EXPLICITLY UNCOUPLED VMS STABILIZATION OF FLUID FLOW

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Abstract. Variational Multiscale Methods have proven to be an accurate and systematic approach to the simulation of turbulent flows. Many turbulent flows are solved by legacy codes or by ones written by a team of programmers and of great complexity so implementing a new approach to turbulence in such cases can be daunting. We propose a new approach to inducing a VMS treatment of turbulence in a legacy code (or any laminar flow code even). The method adds a separate, uncoupled and modular postprocessing step to each time step of the (possibly black box) flow code. We prove stability and convergence for the combination and quantify the VMS dissipation induced. Numerical experiments confirming the theory are given. In particular, the performance of the two step, modular VMS method is fully comparable to a monolithic (fully coupled) VMS method for the benchmark problem of decaying homogeneous turbulence.

1. Introduction. This report develops a modular, postprocessing method to implement a variational multiscale method in complex (possibly legacy, possibly black box and possibly laminar) flow codes. The basic function of both eddy viscosity and numerical dissipation is to truncate scales. When the smallest persistent, energetically significant scale (the flows microscale) is significantly larger than the meshwidth, the simulation is usually over-diffused. When the smallest such scale is smaller than the meshwidth, typically nonphysical wiggles are observed in the numerical solution. A successfully tuned model and numerical method will yield a simulation of a complex flow for which

$$\text{effective microscale} = \text{filter radius} = \text{spacial mesh-width}.$$

Clearly, the optimal place to truncate scales is locally (in scale space) at the marginally resolved scales, i.e., at $u' := u - \overline{u}$ and the optimal definition of the resolved scales $\overline{u}$ is that which is computable on a given grid or in a given finite dimensional subspace of approximate velocities. These two observations are motivation for and explain the remarkable success of the Variational Multiscale (VMS) method of Hughes and collaborators, which was introduced in [Hug95] and used first in turbulence modeling in [HMJ00]. Given the remarkable success of the VMS approach, there is a natural need to introduce a VMS treatment of turbulence within legacy codes, in complex multi-physics applications and in other settings where reprogramming a new method from scratch is not palatable. We propose, analyze and test herein a method to induce a VMS treatment of turbulence in an existing NSE discretization through an additional, modular and uncoupled projection step.

To introduce ideas, suppressing the pressure and spacial discretization, suppose the Navier-Stokes equations are written as

$$\frac{\partial u}{\partial t} + N(u) + \nu A u = f(t)\,.$$

Add one uncoupled, modular, projection-like step (Step 2) to the standard CN method, Step 1: given $u^n \simeq u(t^n)$, compute u^{n+1} by

$$\text{Step 1: Compute } w^{n+1} \text{ via: } \quad \frac{w^{n+1} - u^n}{\Delta t} + N\left(\frac{w^{n+1} + u^n}{2}\right) + \nu A \frac{w^{n+1} + u^n}{2} = f^{n+1/2}\,,$$

$$\text{Step 2: Postprocess } w^{n+1} \text{ to obtain } u^{n+1}\text{: } u^{n+1} = \Pi w^{n+1},$$

where $f^{n+1/2} = (f^n + f^{n+1})/2$. We will show our theoretical results for the Crank-Nicolson time discretization scheme in Step 1, but *the setting of Step 2 is independent of the time discretization, see Remark 3.4.* The deviations from previous work considered herein are that (1) the projection based stabilization is an uncoupled, independent second step and thus amenable to implementation in legacy codes, and (2) the projection in Step 2 is not a filter but constructed to recover the VMS eddy viscosity term.

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Eliminating Step 2 gives

$$\frac{u^{n+1}-u^n}{\Delta t} + N\left(\frac{w^{n+1}+u^n}{2}\right) + \nu A \frac{w^{n+1}+u^n}{2} + \frac{\mathbf{1}}{\mathbf{\Delta t}}(\mathbf{w^{n+1}} - \mathbf{\Pi w^{n+1}}) = f^{n+1/2}\,, \tag{1.1}$$

which is a time relaxation discretization of the original problem with time relaxation coefficient $1/\Delta t$. We define the operator in Step 2 so that (see Section 4 for details) the extra, bold term in (1.1) is exactly a VMS eddy viscosity term acting on marginally resolved scales:

$$u^{n+1} = \Pi w^{n+1} \text{ satisfies, for all } v_h \text{ in the discrete velocity space,}$$
$$(\text{Extra Term}, v_h) = \frac{1}{\Delta t}(w^{n+1} - u^{n+1}, v_h) = (\nu_T(x,h)\,[I-P]\,\nabla \frac{w^{n+1}+u^{n+1}}{2}, [I-P]\,\nabla v_h).$$

The subscript $h = h(x)$ denotes the local meshwidth of a FEM mesh and P, defined precisely in (2.4), is an L^2 projection that defines the VMS velocity gradient averages (so $[I-P]$ defines the fluctuations). Full details are given in Section 3. Also $\nu_T(x,h)$ is the chosen eddy viscosity coefficient. We shall assume (motivated by the nonlinear case in which its value is often extrapolated from previous time levels) in this report that:

CONDITION 1.1. *$\nu_T = \nu_T(x,h)$ is a known, positive, bounded function which is constant elementwise.*

A complete stability and convergence analysis of this method is given in Section 3 and Appendix A. In Section 4 we turn to the problem of actually computing the operator in Step 2 efficiently. A related VMS method (adding ideas from [ALP04]) which is slightly less accurate but more efficient is given in Section 5. Numerical experiments are given in Section 6.

1.1. Previous work. This report presents an explicitly uncoupled variant of the VMS method, which is based on ideas from filter based stabilization. A numerical analysis of the stability, dissipation and error behavior in linear filter based stabilization of the CN-FEM was performed in [ELN10], including effects of deconvolution and relaxation. The case of BDF2 time discretization plus nonlinear filtering, and relaxation was considered in [LRT10]. The case of higher order methods in time is also an important open problem. It seems to be more difficult to give a precise description of the numerical dissipation in higher order methods.

There is a wide range of methods adding numerical dissipation on all scales of a flow, like e.g. the residual based stabilization techniques [BH82]. One can find an overview in [RST08]. In turbulence modeling strategies the aim is to simulate at least the large scales of a flow accurately, which has been considered in the classical Large-Eddy simulation (LES), see [BIL06, Joh04]. Based on ideas in [Hug95, Gue99], the class of VMS methods was developed and used as an alternative to LES since [HMJ00]. We will perform a projection-based VMS method from [Lay02]. For this method there are many different variants to be found in [Gra06, JK10].

2. Notation and Preliminaries. Let Ω be an open, regular domain in $\mathbb{R}^d$ ($d = 2$ or 3). The Navier-Stokes equation with boundary and initial condition are: Given time $T > 0$, body force f, find velocity $u : [0,T] \times \Omega \to \mathbb{R}^d$, pressure $p : [0,T] \times \Omega \to \mathbb{R}$ satisfying

$$\begin{aligned} u_t + u \cdot \nabla u \,-\, \nu \Delta u \,+\, \nabla p = f(x) \text{ and } \nabla \cdot u = 0 \text{ in } \Omega, \text{ for } 0 < t \leq T \\ u(x,0) = u^0(x) \text{ in } \Omega,\, u = 0 \text{ on } \partial\Omega, \text{ for } 0 < t \leq T. \end{aligned} \tag{2.1}$$

We consider our analysis on the finite element method (FEM) for the spacial discretization. (The results extend to many other variational methods.) The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space $W_2^k(\Omega)$, and $\|\cdot\|_k$ denotes the norm in H^k. The space H^{-k} denotes the dual space of H_0^k. For functions $v(x,t)$ defined on the entire time interval $(0,T)$, we define $(1 \leq m < \infty)$

$$\|v\|_{\infty,k} \;:=\; EssSup_{[0,T]}\|v(t,\cdot)\|_k\,, \text{ and } \|v\|_{m,k} \;:=\; \left(\int_0^T \|v(t,\cdot)\|_k^m\,dt\right)^{1/m}.$$

The velocity and pressure spaces are

$$X := (H_0^1(\Omega))^d,\ Q := L_0^2(\Omega), \text{ with } \|v\|_X := \|\nabla v\|.$$

The space of divergence free functions is given by

$$V := \{v \in X :\ (\nabla \cdot v, q) = 0 \ \ \forall q \in Q\}.$$

A weak formulation of (2.1) is: Find $u : [0,T] \to X$, $p : [0,T] \to Q$ for a.e. $t \in (0,T]$ satisfying

$$(u_t, v) + (u \cdot \nabla u, v) - (p, \nabla \cdot v) + \nu(\nabla u, \nabla v) = (f, v) \quad \forall v \in X \tag{2.2}$$

$$u(x,0) = u^0 \text{ in } X \text{ and } (\nabla \cdot u, q) = 0 \qquad \forall q \in Q. \tag{2.3}$$

