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METRIZABILITY OF TOPOLOGICAL SEMIGROUPS ON LINEARLY ORDERED TOPOLOGICAL SPACES

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Abstract. The authors use techniques and results from the theory of generalized metric spaces to give a new, short proof that every connected, linearly ordered topological space that is a cancellative topological semigroup is metrizable, and hence embeddable in $\mathbb{R}$. They also prove that every separable, linearly ordered topological space that is a cancellative topological semigroup is metrizable, so embeddable in $\mathbb{R}$.

1. Background and Introduction

A linearly ordered topological space (LOTS) L is a linearly ordered set L with the open interval topology. A cancellative topological semigroup on L is a semigroup with a continuous semigroup operation such that $ab = ac$, $ba = ca$ and $b = c$ are equivalent for any $a, b, c \in L$. A question that can be traced to Abel and Lie, and was listed as the second half of Hilbert's fifth problem, essentially asks whether a cancellative topological semigroup on a connected, linearly ordered topological space can be embedded in the real line. The history of the problem, and the various solutions and partial solutions and related questions, are most thoroughly documented by Hofmann and Lawson [7]. A very abbreviated excerpt from their exposition might be the following.

There were early contributions by Hölder (1901) and then solutions to restricted version of the question by Aczel (1948) and Tamari (1949); in [5] in 1958, Clifford pointed out that the arguments and results are further refined and expanded by theorems of Hofmann, Aczel, Criagan and Pales, among others (see [7]). Also a proof, using generalized metric techniques is given by Barnhart in [3]. That proof, however, is restricted to the abelian case. Here we give a fairly short proof using only generalized metric techniques. We also show that the theorem still holds if 'connected' is replaced by 'separable'. One might assume that would be a corollary of the theorem in [7] that 'a totally ordered set can be embedded in $\mathbb{R}$ if and only if it contains a countable subset C such that, for any $x < y$ there is a $c \in C$ with $x \leq c \leq y$' [with no mention of a semigroup]. The assumption that that hypothesis follows

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from separability is seen to been false. The space obtained from $[0,1]$ by replacing each point by a pair of adjacent points (the 'Double Arrow' space) is a compact, separable, linearly ordered topological space that can't be embedded in $\mathbb{R}$, and of course does not have the aforementioned property.

2. Metrizability

Theorem 1. *Every connected linearly ordered topological space L which is a cancellative topological semigroup is metrizable, and hence embeddable in $\mathbb{R}$.*

The theorem follows from the following propositions. Below L satisfies the conditions of the theorem. Note that, every closed and bounded subset of L is compact.

Proposition 2. *For any $a \in L$, the maps f_a given by $f_a(x) = ax$ and g_a given by $g_a(x) = xa$ are autohomeomorphisms of L.*

Proof. Fix $a \in L$. Since L is topological semigroup, f_a and g_a are continuous by the properties of topological semigroups. Also by the cancellativity of L, f_a and g_a are both one-to-one. Take $b, c \in L$ with $b < c$. Then $[b, c]$ is compact and connected, so f_a maps $[b, c]$ into the closed interval with endpoints $f_a(b)$ and $f_a(c)$. Therefore, f_a maps open intervals to open intervals, and so f_a^{-1} is continuous. Similarly, g_a^{-1} is continuous. □

From Proposition 2, we know directly that f_a and g_a are both either order-preserving or order-reversing.

Proposition 3. *The space L is first countable.*

Proof. Pick $a \in L$ and an increasing sequence $\{a_\alpha : \alpha < \gamma\}$ which converges to a from the left. Pick a countable subsequence $\{a_n : n \in \omega\}$ of $\{a_\alpha : \alpha < \gamma\}$. Then since L is connected and $\{a_n\}$ has an upper bound, $b = \sup .\{a_n : n \in \omega\}$ exists. If $b = a$, then we have nothing to do from the left. If not, consider f_a. Since f_a is a homeomorphism, $f_a(a_n)$ converges to ab. Also g_b is a homeomorphism which maps a to ab. Therefore the preimage of $\{a_n b : n \in \omega\}$ is a sequence which converges to a from the left.

