UNCONDITIONAL STABILITY OF A PARTITIONED IMEX METHOD FOR MAGNETOHYDRODYNAMIC FLOWS

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Abstract. The MHD flows are governed by the Navier-Stokes equations coupled with the Maxwell equations through coupling terms. We prove the unconditional stability of a partitioned method for the evolutionary full MHD equations, at *high* magnetic Reynolds number, in the Elsässer variables. The method we analyze is a first order, one step scheme, which consists of implicit discretization of the subproblem terms and explicit discretization of coupling terms.

1. Introduction. The equations of magnetohydrodynamics (MHD) describe the motion of electrically conducting, incompressible flows in the presence of a magnetic field. If an electrically conducting fluid moves in a magnetic field, the magnetic field exerts forces which may substantially modify the flow. Conversely, the flow itself gives rise to a second, induced field and thus modifies the magnetic field. Initiated by Alfvèn in 1942 [1], MHD is widely exploited in numerous branches of science including astrophysics and geophysics [18, 25, 13, 10, 9, 3, 6, 12], as well as engineering. Understanding MHD flows is central to many important applications, e.g., liquid metal cooling of nuclear reactors [2, 17, 27], process metallurgy [7], sea water propulsion [23].

The MHD flows involve different physical processes: the motion of fluid is governed by hydrodynamics equations and the magnetic field is governed by Maxwell equations. One approach to coupled problem is by monolithic methods. In these methods, the globally coupled problem is assembled at each time step and then solved iteratively. Partitioned methods, which solve the coupled problem by successively solving the sub-physics problems, are another attractive and promising approach for solving MHD system.

Most terrestrial applications, in particular most industrial and laboratory flows, involve *small* magnetic Reynolds number. In this cases, while the magnetic field considerably alters the fluid motion, the induced field is usually found to be negligible by comparison with the imposed field [7]. Neglecting the induced magnetic field one can reduce the MHD systems to the significantly simpler Reduced MHD (RMHD), for which several IMplicit-EXplicit schemes were studied in [22].

In this report, we prove the unconditional stability of a partitioned method for the evolutionary full MHD equations, at *high* magnetic Reynolds number, in the Elsässer variables. The method we study herein is a first order, one step scheme, which consists of implicit discretization of the subproblem terms and explicit discretization of coupling terms.

2. Magnetohydrodynamics. The equations of magnetohydrodynamics describing the motion of an incompressible fluid flow in presence of a magnetic field are the following (see, e.g. [21, 4, 5])

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nu\Delta\boldsymbol{u} + \nabla p = 0, \qquad \nabla\cdot\boldsymbol{u} = 0,$$
$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - \nu_m\Delta\boldsymbol{B} = 0, \qquad \nabla\cdot\boldsymbol{B} = 0,$$

in $\Omega\times(0,T)$, where Ω is the fluid domain, $\boldsymbol{u} = (u_1(\boldsymbol{x},t), u_2(\boldsymbol{x},t), u_3(\boldsymbol{x},t))$ is the fluid velocity, $p(\boldsymbol{x},t)$ is the pressure, $\boldsymbol{B} = (B_1(\boldsymbol{x},t), B_2(\boldsymbol{x},t), B_3(\boldsymbol{x},t))$ is the magnetic field, ν is

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the kinematic viscosity and ν_m is the magnetic resistivity. On the boundary we prescribe homogeneous Dirichlet boundary conditions.

The total magnetic field can be split in two parts $\boldsymbol{B} = \boldsymbol{B}_\circ + \boldsymbol{b}$ (mean and fluctuations). Then the Elsässer fields [11]

$$\boldsymbol{z}^+ = \boldsymbol{u} + \boldsymbol{b}, \quad \boldsymbol{z}^- = \boldsymbol{u} - \boldsymbol{b}, \tag{2.1}$$

merging the physical properties of the Navier-Stokes and Maxwell equations, suggest stable time-splitting schemes for the full MHD equations. The momentum equations, in the Elsässer variables, are