The finite element spaces considered are built on a conforming, edge to edge triangulation with maximum triangle parameter denoted by a subscript "h". We shall denote conforming velocity, pressure finite element spaces by

$$X_h \subset X,\ Q_h \subset Q.$$

We also must select a space of "well resolved" velocities and pressures, denoted by

$$X_H \subset X,\ Q_H \subset Q.$$

Two commonly seen examples of the definition of the well resolved spaces are:

- A coarse mesh velocity and pressure space $X_H,\ Q_H$ (with meshwidth denoted by subscript $H \leq \sqrt{h}$) is constructed. If the meshes are nested and the space uses the same elements as the fine mesh space then $X_H \subset X_h \subset X,\ Q_H \subset Q_h \subset Q$; see [GGKW10, LRL08] for examples.
- The space of well refined velocities and pressures are defined on the same mesh but using finite element spaces of lower polynomial degree. In this case also $X_H \subset X_h \subset X,\ Q_H \subset Q_h \subset Q$; see [JK10, RL10] for examples.

The first approach requires a code with only one element but pointers between the two meshes (as are commonly found with $h-$adaptive codes) while the second works only on one mesh but requires at least two velocity elements (such as in $p-$adaptive codes). We shall assume that $X_{H/h}$, $Q_{H/h}$ satisfy the usual inf-sup condition necessary for the stability of the pressure, e.g. [Gun89]. The discretely divergence free subspace of $X_{H/h}$ is

$$V_{H/h} = \{v_{H/h} \in X_{H/h} : (\nabla \cdot v_{H/h}, q_{H/h}) = 0 \ \ \forall q_{H/h} \in Q_{H/h}\}.$$

Note that $V_H \not\subseteq V_h$ in general. Taylor-Hood elements (see [BS94, Gun89]) are one common example of such a choice for $(X_h,\ Q_h)$, and are also the elements we use in our numerical experiments. Further, we denote the space of (typically discontinuous) coarse mesh velocity gradient tensors by

$$L_H := \nabla X_H = \{\nabla v_H :\ \text{for all } v_H \in X_H\},$$

and analogously for L_h. The weighted L^2 and elliptic projections are defined as usual (in general and in this specific case following [BIL06], Section 11.6) by

$$\begin{aligned} P_H \nabla u = G_H \in L_H \text{ satisfies } (\nu_T(x,h)\left[G_H - \nabla u\right], l_H) = 0, \forall\, l_H \in L_H, \\ E_H u = \widetilde{u} \in X_H \text{ satisfies } (\nu_T(x,h)\left[\nabla \widetilde{u} - \nabla u\right], \nabla v_H) = 0, \forall\, v_H \in X_H. \end{aligned} \tag{2.4}$$

The motivation for the definition in (2.4) is that means (and thus fluctuations) defined by elliptic projection are equivalent to means of deformations defined by L^2 projection (see [BIL06], Lemma 11.10)

$$\overline{u} := E_H u,\ P_H \nabla u = \nabla E_H u.$$

Further, while computation of velocity means is global, when the means of deformation are defined by L^2 projection into a C^0 finite element space, $P_H \nabla u$ can be computed in parallel element by element.

Define the usual, explicitly skew symmetrized trilinear form

$$b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v).$$

Let $v(t^{n+1/2}) = v((t^{n+1} + t^n)/2)$ for a continuous function in time and $v^{n+1/2} = (v^{n+1} + v^n)/2$ for functions of time that are both continuous and discrete.

3. The Postprocessed VMS Method. The method we propose, analyse and test adds one uncoupled projection step to the Navier-Stokes equations. Within this section we will prove the stability and an á priori error estimate for Algorithm 3.2.

DEFINITION 3.1. *Given w_h^{n+1}, $u_h^{n+1} = \Pi w_h^{n+1} \in V_h$ is the (unique) solution of*

$$\begin{aligned} \left(\frac{w_h^{n+1}-u_h^{n+1}}{\Delta t}, v_h\right) &= \left(\nu_T[I-P_H]\nabla\frac{w_h^{n+1}+u_h^{n+1}}{2}, [I-P_H]\nabla v_h\right) + \left(\lambda_h^{n+1}, \nabla\cdot v_h\right) \text{ for all } v_h \in X_h \\ \left(\nabla\cdot u_h^{n+1}, q_h\right) &= 0 \text{ for all } q_h \in Q_h \end{aligned} \tag{3.1}$$

The theory will be given with a Crank-Nicolson time discretization scheme. We will see that the setup of the uncoupled projection step does not depend on the time discretization scheme.

ALGORITHM 3.2. *Given u_h^n compute u_h^{n+1} by*
Step 1: Compute $w_h^{n+1} \in V_h$ satisfying: for all $v_h \in V_h$

$$\left(\frac{w_h^{n+1}-u_h^n}{\Delta t}, v_h\right) + b^*\left(\frac{w_h^{n+1}+u_h^n}{2}, \frac{w_h^{n+1}+u_h^n}{2}, v_h\right) + \nu\left(\nabla\frac{w_h^{n+1}+u_h^n}{2}, \nabla v_h\right) = \left(f^{n+1/2}, v_h\right)$$

Step 2: Apply projection Π on w_h^{n+1} to obtain u_h^{n+1}

$$u_h^{n+1} = \Pi w_h^{n+1}.$$

Eliminating Step 2 gives

$$\begin{aligned} \left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h\right) + b^*\left(\frac{w_h^{n+1}+u_h^n}{2}, \frac{w_h^{n+1}+u_h^n}{2}, v_h\right) + \nu\left(\nabla\frac{w_h^{n+1}+u_h^n}{2}, \nabla v_h\right) + \left(\frac{w_h^{n+1}-\Pi w_h^{n+1}}{\Delta t}, v_h\right) \\ = \left(f^{n+1/2}, v_h\right), \end{aligned}$$

where the last term on the left hand side is the additional term coming into play because of the projection step. By Definition 3.1, this term recovers the VMS eddy viscosity term and the projected velocity is discretely divergence free. Also, the following lemma quantifies the eddy viscosity induced by Step 2 processing between w_h^{n+1} to u_h^{n+1}.

LEMMA 3.3. **[Numerical Dissipation induced by Step 2]** *Let ν_T fulfill Condition 1.1. Then, there holds*

$$\left\|w_h^{n+1}\right\|^2 = \left\|u_h^{n+1}\right\|^2 + 2\Delta t\left\|\sqrt{\nu_T}[I-P_H]\nabla\frac{w_h^{n+1}+u_h^{n+1}}{2}\right\|^2.$$

Proof. Set $v_h = \frac{w_h^{n+1}+\ u_h^{n+1}}{2}$ and $q_h = \lambda_h^{n+1}$ in (3.1) and obtain

$$\frac{1}{2\Delta t}\left(\|\ w_h^{n+1}\|^2 - \|\ u_h^{n+1}\|^2\right) = \left\|\sqrt{\nu_T}[I-P_H]\nabla\frac{w_h^{n+1}+\ u_h^{n+1}}{2}\right\|^2,$$

where we used that $w_h^{n+1} \in V_h$. This already proves the claim after rearranging. ☐

REMARK 3.4. *The special choice in* (3.1) *used in Step 2 with the argument of the form*

$$\frac{w_h^{n+1}+u_h^{n+1}}{2} (\text{and not } \frac{w_h^{n+1}+u_h^n}{2})$$

does not depend on the time discretization scheme in Step 1. With a different discretization used in Step 1 we would get the same induced eddy dissipation terms in Step 2 within the proof of Lemma 3.3. This is why the explicitly uncoupled Step 2 of Algorithm 3.2 does not depend on the time discretization scheme in Step 1 and why Step 2 can be used with an arbitrary (possibly black box) CFD code.

Lemma 3.3 is one key to prove stability of Algorithm 3.2.