By similar reasoning, we can get a countable sequence $\{b_n : n \in \omega\}$ converges to a from the right. Then we get that $\{(a_n, b_n) : n \in \omega\}$ is a countable local base at a. □

Proposition 4. *The sequence $\{a^n : n \in \omega\}$ is either constant or strictly monotone and unbounded for any $a \in L$.*

Proof. Three cases arise:

Case 1. Assume $a = a^2$. Then the sequence is obviously constant.

Case 2. Assume $a < a^2$. Since f_a is order-preserving, we know we need only to show $a^2 < a^3$. Suppose, for contradiction, $a^3 < a^2$.

If $p, q \in [a, a^2]$ and $p < q$, then we have the following two conditions:

i) if $ap < p$, then $aq < ap < p < q$. Therefore, $aq < q$.

ii) if $aq > q$, then $ap > aq > q > p$. Therefore, $ap > p$.

Thus we can take $I = \inf\{p : p \in [a, a^2], ap < p\}$ and $S = \sup\{p : p \in [a, a^2], ap > p\}$. It is obvious that $I, S \in [a^3, a^2]$ and $I \leq S$.

Consider the relationship between I and S. If $I = S$, then $aI = I$, and this contradicts $a < a^2$. If $I < S$, then for any m with $I \leq m \leq S$, we have $am = m$, and again we get a contradiction.

Case 3. Assume $a^2 < a$. Using a similar argument as in Case 2, we can show $a^3 < a^2$, as required.

Thus $\{a^n : n \in \omega\}$ is strictly monotone in Case 2 and Case 3. Unboundedness is easy to prove by contradiction. □

Note that from the above: if $a < a^2$, then $\{x \in L : a \leq x\}$ is a union of almost disjoint homeomorphic closed intervals. Also note that we now know f_a and g_a are both order-preserving.

Proposition 5. *Take $a \in L$ and assume without loss of generality $a < a^2$ and $a \neq \min L$. Then $L_a = [a, \infty)$ is metrizable.*

Proof. By Proposition 2, there exists sequences $x_1 < x_2 < \cdots$ and $y_1 > y_2 > \cdots$ that both converge to a. Define $g_n(a) = (x_n, y_n)$. For each $p \in [a^3, a^4]$, take $q \in [a^2, a^3]$ with $p = aq$ and let $g_n(p) = (x_n q, y_n q)$. Now we show that the neighborhood system $\{g_n(p), n \in \omega, p \in (a^3, a^4)\}$ satisfies the requirements of semi-metrizability.

Suppose $y \in [a^3, a^4]$ and for each n, $y \in g_n(p_n) = g_n(aq_n) = (x_n q_n, y_n q_n)$. Without loss of generality, assume $q_n \longrightarrow z$, then $x_n q_n \longrightarrow az$ and $y_n q_n \longrightarrow az$. Thus since $y \in (x_n q_n, y_n q_n)$ for each n, we have that $y = az$. Therefore $p_n \longrightarrow y$. It follows that $[a^3, a^4]$ is semi-metrizable and hence it is metrizable by the equivalence of the semi-metrizability and metrizability in LOTS [4]. Hence $L_a = \{x \in L : a \leq x\}$ is metrizable. And since L_a is connected and locally compact, it is separable. Hence L_a is embeddable in $\mathbb{R}$. □

Proof of Theorem 1. Here, without loss of generality we can assume there is an $a \in L$ with $a < a^2$. Next we will prove the theorem in three cases.

Case 1. $\min L = m$. Then it is easy to see that $m \leq m^2$. If $m < m^2$, then L is metrizable by Proposition 5. Otherwise, by Proposition 3, we can find $\{x_n : n \in \omega\}$ which converges to m from the right. Then for each $n \in \omega$, $x_n < (x_n)^2$. Hence L_{x_n} is metrizable for each n by Proposition 5. Therefore L is metrizable.

Case 2. $\min L$ does not exist and there is some $b \in L$ with $b^2 < b$. Then we can take $m = \inf\{a : a < a^2\}$. This follows because we can get $c < c^2$ from $a < c$ and $a < a^2$ from Proposition 4. Then it is easy to check $m = m^2$. Let $x_1 < x_2 < \cdots$ and $y_1 > y_2 > \cdots$ both converge to m. And let $R_{x_i} = \{a \in L : a \leq x_i\}$. A proof similar to that of Proposition 5, shows that R_{x_i} is metrizable for each i. Since L_{y_i} is also metrizable, we get that L is metrizable.