$$\frac{\partial \boldsymbol{z}^\pm}{\partial t} \mp (\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^\pm + (\boldsymbol{z}^\mp \cdot \nabla)\boldsymbol{z}^\pm - \frac{\nu+\nu_m}{2}\Delta \boldsymbol{z}^\pm - \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^\mp + \nabla p = 0, \tag{2.2}$$

while the continuity equations are $\nabla \cdot \boldsymbol{z}^\pm = 0$. We note that the nonlinear interactions occur between the Alfvènic fluctuations $\boldsymbol{z}^\pm$. The mean magnetic field plays an important role in MHD turbulence, for example it can make the turbulence anisotropic; suppress the turbulence by decreasing energy cascade, etc. In the presence of a strong mean magnetic field, $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ wavepackets travel in opposite directions with the phase velocity of $\boldsymbol{B}_\circ$, and interact weakly. For Kolmogorov's and Iroshnikov/Kraichnan's phenomenological theories of MHD isotropic and anisotropic turbulence, see [19, 20, 8, 24, 28, 26, 14, 15, 29].

3. First order unconditionally stable IMEX partitioned scheme. One approach to coupled MHD problem is monolithic methods, or implicit (fully coupled) algorithms, that are robust and stable, but quite demanding in computational time and resources. The method we propose and analyze herein has the coupling terms lagged, thus the system uncouples into two subproblem solves. Let approximate the momentum equations (2.2) and continuity equations in the Elsässer variables by the following first-order IMEX scheme (backward-Euler forward-Euler)

$$\begin{aligned} &\frac{\boldsymbol{z}^\pm_{n+1} - \boldsymbol{z}^\pm_n}{\Delta t} \mp (\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^\pm_{n+1} + (\boldsymbol{z}^\mp_n \cdot \nabla)\boldsymbol{z}^\pm_{n+1} \\ &\qquad\qquad - \frac{\nu+\nu_m}{2}\Delta \boldsymbol{z}^\pm_{n+1} - \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^\mp_n + \nabla p^{n+1} = 0, \end{aligned} \tag{3.1}$$

$$\nabla \cdot \boldsymbol{z}^\pm_{n+1} = 0. \tag{3.2}$$

The scheme (3.1)-(3.2) has the following appealing features:

(i) Unconditional absolute stability.

(ii) Modularity: the variables $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ are **decoupled**.

Remark 3.1. *Using the defect-correction method [16] a second order scheme can be constructed from* (3.1)-(3.2). In the remainder we denote by $|\cdot|$ the usual $L^2(\Omega)$ norm.

Theorem 3.1. *Let $\boldsymbol{z}^+_n, \boldsymbol{z}^-_n, p_n$ satisfy* (3.1)-(3.2) *for each $n \in \{1, 2, \ldots, \frac{T}{\Delta t}\}$. Then the following energy estimate holds*

$$\begin{aligned} &\frac{|\boldsymbol{z}^+_N|^2 + |\boldsymbol{z}^-_N|^2}{2\Delta t} + \frac{1}{2\Delta t}\sum_{n=1}^{N}\left(|\boldsymbol{z}^+_n - \boldsymbol{z}^+_{n-1}|^2 + |\boldsymbol{z}^-_n - \boldsymbol{z}^-_{n-1}|^2\right) \\ &\quad + \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\left(|\nabla \boldsymbol{z}^+_N|^2 + |\nabla \boldsymbol{z}^-_N|^2\right) + \frac{\nu\nu_m}{\nu+\nu_m}\sum_{n=1}^{N}\left(|\nabla \boldsymbol{z}^-_n|^2 + |\nabla \boldsymbol{z}^+_n|^2\right) \\ &\quad + \frac{|\nu-\nu_m|}{4}\sum_{n=1}^{N}\left(\sqrt{\frac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla \boldsymbol{z}^+_n| + \sqrt{\frac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla \boldsymbol{z}^-_{n-1}|\right)^2 \end{aligned} \tag{3.3}$$

$$+\tfrac{|\nu-\nu_m|}{4}\sum_{n=1}^{N}\Big(\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla \boldsymbol{z}_n^-|+\sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla \boldsymbol{z}_{n-1}^+|\Big)^2$$
$$\le \frac{|\boldsymbol{z}_0^+|^2+|\boldsymbol{z}_0^-|^2}{2\Delta t}+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\big(|\nabla \boldsymbol{z}_0^+|^2+|\nabla \boldsymbol{z}_0^-|^2\big).$$