THEOREM 3.5. *Let ν_T satisfy Condition 1.1, then*

$$\frac{1}{2}\| u_h^N \|^2 + \Delta t \sum_{n=0}^{N-1} \left[\frac{\nu}{2} \left\| \nabla \frac{w_h^{n+1} + u_h^n}{2} \right\|^2 + \left\| \sqrt{\nu_T}[I - P_H]\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2} \right\|^2 \right] \leq \frac{1}{2}\| u_h^0 \|^2 + \frac{\Delta t}{2\nu} \sum_{n=0}^{N-1} \left\| f^{n+1/2} \right\|_{-1}^2 .$$

Proof. Set $v_h = \frac{w_h^{n+1} + u_h^n}{2}$ in Step 1 and obtain

$$\frac{1}{2\Delta t}\left(\| w_h^{n+1} \|^2 - \| u_h^n \|^2 \right) + \nu \left\| \nabla \frac{w_h^{n+1} + u_h^n}{2} \right\|^2 = \left(f^{n+1/2}, \frac{w_h^{n+1} + u_h^n}{2} \right).$$

Application of Lemma 3.3 to this equation gives

$$\frac{1}{2\Delta t}\left(\| u_h^{n+1} \|^2 - \| u_h^n \|^2 \right) + \left[\nu \left\| \nabla \frac{w_h^{n+1} + u_h^n}{2} \right\|^2 + \left\| \sqrt{\nu_T}[I - P_H]\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2} \right\|^2 \right] = \left(f^{n+1/2}, \frac{w_h^{n+1} + u_h^n}{2} \right).$$

Summing this up from $n = 0$ to $n = N - 1$ results in

$$\frac{1}{2}\| u_h^N \|^2 + \Delta t \sum_{n=0}^{N-1} \left[\nu \left\| \nabla \frac{w_h^{n+1} + u_h^n}{2} \right\|^2 + \left\| \sqrt{\nu_T}[I - P_H]\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2} \right\|^2 \right] = \frac{1}{2}\| u_h^0 \|^2 + \Delta t \sum_{n=0}^{N-1} \left(f^{n+1/2}, \frac{w_h^{n+1} + u_h^n}{2} \right), \quad (3.2)$$

where we can apply Young's inequality to the right hand side inside the sum to see

$$\Delta t \left(f^{n+1/2}, \frac{w_h^{n+1} + u_h^n}{2} \right) \leq \frac{\Delta t}{2\nu} \left\| f^{n+1/2} \right\|_{-1}^2 + \frac{\nu \Delta t}{2} \left\| \nabla \frac{w_h^{n+1} + u_h^n}{2} \right\|^2 .$$

Hiding the last term on the left hand side of (3.2) proves the claim. ☐

Theorem 3.5 also gives a stability estimate for w_h^N. In particular, Corollary 3.6 shows that w_h^N is also not the usual CN approximation.

COROLLARY 3.6. *Let ν_T fulfill Condition 1.1, then*

$$\frac{1}{2}\| w_h^N \|^2 + \Delta t \sum_{n=0}^{N-1} \frac{\nu}{2} \left\| \nabla \frac{w_h^{n+1} + u_h^n}{2} \right\|^2 + \Delta t \sum_{n=0}^{N-2} \left\| \sqrt{\nu_T}[I - P_H]\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2} \right\|^2 \leq \frac{1}{2}\| u_h^0 \|^2 + \frac{\Delta t}{2\nu} \sum_{n=0}^{N-1} \left\| f^{n+1/2} \right\|_{-1}^2 .$$

Proof. Apply Lemma 3.3 for $\|w_h^N\|^2$ to Theorem 3.5. ☐

As a next step we will give an á priori error estimate for the approximation scheme, Algorithm 3.2. Let $t^n = n\Delta t$, $n = 0, 1, 2, \ldots, N_T$, and $T := N_T \Delta t$. Also introduce the following discrete norms:

$$|||v|||_{\infty,k} := \max_{0 \leq n \leq N_T} \|v^n\|_k , \qquad |||v_{1/2}|||_{\infty,k} := \max_{1 \leq n \leq N_T} \|v(t^{n-1/2})\|_k ,$$

$$|||v|||_{m,k} := \left(\sum_{n=0}^{N_T} \|v^n\|_k^m \Delta t \right)^{1/m} , \qquad |||v_{1/2}|||_{m,k} := \left(\sum_{n=1}^{N_T} \|v(t^{n-1/2})\|_k^m \Delta t \right)^{1/m} .$$

In order to establish the optimal asymptotic error estimates for the approximation we need to assume the following regularity of the true solution:

$$\begin{aligned} &u \in L^{\infty}(0,T;H^{k+1}(\Omega)) \cap H^1(0,T;H^{k+1}(\Omega)) \cap H^3(0,T;L^2(\Omega)) \cap W_4^2(0,T;H^1(\Omega)), \\ &p \in L^{\infty}(0,T;H^{s+1}(\Omega))\,, \text{ and } f \in H^2(0,T;L^2(\Omega)). \end{aligned} \tag{3.3}$$

For the error between $u^n - u_h^n$ we have the following theorem and corollary, which are proven in the Appendix A.

THEOREM 3.7. *For u, p, and f satisfying (3.3), (2.2) and (2.3), and u_h^n, w_h^n given by Algorithm 3.2 we have that, for Δt sufficiently small,*

$$\begin{aligned} \frac{1}{2}\|u^N - u_h^N\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{N-1} \Big(\nu\|\nabla(u(t^{n+1/2}) - (w_h^{n+1}+u_h^n)/2)\|^2 \\ + \|\sqrt{\nu_T}[I-P_H]\nabla(u(t^{n+1}) - (w_h^{n+1}+u_h^{n+1})/2)\|^2\Big) \\ \le Ch^{2k+2}|||u|||^2_{\infty,k+1} + C\nu h^{2k}|||u|||^2_{2,k+1} + C\nu_T h^{2k}|||u|||^2_{2,k+1} + C\nu_T H^{2k}|||u|||^2_{2,k+1} \\ + C\frac{h^{2k}}{\nu^2}|||u|||^2_{\infty,k+1} + C\frac{h^{2k+1}}{\nu}\left(|||u|||^4_{4,k+1} + |||\nabla u|||^4_{4,0}\right) + C\frac{h^{2s+2}}{\nu}|||p_{1/2}|||^2_{2,s+1} \\ + Ch^{2k+2}|||u_t|||^2_{2,k+1} + C(\Delta t)^4\Big(\frac{1}{\nu}|||\nabla u|||^4_{4,0} + \frac{1}{\nu}|||\nabla u_{1/2}|||^4_{4,0} \\ + |||u_{ttt}|||^2_{2,0} + \nu|||\nabla u_{tt}|||^2_{2,0} + \frac{1}{\nu}|||\nabla u_{tt}|||^4_{4,0} + |||f_{tt}|||^2_{2,0}\Big) \end{aligned}$$

For $k = 2$, $s = 1$ Taylor-Hood elements, i.e. C^0 piecewise quadratic velocity space X_h and C^0 piecewise linear pressure space Q_h, we have the following asymptotic estimate.

COROLLARY 3.8. *Under the assumptions of Theorem 3.7, with $\Delta t = Ch$, $\nu_T = h^2$, $H = \sqrt{h}$ and (X_h, Q_h) given by the Taylor-Hood approximation elements, we have*

$$\begin{aligned} \|u^N - u_h^N\|^2 + \frac{\Delta t}{2}\sum_{n=0}^{N-1} \Big(\nu\|\nabla(u(t^{n+1/2}) - (w_h^{n+1}+u_h^n)/2)\|^2 \\ + \|\sqrt{\nu_T}[I-P_H]\nabla(u(t^{n+1}) - (w_h^{n+1}+u_h^{n+1})/2)\|^2\Big) \le\; C\left((\Delta t)^4 + h^4\right). \end{aligned}$$

3.1. Growth of Pertubations in the discrete scheme. The question naturally arises of dependence of the constant C in Theorem 3.7 upon the final time T. This dependence is exponential (reflecting exponential stretching in the continuous NSE) and inevitably arising from the discrete Gronwall inequality. It is related to the maximal Lyapunov exponent in the discrete model given by Algorithm 3.2. In this subsection we derive an estimate for the Lyapunov exponent of this model. To simplify the notation we will suppress the index h, although we only consider discrete solutions here. Let (u_1, w_1, f_1) and (u_2, w_2, f_2) be two solutions with different problem data from Algorithm 3.2. By subtracting the two corresponding equations in Step 1, we obtain

$$\begin{aligned} \frac{1}{\Delta t}\left((w_1^{n+1} - w_2^{n+1}) - (u_1^n - u_2^n), v\right) + \nu\left(\nabla\frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}, \nabla v\right) \\ + b^*\left(\frac{w_1^{n+1}+u_1^n}{2}, \frac{w_1^{n+1}+u_1^n}{2}, v\right) - b^*\left(\frac{w_2^{n+1}+u_2^n}{2}, \frac{w_2^{n+1}+u_2^n}{2}, v\right) = \left(f_1^{n+1/2} - f_2^{n+1/2}, v\right) \end{aligned}$$

for all functions $v \in V_h$. Setting $v = \frac{1}{2}[(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)]$ gives

$$\begin{aligned}
\frac{1}{2\Delta t}\left(\|w_1^{n+1} - w_2^{n+1}\|^2 - \|u_1^n - u_2^n\|^2\right) + \nu \left\|\nabla \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right\|^2 = \\
b^*\left(\frac{w_2^{n+1} + u_2^n}{2}, \frac{w_2^{n+1} + u_2^n}{2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right) \\
-b^*\left(\frac{w_1^{n+1} + u_1^n}{2}, \frac{w_1^{n+1} + u_1^n}{2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right) \\
+\left(f_1^{n+1/2} - f_2^{n+1/2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right).
\end{aligned} \tag{3.4}$$