Case 3. $\min L$ does not exist and $a < a^2$ for any $a \in L$. If there is a countable co-initial decreasing sequence $\{x_n : n \in \omega\}$ which is unbounded, then $L = \bigcup_{n\in\omega} L_{x_n}$ is metrizable because L_{x_n} is metrizable for each $n \in \omega$. If not, we take $\{x_\alpha : \alpha \in \omega_1\}$ which is strictly decreasing and unbounded below without countable co-initial subsequence. Then we take $a \in L$. Consider the set $\{(x_\alpha)^m : m \in \omega, \alpha \in \omega_1\}$. Then we can find n_0, m_0 and α_0 such that $(x_\alpha)^{m_0} \in [a^{n_0}, a^{n_0+1}]$ for $\alpha > \alpha_0$. This is because $x < y \Rightarrow x^n < y^n$. Then we can suppose $\{(x_\alpha)^{m_0} : \alpha > \alpha_0\}$ converges to $b \in [a^{n_0}, a^{n_0+1}]$. This contradicts the first countability of L. So we get a contradiction. □

Next, we have another nice theorem about the metrizability of a separable linearly ordered topological space.

Theorem 6. *Every separable linearly ordered topological space which is also a cancellative topological semi-group is metrizable.*

Proof. Recall that a separable LOTS is metrizable if and only if its set of endpoints (points with either an immediate predecessor or an immediate successor) is at most countable. Also, note that in a separable LOTS every uncountable set contains a limit point, since every LOTS is monotonically normal

Assume L is a separable LOTS with uncountably many endpoints, and let L be a cancellative topological semi-group. Since L is separable, L has at most countably many isolated points. So we may assume, without loss of generality, that L has no isolated points (because the sum of nonisolated points can not be isolated, and L is semigroup). Then the endpoints occur in pairs of adjacent points. Let E be the set of all such adjacent point pairs, $x = (x_1, x_2)$ with $x_1 < x_2$.

Let L^* be the space obtained by identifying each pair (x_1, x_2) to a point x. Then L^* is metrizable with metric δ which induces a pseudometric d on L. Let $E = \{(x_1, x_2) : d(x_1, x_2) = 0\}$.

For each $x = (x_1, x_2) \in E$, we know $x_1 + x_2 \notin \{2x_1, 2x_2\}$ by the cancellativity of addition. Notice that $2x_1 = 2x_2$ is possible. So if $diam_d\{x_1 + x_2, 2x_1, 2x_2\} = 0$, then we can get $2x_1 = 2x_2$, and (p, q) or (q, p) is in E if $x_1 + x_2 = p$ and $2x_1 = q$.

Now consider the subset, $F = \{x \in E : diam_d\{x_1 + x_2, 2x_1, 2x_2\} > 0\}$, of L^*. Two cases arise:

Case 1. F is uncountable. Then since L is separable and monotonically normal, F has cluster points in L. That leads to a contradiction to the continuity of the operation and the distance function.

Case 2. F is countable. Then without loss of generality we can assume $B' = \{x \in E : (2x_1, x_1 + x_2) \in E\}$ is uncountable [otherwise, $\{x \in E : (x_1+x_2, 2x_1) \in E\}$ is uncountable]. Then, by separability, all but countably many points of B' are limit points of B'. By the continuity of the semi-group operation, for each $x \in B'$, there is $n_x \in \omega$ such that if $\delta(x, t) < 1/n_x$, then $t_1 + t_2 \leq 2x_1 = 2x_2 < x_1 + x_2$. Then there is $\varepsilon > 0$ such that

$G = \{x \in B' : 1/n_x > \varepsilon\}$ is uncountable. Now we can pick $x, z \in G$ such that $\delta(x, z) < \varepsilon$. It follows that

$$z_1 + z_2 < x_1 + x_2 < z_1 + z_2$$

which is a contradiction.

Thus L is metrizable. □

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