Proof. First we multiply the momentum equations (3.1) with $\boldsymbol{z}_{n+1}^+, \boldsymbol{z}_{n+1}^-$, respectively, use the continuity equations and the polarized identity to obtain

$$\frac{|\boldsymbol{z}_{n+1}^+|^2-|\boldsymbol{z}_n^+|^2+|\boldsymbol{z}_{n+1}^+-\boldsymbol{z}_n^+|^2}{2\Delta t}+\frac{\nu+\nu_m}{2}|\nabla \boldsymbol{z}_{n+1}^+|^2+\frac{\nu-\nu_m}{2}\langle\nabla \boldsymbol{z}_n^-,\nabla \boldsymbol{z}_{n+1}^+\rangle=0,$$
$$\frac{|\boldsymbol{z}_{n+1}^-|^2-|\boldsymbol{z}_n^-|^2+|\boldsymbol{z}_{n+1}^--\boldsymbol{z}_n^-|^2}{2\Delta t}+\frac{\nu+\nu_m}{2}|\nabla \boldsymbol{z}_{n+1}^-|^2+\frac{\nu-\nu_m}{2}\langle\nabla \boldsymbol{z}_n^+,\nabla \boldsymbol{z}_{n+1}^-\rangle=0.$$

Then add up the two relations to get

$$\begin{aligned}&\frac{|\boldsymbol{z}_{n+1}^+|^2+|\boldsymbol{z}_{n+1}^-|^2-|\boldsymbol{z}_n^+|^2-|\boldsymbol{z}_n^-|^2}{2\Delta t}+\frac{|\boldsymbol{z}_{n+1}^+-\boldsymbol{z}_n^+|^2+|\boldsymbol{z}_{n+1}^--\boldsymbol{z}_n^-|^2}{2\Delta t}\\&+\frac{\nu+\nu_m}{2}\big(|\nabla \boldsymbol{z}_{n+1}^+|^2+|\nabla \boldsymbol{z}_{n+1}^-|^2\big)+\frac{\nu-\nu_m}{2}\big(\langle\nabla \boldsymbol{z}_n^-,\nabla \boldsymbol{z}_{n+1}^+\rangle+\langle\nabla \boldsymbol{z}_n^+,\nabla \boldsymbol{z}_{n+1}^-\rangle\big)=0.\end{aligned}\tag{3.4}$$

Secondly, the dissipation terms can be estimated, using the Cauchy-Schwarz inequality and the polarized identity, as follows

$$\begin{aligned}&\frac{\nu+\nu_m}{2}|\nabla \boldsymbol{z}_{n+1}^+|^2+\frac{\nu+\nu_m}{2}|\nabla \boldsymbol{z}_{n+1}^-|^2+\frac{\nu-\nu_m}{2}\langle\nabla \boldsymbol{z}_n^-,\nabla \boldsymbol{z}_{n+1}^+\rangle+\frac{\nu-\nu_m}{2}\langle\nabla \boldsymbol{z}_n^+,\nabla \boldsymbol{z}_{n+1}^-\rangle\\&\ge\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\big(|\nabla \boldsymbol{z}_{n+1}^+|^2+|\nabla \boldsymbol{z}_{n+1}^-|^2-|\nabla \boldsymbol{z}_n^-|^2-|\nabla \boldsymbol{z}_n^+|^2\big)+\frac{\nu\nu_m}{\nu+\nu_m}\big(|\nabla \boldsymbol{z}_{n+1}^-|^2+|\nabla \boldsymbol{z}_{n+1}^+|^2\big)\\&+\tfrac{|\nu-\nu_m|}{4}\Big(\sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla \boldsymbol{z}_n^-|+\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla \boldsymbol{z}_{n+1}^+|\Big)^2+\tfrac{|\nu-\nu_m|}{4}\Big(\sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla \boldsymbol{z}_n^+|+\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla \boldsymbol{z}_{n+1}^-|\Big)^2.\end{aligned}$$