As a next step we estimate all terms on the RHS and start with the easy one

$$\begin{aligned}
&\left(f_1^{n+1/2} - f_2^{n+1/2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right) \\
&\qquad \le \frac{\nu}{8}\left\|\nabla \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right\|^2 + \frac{2}{\nu}\left\|(f_1^{n+1/2} - f_2^{n+1/2})\right\|_{-1}^2.
\end{aligned}$$

To bound the nonlinear term we use $b^*(u, v, w) \le C_* \sqrt{\|u\|\|\nabla u\|}\|\nabla v\|\|\nabla w\|$

$$\begin{aligned}
&\Big| b^*\left(\frac{w_2^{n+1} + u_2^n}{2}, \frac{w_2^{n+1} + u_2^n}{2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right) \\
&\quad - b^*\left(\frac{w_1^{n+1} + u_1^n}{2}, \frac{w_1^{n+1} + u_1^n}{2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right)\Big| \\
&= \Big| b^*\left(\frac{w_2^{n+1} + u_2^n}{2}, \frac{(w_2^{n+1} - w_1^{n+1}) + (u_2^n - u_1^n)}{2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right) \\
&\quad + b^*\left(\frac{w_2^{n+1} + u_2^n}{2}, \frac{w_1^{n+1} + u_1^n}{2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right) \\
&\quad - b^*\left(\frac{w_1^{n+1} + u_1^n}{2}, \frac{w_1^{n+1} + u_1^n}{2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right)\Big| \\
&= \Big| b^*\left(\frac{(w_2^{n+1} - w_1^{n+1}) + (u_2^n - u_1^n)}{2}, \frac{w_1^{n+1} + u_1^n}{2}, \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right)\Big| \\
&\le C_* \sqrt{\left\|\frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right\|}\left\|\nabla \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right\|^{3/2}\left\|\nabla \frac{w_1^{n+1} + u_1^n}{2}\right\| \\
&\le \frac{3\nu}{4}\left\|\nabla \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right\|^2 + \frac{C_*^4}{4\nu^3}\left\|\nabla \frac{w_1^{n+1} + u_1^n}{2}\right\|^4\left\|\frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right\|^2,
\end{aligned}$$

where the factor $\left\|\nabla \frac{w_1^{n+1} + u_1^n}{2}\right\|^4$ can also be replaced by $\min_{i=1,2}\left\|\nabla \frac{w_i^{n+1} + u_i^n}{2}\right\|^4$ when we apply the same steps for $w_2^{n+1} + u_2^n$ again and use both estimates. With this in mind (3.4) becomes

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\|w_1^{n+1} - w_2^{n+1}\|^2 - \|u_1^n - u_2^n\|^2\right) + \frac{\nu}{8}\left\|\nabla \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2}\right\|^2 \\
&\qquad \le \frac{C_*^4}{8\nu^3} \min_{i=1,2}\left\|\nabla \frac{w_i^{n+1} + u_i^n}{2}\right\|^4 \left(\left\|w_1^{n+1} - w_2^{n+1}\right\|^2 + \|u_1^n - u_2^n\|^2\right) + \frac{2}{\nu}\left\|(f_1^{n+1/2} - f_2^{n+1/2})\right\|_{-1}^2.
\end{aligned}$$

To get a connection between u and w, we use a variant of Lemma 3.3 for the difference of the solutions and get

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\|u_1^{n+1}-u_2^{n+1}\|^2-\|u_1^n-u_2^n\|^2\right)+\frac{\nu}{8}\left\|\nabla\frac{(w_1^{n+1}-w_2^{n+1})+(u_1^n-u_2^n)}{2}\right\|^2\\
&\qquad+\left\|\sqrt{\nu_T}[I-P_H]\nabla\frac{(w_1^{n+1}-w_2^{n+1})+(u_1^{n+1}-u_2^{n+1})}{2}\right\|^2\\
&\le\frac{C_*^4}{8\nu^3}\min_{i=1,2}\left\|\nabla\frac{w_i^{n+1}+u_i^n}{2}\right\|^4\left(\left\|u_1^{n+1}-u_2^{n+1}\right\|^2+\|u_1^n-u_2^n\|^2\right)+\frac{2}{\nu}\left\|(f_1^{n+1/2}-f_2^{n+1/2})\right\|_{-1}^2\\
&\qquad+\frac{\Delta t C_*^4}{4\nu^3}\min_{i=1,2}\left\|\nabla\frac{w_i^{n+1}+u_i^n}{2}\right\|^4\left\|\sqrt{\nu_T}[I-P_H]\nabla\frac{(w_1^{n+1}-w_2^{n+1})+(u_1^{n+1}-u_2^{n+1})}{2}\right\|^2.
\end{aligned}$$

At this point let us assume that

$$\Delta t\le\left(\frac{C_*^4}{3\nu^3}\min_{i=1,2}\left\|\nabla\frac{w_i^{n+1}+u_i^n}{2}\right\|^4\right)^{-1}$$

to get

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\|u_1^{n+1}-u_2^{n+1}\|^2-\|u_1^n-u_2^n\|^2\right)+\frac{\nu}{8}\left\|\nabla\frac{(w_1^{n+1}-w_2^{n+1})+(u_1^n-u_2^n)}{2}\right\|^2\\
&\qquad+\frac{1}{4}\left\|\sqrt{\nu_T}[I-P_H]\nabla\frac{(w_1^{n+1}-w_2^{n+1})+(u_1^{n+1}-u_2^{n+1})}{2}\right\|^2\\
&\le\frac{C_*^4}{8\nu^3}\min_{i=1,2}\left\|\nabla\frac{w_i^{n+1}+u_i^n}{2}\right\|^4\left(\left\|u_1^{n+1}-u_2^{n+1}\right\|^2+\|u_1^n-u_2^n\|^2\right)+\frac{2}{\nu}\left\|(f_1^{n+1/2}-f_2^{n+1/2})\right\|_{-1}^2
\end{aligned}$$

and sum up the inequalities from $n=0$ to $n=N-1$. It holds

$$\begin{aligned}
&\frac{1}{2\Delta t}\|u_1^N-u_2^N\|^2+\sum_{n=0}^{N-1}\Big(\frac{\nu}{8}\left\|\nabla\frac{(w_1^{n+1}-w_2^{n+1})+(u_1^n-u_2^n)}{2}\right\|^2\\
&\qquad+\frac{1}{4}\left\|\sqrt{\nu_T}[I-P_H]\nabla\frac{(w_1^{n+1}-w_2^{n+1})+(u_1^{n+1}-u_2^{n+1})}{2}\right\|^2\Big)\\
&\le\frac{1}{2\Delta t}\|u_1^0-u_2^0\|^2+\frac{C_*^4}{8\nu^3}\sum_{n=0}^{N-1}\min_{i=1,2}\left\|\nabla\frac{w_i^{n+1}+u_i^n}{2}\right\|^4\left(\left\|u_1^{n+1}-u_2^{n+1}\right\|^2+\|u_1^n-u_2^n\|^2\right)\\
&\qquad+\frac{2}{\nu}\sum_{n=0}^{N-1}\left\|(f_1^{n+1/2}-f_2^{n+1/2})\right\|_{-1}^2\\
&=\sum_{n=0}^{N}\frac{\kappa_n}{2}\|u_1^n-u_2^n\|^2+\frac{2}{\nu}\sum_{n=0}^{N-1}\left\|(f_1^{n+1/2}-f_2^{n+1/2})\right\|_{-1}^2,
\end{aligned}$$

where

$$\kappa_n=\frac{C_*^4}{4\nu^3}\begin{cases}\frac{4\nu^3}{C_*^4\Delta t}+\min_{i=1,2}\left\|\nabla\frac{w_i^1+u_i^0}{2}\right\|^4 & \text{for } n=0\\ \min_{i=1,2}\left(\left\|\nabla\frac{w_i^n+u_i^{n-1}}{2}\right\|^4+\left\|\nabla\frac{w_i^{n+1}+u_i^n}{2}\right\|^4\right) & \text{for } n=1,\dots,N-1\\ \min_{i=1,2}\left\|\nabla\frac{w_i^N+u_i^{N-1}}{2}\right\|^4 & \text{for } n=N.\end{cases}$$

