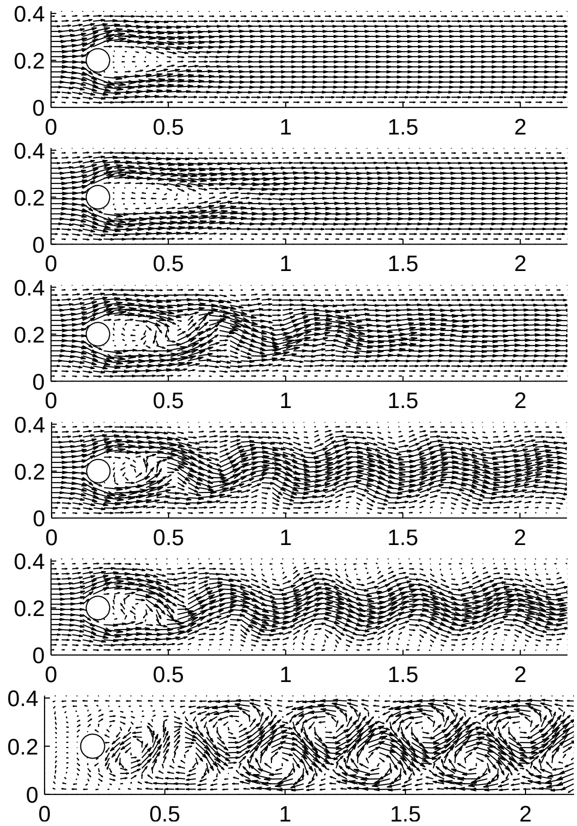

FIG. 6.2. *The velocity at t = 2, 4, 5, 6, 7, and 8 of Algorithm 4.1, with $N = 1$ and $\chi = \Delta t = 0.01$.*

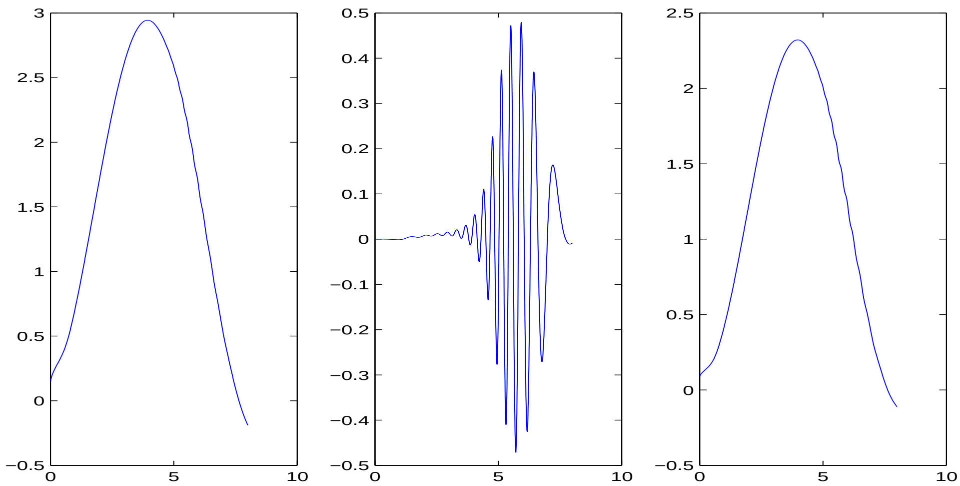

FIG. 6.3. *The development of $c_d(t)$, $c_l(t)$ and $\Delta p(t)$ of Algorithm 4.1 with $N = 1$ and $\chi = \Delta t = 0.01$.*

For this setting, it is expected that as the flow increases from time $t = 2$ to $t = 4$ two vortices start to develop behind the cylinder. Between $t = 4$ and $t = 5$, the vortices separate from the cylinder, so that a vortex street develops, and they continue to be visible through the final time $t = 8$. This can be seen in Figure 6.2.

The evolutions of $c_{d,max}$, $c_{l,max}$ and Δp in time are presented in Figure 6.3.

