Linear Algebra Preliminary Exam

May 5 2023

1. Let V be an n-dimensional real vector space and $\mathfrak{B}$ be a basis for V. Let S be a collection of subspaces of V such that any subspace in S is spanned by a subset of the basis $\mathfrak{B}$. Show that for any subspaces $U, W_1, W_2 \in S$, we have
$$U \cap (W_1 + W_2) = U \cap W_1 + U \cap W_2.$$

2. Let v_1, v_2, v_3 be nonzero eigenvectors of an operator A such that $v_1 + v_2 = v_3$. Show that they all have the same eigenvalue.

3. Let A, B be two **anti-self-adjoint** matrices. Show that $A^2 + B^2$ is invertible if and only if $N_A \cap N_B = \{0\}$.

4. Let A be an $n \times n$ complex matrix which commutes with all positive definite matrices. Show that $A = kI$ for some scalar k.

5. Let V be a **complex** Euclidean space and $P : V \to V$ be a linear map satisfying $P^2 = P$. Suppose additionally that
$$\|Px\| \leq \|x\|$$
holds for every $x \in V$. Show that P is an orthogonal projection.

6. Let A, B be two real symmetric matrices, show that
$$\operatorname{tr}(AB)^2 \leq \operatorname{tr} A^2 \operatorname{tr} B^2.$$