When we now apply the discrete Gronwall inequality from Lemma A.3, we get

$$\|u_1^N - u_2^N\|^2 + \Delta t \sum_{n=0}^{N-1} \Big(\frac{\nu}{4} \left\| \nabla \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^n - u_2^n)}{2} \right\|^2 + \frac{1}{2} \left\| \sqrt{\nu_T}[I - P_H] \nabla \frac{(w_1^{n+1} - w_2^{n+1}) + (u_1^{n+1} - u_2^{n+1})}{2} \right\|^2 \Big)$$
$$\leq \exp\left(\Delta t \sum_{n=1}^{N} g_n \kappa_n\right) \left\{ \Delta t \kappa_0 \left\|(u_1^0 - u_2^0)\right\|^2 + \frac{4\Delta t}{\nu} \sum_{n=0}^{N-1} \left\|(f_1^{n+1/2} - f_2^{n+1/2})\right\|_{-1}^2 \right\},$$

where $g_n = (1 - \Delta t \kappa_n)^{-1}$ under the assumption that $\Delta t \kappa_n < 1$. Now, we will look at the exponetial multiplier. For clarity, let us define

$$|w + u|_{1,\infty}^4 : = \max_{n=0,\dots,N} \min_{i=1,2} \left\| \nabla \frac{w_i^n + u_i^{n-1}}{2} \right\|^4 .$$

Given in addition that $\Delta t \leq \left(\frac{C_*^4}{\nu^3} \min_{i=1,2} \left\| \nabla \frac{w_i^{n+1} + u_i^n}{2} \right\|^4 \right)^{-1}$ we can estimate

$$\exp\left(\Delta t \sum_{n=1}^{N} g_n \kappa_n\right) \leq \exp\left(\Delta t \frac{C_*^4}{2\nu^3} |w + u|_{1,\infty}^4 (1 - \Delta t \frac{C_*^4}{2\nu^3} |w + u|_{1,\infty}^4)^{-1} \sum_{n=1}^{N} 1\right)$$
$$\leq \exp\left(N \Delta t \frac{C_*^4}{\nu^3} |w + u|_{1,\infty}^4\right) \leq \exp\left(T \frac{C_*^4}{\nu^3} |w + u|_{1,\infty}^4\right). \tag{3.5}$$

Remark 3.9. *The result in* (3.5) *is what one can expect from the discrete Gronwall inequality. Nevertheless it would be better to have an improvement of the factor* $\frac{C_*^4}{\nu^3}$ *to* $\frac{C_*^4}{(\nu+\nu_T)^3}$. *The analysis herein failed to produce this because of the mismatch in the arguments of the usual Galerkin terms in comparison to the term steming from the VMS projection step. The Galerkin terms had an argument of the Crank-Nicolson time discretization scheme, i.e.* $w_i^n + u_i^{n-1}$, *where the terms from the VMS projection step had an argument* $w_i^n + u_i^n$. *Recall that the projection step does not depend on the time discretization, Remark 3.4.*

4. Computing the Projection. In Step 2 the action of Π must be computed. We consider two approaches to solving the linear system to compute the projection Πu. (In Section 5 we prove viability of the method where the difficult term in (3.1) involving the operator P_H is simply lagged to the previous time level, completely circumventing this possible difficulty). The simplest method is a fixed point iteration in which the terms involving P_H are in the RHS residual calculation. We prove convergence in Theorem 4.5. This method was used in our computable experiments in which convergence was seen in 15 steps or less. The proof of Theorem 4.5 can be adapted to give an estimate of the number of steps that is not in accord with the rapid convergence observed in our experiments. Step 2 involves solving a linear system with a mixed structure. Let RHS denote a right hand side known from previous values and let $\left\{\phi_1^h \cdots, \phi_N^h\right\}$ denote a basis for the velocity space X^h. Then we have the system

$$\begin{bmatrix} M + \frac{\Delta t}{2} A & C \\ C^t & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} RHS \\ 0 \end{bmatrix}, \text{ where} \tag{4.1}$$
$$(M + \frac{\Delta t}{2} A)_{ij} = B(\phi_i^h, \phi_j^h) := (\phi_i^h, \phi_j^h) + \frac{\Delta t}{2} (\nu_T [I - P_H] \nabla \phi_i^h, [I - P_H] \nabla \phi_j^h).$$

The $1,1$ block $M + \frac{\Delta t}{2} A$ is SPD. However, the difficulty in this system is that (for some common choices of P_H) if it is assembled it has a large bandwidth. For example, if P_H is the (weighted) L^2 projection onto a coarse mesh space, then it is very easy to compute it in a residual term but it couples fine mesh basis functions across the coarse mesh macro element. Our standard approach to mixed type systems is to solve the Schur complement system

$$C^T (M + \frac{\Delta t}{2} A)^{-1} C \lambda = C^T (RHS)$$

by an iterative method in which the inner action of $(M+\frac{\Delta t}{2}A)^{-1}$ is evaluated by another iterative method. We show in Proposition 4.3 that $cond(M+\frac{\Delta t}{2}A)=O(1)$ so this inner iteration is not challenging (and the action of P_H is computed in the residual calculation at each step). This suggests that alternate approaches (whose delineation is still an open question) are feasible.

To study the condition number of the $1,1$ block of (4.1), we make the following two assumptions on the velocity space which hold for many spaces on shape-regular meshes.

CONDITION 4.1 (Inverse estimate). *There is a C_{INV} such that for every $v^h \in X^h$ we have*

$$||\nabla v^h|| \le C_{INV} h^{-1} ||v^h||.$$

CONDITION 4.2 (Norm Equivalence). *There are positive constants C_1, C_2 such that for every $v^h \in X^h, v^h = \sum_{i=1}^N \alpha_i \phi_i^h$, we have*

$$C_1 h^{-d} ||v^h||^2 \le \sum_{i=1}^N \alpha_i^2 \le C_2 h^{-d} ||v^h||^2.$$

PROPOSITION 4.3. *Suppose the velocity space satisfies the inverse estimate and norm equivalence conditions above and $\nu_T = \nu_T(x,h)$. Then*

$$cond_2(M+\frac{\Delta t}{2}A) \le \frac{C_2}{C_1}\left[1 + C_{INV}^2 \frac{\triangle t}{2h^2}\left(\max_x \nu_T(x,h)\right)\right].$$

Proof. First note that $M+\frac{\Delta t}{2}A$ is clearly SPD. Let $\overrightarrow{\alpha} = (\alpha_1, \cdots, \alpha_N)^t$ be an eigenvector of $M+\frac{\Delta t}{2}A$ and define $v^h := \sum_{i=1}^N \alpha_i \phi_i^h$. We have

$$\begin{aligned}\lambda|\overrightarrow{\alpha}|^2 = \overrightarrow{\alpha}^t(M+\frac{\Delta t}{2}A)\overrightarrow{\alpha} = B(v^h, v^h) = \\ = ||v^h||^2 + \frac{\Delta t}{2}(\nu_T[I-P_H]\nabla v^h, [I-P_H]\nabla v^h).\end{aligned}$$

If $\lambda = \lambda_{\min}$ then by dropping the term $\frac{\Delta t}{2}(\nu_T[I-P_H]\nabla v^h, [I-P_H]\nabla v^h)$ and using norm equivalence we have

$$\lambda_{\min} = \frac{||v^h||^2 + \frac{\Delta t}{2}(\nu_T[I-P_H]\nabla v^h, [I-P_H]\nabla v^h)}{|\overrightarrow{\alpha}|^2} \ge C_2^{-1} h^d.$$

If $\lambda = \lambda_{\max}$ then by majorizing the term $\frac{\Delta t}{2}(\nu_T[I-P_H]\nabla v^h, [I-P_H]\nabla v^h)$ and using the inverse estimate and norm equivalence we have

$$\begin{aligned}\lambda_{\max} &= \frac{||v^h||^2 + \frac{\Delta t}{2}(\nu_T[I-P_H]\nabla v^h, [I-P_H]\nabla v^h)}{|\overrightarrow{\alpha}|^2} \le \\ &\le \frac{||v^h||^2 + \frac{\Delta t}{2}(\nu_T \nabla v^h, \nabla v^h)}{|\overrightarrow{\alpha}|^2} \le C_1^{-1} h^d \frac{||v^h||^2 + \frac{\triangle t}{2}\left(\max_x \nu_T(x,h)\right)||\nabla v^h||^2}{||v^h||^2} \\ &\le C_1^{-1} h^d \frac{||v^h||^2 + \frac{\triangle t}{2}\left(\max_x \nu_T(x,h)\right) C_{INV}^2 h^{-2} ||v^h||^2}{||v^h||^2} \le C_1^{-1} h^d \left[1 + C_{INV}^2 \frac{\triangle t}{2h^2}\left(\max_x \nu_T(x,h)\right)\right].\end{aligned}$$

The result follows by dividing these two estimates. ∎

In many cases the dependence of $\nu_T(x,h)$ upon h scales like $O(h^2)$, implying (in these cases) that $\nu_T(x,h) = O(1)$.