The substitution of the above estimate in (3.4) implies

$$\begin{aligned}&\frac{|\boldsymbol{z}_{n+1}^+|^2+|\boldsymbol{z}_{n+1}^-|^2-|\boldsymbol{z}_n^+|^2-|\boldsymbol{z}_n^-|^2}{2\Delta t}+\frac{|\boldsymbol{z}_{n+1}^+-\boldsymbol{z}_n^+|^2+|\boldsymbol{z}_{n+1}^--\boldsymbol{z}_n^-|^2}{2\Delta t}\\&+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\big(|\nabla \boldsymbol{z}_{n+1}^+|^2+|\nabla \boldsymbol{z}_{n+1}^-|^2-|\nabla \boldsymbol{z}_n^-|^2-|\nabla \boldsymbol{z}_n^+|^2\big)+\frac{\nu\nu_m}{\nu+\nu_m}\big(|\nabla \boldsymbol{z}_{n+1}^-|^2+|\nabla \boldsymbol{z}_{n+1}^+|^2\big)\\&+\tfrac{|\nu-\nu_m|}{4}\Big(\sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla \boldsymbol{z}_n^-|+\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla \boldsymbol{z}_{n+1}^+|\Big)^2\\&+\tfrac{|\nu-\nu_m|}{4}\Big(\sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla \boldsymbol{z}_n^+|+\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla \boldsymbol{z}_{n+1}^-|\Big)^2\le 0.\end{aligned}$$

Finally, summation from $n=0$ to $N-1$ gives the energy estimate (3.3), which yields the unconditional stability of scheme (3.1)-(3.2). ∎

REMARK 3.2. *In the original velocity and magnetic field variables* $\boldsymbol{u}$ *and* $\boldsymbol{B}$*, the method* (3.1)-(3.2) *writes:*

$$\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{\Delta t}+(\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_{n+1}-(\boldsymbol{B}_n\cdot\nabla)\boldsymbol{B}_{n+1}-\frac{\nu+\nu_m}{2}\Delta\boldsymbol{u}_{n+1}-\frac{\nu-\nu_m}{2}\Delta\boldsymbol{u}_n+\nabla p_{n+1}=0,$$
$$\frac{\boldsymbol{B}_{n+1}-\boldsymbol{B}_n}{\Delta t}+(\boldsymbol{u}_n\cdot\nabla)\boldsymbol{B}_{n+1}-(\boldsymbol{B}_n\cdot\nabla)\boldsymbol{u}_{n+1}-\frac{\nu+\nu_m}{2}\Delta\boldsymbol{B}_{n+1}+\frac{\nu-\nu_m}{2}\Delta\boldsymbol{B}_n=0,$$

and $\nabla \cdot \boldsymbol{u}_n = 0, \nabla \cdot \boldsymbol{B}_n = 0$. *The local truncation error is*

$$
\begin{aligned}
\tau^u_{n+1}(\Delta t) &= \frac{\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)}{\Delta t} + (\boldsymbol{u}(t_n)\cdot\nabla)\boldsymbol{u}(t_{n+1}) - (\boldsymbol{B}(t_n)\cdot\nabla)\boldsymbol{B}(t_{n+1}) \\
&\quad - \frac{\nu+\nu_m}{2}\Delta\boldsymbol{u}(t_{n+1}) - \frac{\nu-\nu_m}{2}\Delta\boldsymbol{u}(t_n) + \nabla p(t_{n+1}), \\
\tau^B_{n+1}(\Delta t) &= \frac{\boldsymbol{B}(t_{n+1}) - \boldsymbol{B}(t_n)}{\Delta t} + (\boldsymbol{u}(t_n)\cdot\nabla)\boldsymbol{B}(t_{n+1}) - (\boldsymbol{B}(t_n)\cdot\nabla)\boldsymbol{u}(t_{n+1}) \\
&\quad - \frac{\nu+\nu_m}{2}\Delta\boldsymbol{B}(t_{n+1}) + \frac{\nu-\nu_m}{2}\Delta\boldsymbol{B}(t_n).
\end{aligned}
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