For the computation of drag and lift coefficients we used the one dimensional method described by John in [J04]. Results on the computations of maximal drag and lift coefficients and pressure drop, for $N = 1$, are presented in Table 6.4. The following reference intervals are given in [ST96]:

$$c_{d,max}^{ref} \in [2.93,\ 2.97], \quad c_{l,max}^{ref} \in [0.47,\ 0.49], \quad \Delta p^{ref} \in [-0.115,\ -0.105]$$

relax. coeff.	Δt	t($c_{d,max}$)	$c_{d,max}$	t($c_{l,max}$)	$c_{l,max}$	Δp(8s)
$\chi = 0$	0.04	3.96	2.87862	6.12	0.36572	-0.103274
	0.02	3.94	2.85123	5.98	0.420198	-0.103153
	0.01	3.94	2.80214	5.96	0.409368	-0.106276
$\chi = \Delta t$	0.04	3.96	2.94233	6.12	0.392735	-0.102161
	0.02	3.94	2.94317	5.98	0.458007	-0.106762
	0.01	3.94	2.94352	5.93	0.479286	-0.110899

TABLE 6.4

Results for drag/lift coefficients and pressure difference for deconvolution order $N = 1$.

The computations in Table 6.4 show that *evolve-filter-deconvolve-relax*, Algorithm 4.1, computes the drag and lift coefficients, and the pressure difference, within the benchmark intervals, and illustrates the positive role of using relaxation in the approximation algorithm. Computations were also performed for higher Reynolds number corresponding to $\nu = 10^{-4}$. For this case the direct approximation approach (no regularization) failed (i.e. the fixed point iteration for the discretization of the nonlinear term of the Navier-Stokes equations stopped to converge around $T = 6$), whereas computations using Algorithm 4.1 were successful for $N = 1$, see Figure 6.4. There are no benchmark intervals for lift and drag coefficients for this case. Some quality of the approximation can be assessed by the appropriate appearance and evolution of the vortex street in the simulation. The direct approximation without regularization is much more sensitive to the instabilities that appear in the flow at higher Re than Algorithm 4.1, which produced a clear vortex street given in Figure 6.4 for $N = 1$.

7. Conclusions. In 2002 R. Peyret wrote in the fundamental book on spectral methods for the NSE [P02] that

> "... filters must be used with care and their effect evaluated with precision when tuning the parameters...The application of a filter in the course of time integration (especially if it is often applied) may be dangerous."

We have found that simple filtering at each step is indeed dangerous as it introduces large amounts of numerical diffusion. On the other hand, our precise analysis has also shown that with care, filtering plus deconvolution plus relaxation stabilizes marginally resolved scales and does *not* over diffuse. This combination can be an invaluable tool, with many algorithmic advantages, for enhancing stability without degrading accuracy through an extra uncoupled and modular step.

We have only considered the simplest filter and deconvolution operators that fit our mathematical tools. There are very many other possible filters and deconvolution operators that can be tested and await analysis. From one point of view, the process *Filter* $\rightarrow$ *Deconvolve* simply generates another filter which is closer to spectral cutoff and which can be applied by a sequence of simpler filter steps. Thus, both investigation of other filters and study of choices of relaxation parameters are important next steps.

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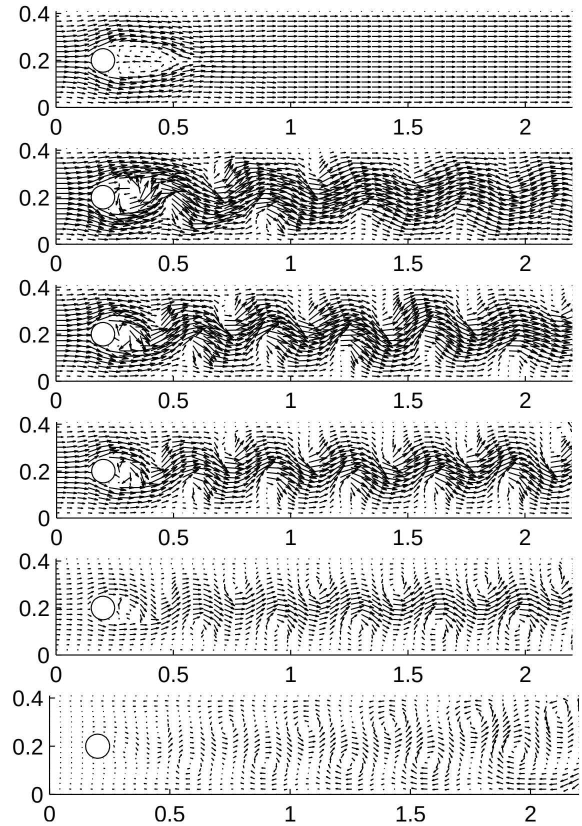

FIG. 6.4. *The velocity at t = 2, 4, 5, 6, 7, and 8 of Algorithm 4.1 for* $\nu = 10^{-4}$ *with* $N = 1$ *and* $\Delta t = 0.01$.

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