Consider next the fixed point iteration for solving (4.1).

ALGORITHM 4.4. *Until convergence criteria are satisfied, given $\underline{u}_j \in V_h$ find $\underline{u}_{j+1} \in V_h$ satisfying*

$$(\underline{u}_{j+1}, v_h) + \frac{\Delta t}{2}(\nu_T \nabla \underline{u}_{j+1}, \nabla v_h) = \frac{\Delta t}{2}(\nu_T P_H \nabla \underline{u}_j, \nabla v_h) + (w_h^{n+1}, v_h) - \frac{\Delta t}{2}(\nu_T[I-P_H]\nabla w_h^{n+1}, \nabla v_h)$$

for all $v_h \in V_h$.

THEOREM 4.5. *Let* $\{\underline{u}_j\}_{j\in N}$ *be determined by Algorithm 4.4. Suppose that* $0 < C_1 \le \nu_T \le C_2 < \infty$. *Then* $\underline{u}_j \to \Pi w_h^{n+1}$ *in* X *as* $j \to \infty$.

Proof. Subtracting the above equalities defining $\underline{u}_j$ and $\underline{u}_{j+1}$ yields

$$(\underline{u}_{j+1} - \underline{u}_j, v_h) + \frac{\Delta t}{2}(\nu_T \nabla(\underline{u}_{j+1} - \underline{u}_j), \nabla v_h) = \frac{\Delta t}{2}(\nu_T P_H \nabla(\underline{u}_j - \underline{u}_{j-1}), \nabla v_h).$$

Set $v_h = \underline{u}_{j+1} - \underline{u}_j$ and applying Young's inequality to the RHS gives

$$\begin{aligned}\|\underline{u}_{j+1} - \underline{u}_j\|^2 + \frac{\Delta t}{2}\|\sqrt{\nu_T}\nabla(\underline{u}_{j+1} - \underline{u}_j)\|^2 &\le \frac{\Delta t}{4}\|\sqrt{\nu_T}P_H\nabla(\underline{u}_j - \underline{u}_{j-1})\|^2 + \frac{\Delta t}{4}\|\sqrt{\nu_T}\nabla(\underline{u}_{j+1} - \underline{u}_j)\|^2 \\ &\le \frac{\Delta t}{4}\|\sqrt{\nu_T}\nabla(\underline{u}_j - \underline{u}_{j-1})\|^2 + \frac{\Delta t}{4}\|\sqrt{\nu_T}\nabla(\underline{u}_{j+1} - \underline{u}_j)\|^2.\end{aligned}$$

Applying the inverse estimate $\|\underline{u}_{j+1} - \underline{u}_j\|^2 \ge C_{INV}^{-2} h^2 \|\nabla(\underline{u}_{j+1} - \underline{u}_j)\|^2$, we obtain

$$C_{INV}^{-2} h^2 \|\nabla(\underline{u}_{j+1} - \underline{u}_j)\|^2 + \frac{\Delta t}{4}\|\sqrt{\nu_T}\nabla(\underline{u}_{j+1} - \underline{u}_j)\|^2 \le \frac{\Delta t}{4}\|\sqrt{\nu_T}\nabla(\underline{u}_j - \underline{u}_{j-1})\|^2.$$

Since ν_T is bounded from above by C_1 we have

$$\frac{h^2}{C_{INV}^2 C_1}\|\sqrt{\nu_T}\nabla(\underline{u}_{j+1} - \underline{u}_j)\|^2 \le C_{INV}^{-2} h^2 \|\nabla(\underline{u}_{j+1} - \underline{u}_j)\|^2.$$

Therefore

$$\left(\frac{h^2}{C_{INV}^2 C_1} + \frac{\Delta t}{4}\right)\|\sqrt{\nu_T}\nabla(\underline{u}_{j+1} - \underline{u}_j)\|^2 \le \frac{\Delta t}{4}\|\sqrt{\nu_T}\nabla(\underline{u}_j - \underline{u}_{j-1})\|^2.$$

This implies (as a consequence of Contraction Mapping Theorem) both existence and uniqueness of a solution u to (4.1) and convergence. ∎

5. A Computationally Attractive Variant. The projector Π in Algorithm 3.2 is the solution of

$$\frac{1}{\Delta t}(w^{n+1} - u^{n+1}, v_h) = (\nu_T [I - P_H] \nabla \frac{w^{n+1} + u^{n+1}}{2}, [I - P_H] \nabla v_h).$$

The difficulty with this system for u^{n+1} is coupling across many fine mesh elements caused by the projection P_H. First note that the above is equivalent to

$$\frac{1}{\Delta t}(w^{n+1} - u^{n+1}, v_h) = (\nu_T \nabla \frac{w^{n+1} + u^{n+1}}{2}, \nabla v_h) - (\nu_T P_H \nabla \frac{w^{n+1} + u^{n+1}}{2}, \nabla v_h).$$

Thus the difficulty is given by the second term alone. We consider the modification of Step 2 in Algorithm 3.2 of just lagging this term as in

Step 2': Given $w^{n+1} \in V_h$, find $u^{n+1} \in V_h$ satisfying

$$\frac{1}{\Delta t}(w^{n+1} - u^{n+1}, v_h) = (\nu_T \nabla \frac{w^{n+1} + u^{n+1}}{2}, \nabla v_h) - (\nu_T P_H \nabla \frac{w^n + u^n}{2}, \nabla v_h), \forall v_h \in V_h. \quad (5.1)$$

In (5.1) the action of P_H is calculated for a known function and goes into the RHS of the linear system (5.1). Surprisingly, we show this to be unconditionally stable and second order accurate.

We thus consider the modification of Algorithm 3.2 below.

ALGORITHM 5.1.

Step 1: Given u_h^n *find* $w_h^{n+1} \in X_h, p_h^{n+1} \in Q_h$ *satisfying*

$$\begin{aligned}(\frac{w_h^{n+1} - u_h^n}{\Delta t}, v_h) + b^*(\frac{w_h^{n+1} + u_h^n}{2}, \frac{w_h^{n+1} + u_h^n}{2}, v_h) + \nu(\nabla \frac{w_h^{n+1} + u_h^n}{2}, \nabla v_h) - (p_h^{n+1/2}, \nabla \cdot v_h) \\ = (f^{n+1/2}, v_h),\ \textit{for all } v_h \in X_h, \quad (5.2) \\ (\nabla \cdot w_h^{n+1}, q_h) = 0,\ \textit{for all } q_h \in Q_h.\end{aligned}$$

Step 2: $u_h^{n+1} := \Pi w_h^{n+1}$ *where* $(u_h^{n+1}, \lambda_h) \in X_h \times Q_h$ *is the unique solution of*

$$\frac{1}{\Delta t}(w_h^{n+1} - u_h^{n+1}, v_h) - (\lambda_h, \nabla \cdot v_h) = (\nu_T \nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}, \nabla v_h) - (\nu_T P_H \nabla \frac{w_h^n + u_h^n}{2}, \nabla v_h), \forall v_h \in X_h\,,$$
$$(\nabla \cdot u_h^{n+1}, q_h) = 0, \forall q_h \in Q_h.$$

THEOREM 5.2. *Assume ν_T is constant in space at each time level. Consider Algorithm 5.1. It satisfies, for any $N > 0$, the following energy equality, implying stability,*

$$\frac{1}{2}\left[||u_h^N||^2 + \Delta t||\sqrt{\nu_T} P_H \nabla \frac{w_h^N + u_h^N}{2}||^2\right] + \Delta t \sum_{n=0}^{N-1} \nu ||\nabla \frac{w_h^{n+1} + u_h^n}{2}||^2$$
$$+ \Delta t \sum_{n=0}^{N-1} ||\sqrt{\nu_T}[I - P_H]\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}||^2 + \Delta t \sum_{n=0}^{N-1} \frac{\Delta t^2}{8} ||\sqrt{\nu_T} P_H \nabla \left[\frac{w_h^{n+1} - w_h^n}{\Delta t} + \frac{u_h^{n+1} - u_h^n}{\Delta t}\right]||^2$$
$$= \frac{1}{2}\left[||u_h^0||^2 + \Delta t||\sqrt{\nu_T} P_H \nabla \frac{w_h^0 + u_h^0}{2}||^2\right] + \Delta t \sum_{n=0}^{N-1} \frac{1}{2}(f^{n+1/2}, w_h^{n+1} + u_h^n).$$

Proof. Take the L^2 inner product of (5.2) with $(w_h^{n+1} + u_h^n)/2$. Rearranging the result gives

$$\frac{1}{2\Delta t}\left[\|u_h^{n+1}\|^2 - \|u_h^n\|^2\right] + \nu \|\nabla \frac{w_h^{n+1} + u_h^n}{2}\|^2 +$$
$$+ \frac{1}{2\Delta t}\left[\|w_h^{n+1}\|^2 - \|u_h^{n+1}\|^2\right] = \frac{1}{2}(f^{n+1/2}, w_h^{n+1} + u_h^n).$$

Now consider Step 2. Set $v_h = \left(w_h^{n+1} + u_h^{n+1}\right)/2$. This gives

$$\frac{1}{2\Delta t}\left[||w_h^{n+1}||^2 - ||u_h^{n+1}||^2\right]$$
$$= ||\sqrt{\nu_T}\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}||^2 - (\nu_T P_H \nabla \frac{w_h^n + u_h^n}{2}, \nabla \frac{w_h^{n+1} + u_h^{n+1}}{2})$$
$$= ||\sqrt{\nu_T}\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}||^2 - (\nu_T P_H \nabla \frac{w_h^n + u_h^n}{2}, P_H \nabla \frac{w_h^{n+1} + u_h^{n+1}}{2})$$
$$= ||\sqrt{\nu_T}\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}||^2 - \left\{\frac{1}{2}||\sqrt{\nu_T} P_H \nabla \frac{w_h^n + u_h^n}{2}||^2 + \right.$$
$$\left. + \frac{1}{2}||\sqrt{\nu_T} P_H \nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}||^2 - \frac{1}{2}\left\|\sqrt{\nu_T} P_H \nabla \left[\frac{w_h^{n+1} + u_h^{n+1}}{2} - \frac{w_h^n + u_h^n}{2}\right]\right\|^2\right\}$$
$$= ||\sqrt{\nu_T}(I - P_H)\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}||^2 + \frac{1}{2}\left\|\sqrt{\nu_T} P_H \nabla \left[\frac{w_h^{n+1} + u_h^{n+1}}{2} - \frac{w_h^n + u_h^n}{2}\right]\right\|^2 +$$
$$+ \left\{\frac{1}{2}||\sqrt{\nu_T} P_H \nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}||^2 - \frac{1}{2}||\sqrt{\nu_T} P_H \nabla \frac{w_h^n + u_h^n}{2}||^2\right\}.$$

Now insert the above RHS in the energy estimate for the term $\frac{1}{2\Delta t}\left[||w_h^{n+1}||^2 - ||u_h^{n+1}||^2\right]$. This gives

$$\frac{1}{2\Delta t}\left[||u_h^{n+1}||^2 - ||u_h^n||^2\right] + \left\{\frac{1}{2}||\sqrt{\nu_T} P_H \nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}||^2 - \frac{1}{2}||\sqrt{\nu_T} P_H \nabla \frac{w_h^n + u_h^n}{2}||^2\right\} +$$
$$+ \nu ||\nabla \frac{w_h^{n+1} + u_h^n}{2}||^2 + ||\sqrt{\nu_T}(I - P_H)\nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}||^2 + \frac{1}{2}||\sqrt{\nu_T} P_H \nabla \left[\frac{w_h^{n+1} + u_h^{n+1}}{2} - \frac{w_h^n + u_h^n}{2}\right]||^2$$
$$= \frac{1}{2}(f^{n+1/2}, w_h^{n+1} + u_h^n).$$

Summing from $n = 0$ to $N-1$ yields the result. □

REMARK 5.3. *The form of the kinetic energy and numerical diffusion induced by Algorithm 5.1 is*

$$Kinetic\ Energy = \frac{1}{2}\left[||u_h^N||^2 + \Delta t||\sqrt{\nu_T}P_H\nabla\frac{w_h^N + u_h^N}{2}||^2\right]$$
$$Viscous\ Diffusion = \nu||\nabla\frac{w_h^{n+1} + u_h^n}{2}||^2$$
$$VMS\ Diffusion = ||\sqrt{\nu_T}[I - P_H]\nabla\frac{w_h^{n+1} + u_h^{n+1}}{2}||^2$$
$$Additional\ Algorithmic\ Diffusion = \frac{\Delta t^2}{8}||\sqrt{\nu_T}P_H\nabla\left[\frac{w_h^{n+1} - w_h^n}{\Delta t} + \frac{u_h^{n+1} - u_h^n}{\Delta t}\right]||^2$$

6. Numerical Experiments. We present numerical experiments to test the algorithms presented herein. Using the Green-Taylor vortex problem, we confirm the predicted convergence rates from the theory. Further testing is then performed using the well-known benchmark of the decaying homogeneous isotropic turbulence to compare the algorithms presented herein to the usual approach where everthing is applied in one step. We used *FreeFEM++* [HP] for the Green-Taylor vortex and *deal.II* [BHK07a, BHK07b] for the decaying homogeneous isotropic turbulence.

6.1. Green-Taylor vortex. For the first test we select the velocity field given by the Green-Taylor vortex, [GT37], [Tay23], which is used as a numerical test in many papers, e.g., Chorin [Cho68], Tafti [Taf96], John and Layton [JL02], Barbato, Berselli and Grisanti [BBG07] and Berselli [Ber05]. The exact velocity field is given by

$$\begin{aligned} u_1(x,y,t) &= -\cos(\omega\pi x)\sin(\omega\pi y)e^{-2\omega^2\pi^2 t/\tau}, \\ u_2(x,y,t) &= \sin(\omega\pi x)\cos(\omega\pi y)e^{-2\omega^2\pi^2 t/\tau}, \\ p(x,y,t) &= -\frac{1}{4}(\cos(2\omega\pi x) + \cos(2\omega\pi y))e^{-4\omega^2\pi^2 t/\tau}. \end{aligned} \tag{6.1}$$

We take the time interval $0 \le t \le 1$ and

$$\omega = 2,\ \tau = Re = 500,\ \Omega = (0,1)^2,\ h = 1/m,\ \Delta t = h/10,\ H^2 = h$$

where m is the number of subdivisions of the interval $(0,1)$. We utilize Taylor-Hood finite elements for the discretization. Newton iterations are applied to solve the nonlinear system with a $\|w_{(j+1)} - w_{(j)}\|_{H^1(\Omega)} < 10^{-10}$ as a stopping criterion. For the fixed point iteration in Algorithm 4.4, the convergence criterion is $\|\underline{u}_{(j+1)} - \underline{u}_{(j)}\|_{H^1(\Omega)} < 10^{-10}$. Convergence rates are calculated from the error at two successive values of h in the usual manner by postulating $e(h) = Ch^\beta$ and solving for β via $\beta = \ln(e(h_1)/e(h_2))/\ln(h_1/h_2)$. The boundary conditions could be taken to be periodic (the easier case). Instead we take the boundary condition on the problem to be inhomogeneous Dirichlet: $u_h = u_{exact}$, on $\partial\Omega$.

The errors and rates of convergence are presented below in Table 6.1 and 6.2.

h	Δt	$\|u - u_h\|_{\infty,0}$	rate	$\|\nabla u - \nabla u_h\|_{2,0}$	rate
1/16	1/160	3.788e-2	−−	4.560e-1	−−
1/25	1/250	1.306e-2	2.39	2.009e-1	1.84
1/36	1/360	4.819e-3	2.73	8.627e-2	2.32
1/49	1/490	1.900e-3	3.02	3.975e-2	2.51
1/64	1/640	8.674e-4	2.94	1.931e-2	2.70
1/81	1/810	4.395e-4	2.89	1.009e-2	2.75
1/100	1/1000	2.642e-4	2.42	5.818e-3	2.61

TABLE 6.1
Error and convergence rate data for Algorithm 4.4

h	Δt	$\|u - u_h\|_{\infty,0}$	rate	$\|\nabla u - \nabla u_h\|_{2,0}$	rate
1/16	1/160	3.776e-2	$--$	4.546e-1	$--$
1/25	1/250	1.303e-2	2.38	2.007e-1	1.83
1/36	1/360	4.811e-3	2.73	8.624e-2	2.32
1/49	1/490	1.897e-3	3.02	3.974e-2	2.51
1/64	1/640	8.657e-4	2.94	1.931e-2	2.70
1/81	1/810	4.387e-4	2.89	1.009e-2	2.75
1/100	1/1000	2.638e-4	2.41	5.817e-3	2.61

TABLE 6.2

Error and convergence rate data for Algorithm 5.1

FE	$\mathbb{Q}_2/\mathbb{Q}_1$, $L_H = \{0\}$	$\mathbb{Q}_2/\mathbb{Q}_1$, $L_H = \mathbb{Q}_0^{disc}$
C_*	0.0942	0.2010

TABLE 6.3

Correspondig C_* for different finite element large scale spaces

From the tables, we see that the rates of convergence of both algorithms confirm the predicted convergence rates from theory. Algorithm 5.1 (in which the projected term in Step 2 is simply lagged to the previous time level) proves itself to be effective. While it does not utilize any iterative method in Step 2, the quality of its errors is as good as full solve VMS algorithm.

6.2. Decaying Homogeneous Isotropic Turbulence. Our next numerical illustration is for the three dimensional flow of the decaying homogeneous isotropic turbulence. The setting is a domain $\Omega = [0, 2\pi]^3$ with periodic boundary conditions on all sides of Ω and right hand side $f = 0$.

For comparison, we consider the experimental results of [CBC71] which provide energy spectra at three different times. We take the first for calculating the turbulent initial data and compare the numerical solution to the remaining two energy spectra. Therefore we apply a Fourier transform $\hat{u}(k,t) = \int_\Omega u(x,t)e^{-ik\cdot x}dx$ and get the values of the energy spectrum of the numerical solution $E(k,t) = \frac{1}{2}\sum_{k-\frac{1}{2}\leq|k|\leq k+\frac{1}{2}} \hat{u}(k,t)\cdot\hat{u}(k,t)$ for a given time t. The experiment in [CBC71] is prescribed by a Taylor scale Reynolds number $Re_\lambda = 150$ and $\nu = 1.494 \times 10^{-5}$ (Reynolds number for air).

For the simulations we apply the FE library *deal.II*, see [BHK07a, BHK07b], with the one legged Crank-Nicolson time discretization scheme. The time-step size is taken as $\Delta t = 0.0174$, since smaller values showed no improvement. We apply the inf-sup stable Taylor-Hood element $\mathbb{Q}_2/\mathbb{Q}_1$ for the discretization of velocity and pressure in space.

We will use this test case to have a fair comparison of the method developed within this paper to the usual VMS approach for this method. Therefore we choose ν_T to be as optimized for the usual VMS approach in [RL10]. We do not tune parameters, because the parameters used were derived analytically based on arguments of Lilly [Lil67]. We set ν_T to be cellwise constant and for every cell $K \subset \Omega$

$$\nu_T = C_*\Delta^2\|[I - P_H]\nabla u\|_{L^2(K)}.$$

The nonlinearity is iterated linearly within the Stokes iteration of Algorithm 4.4, i.e. $\nu_T = C_*\Delta^2\|[I - P_H]\nabla \underline{u}_{(j)}\|_{L^2(K)}$. The filterwidth is taken to be $\Delta = \frac{\min(\Delta x,\Delta y,\Delta z)}{2(q-1)}$, where $q \geq 2$ is the polynomial degree of the finite element space for the velocity and the parameter C_* is chosen like in Table 1 of [RL10], see Table 6.3 herein.

To illustrate the behaviour of the decaying turbulence we show some results on the development of the kinetic energy, approximated by $\|u_h\|^2$, in Figure 6.1. The kinetic energy of the approximated solution is shown with 32^3 degrees of freedom for the pure Galerkin method without any additional stabilization etc. We observe that a turbulence model is really necessary, since the energy does not decay. The other lines correspond to the usual approaches of the variational multiscale method and the Smagorinsky in comparison to the Algorithms with an explicitly uncoupled postprocessing step developed herein. They are denoted by Exp. VMS, respectively Exp. Smagorinsky. We obtain that the additional postprocessing step induces additional numerical diffusion and that the Smagorinsky induces more diffusion than the VMS method.

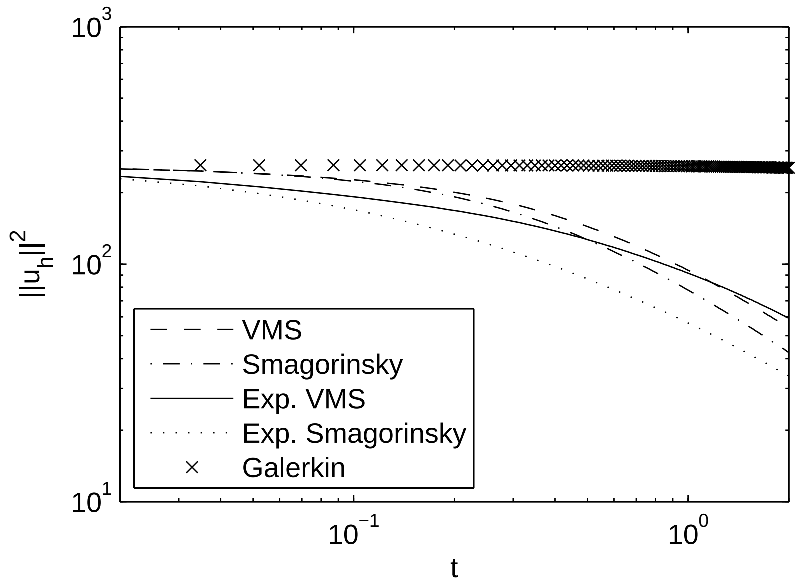

FIG. 6.1. *The decay of energy with time for different schemes*

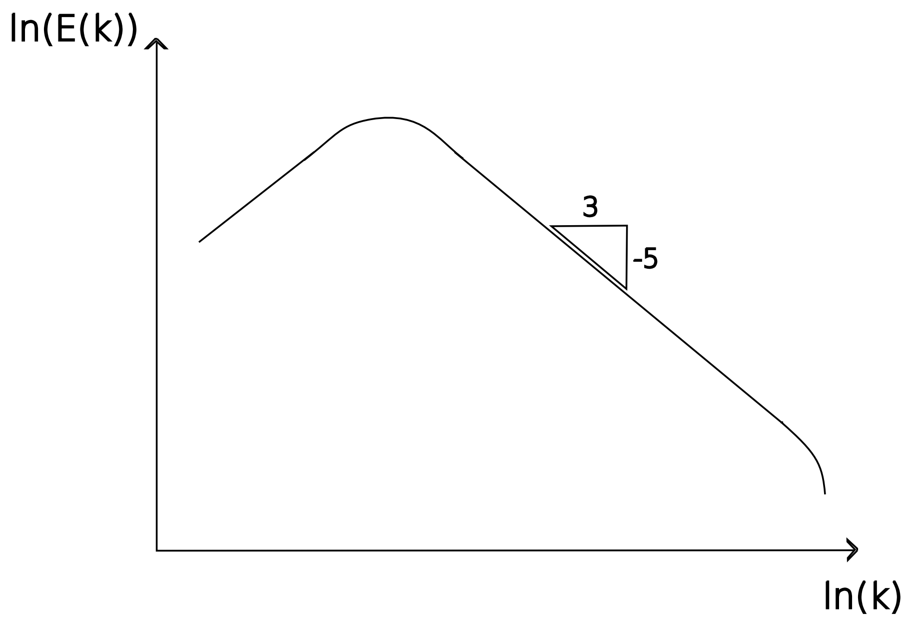

FIG. 6.2. *Kolmogorov energy spectrum*

Next, we look at the energy spectrum at different points in time and we expect spectra satisfying the famous $-5/3$-law of Kolmogorov. This law is illustrated by Figure 6.2, where we obtain how the spectral amplitude of the kinetic energy $E(k)$ depends on the wave number.

In Figure 6.3 we see results for the Smagorinsky model, i.e. $L_H = \{0\}$. With 32^3 degrees of freedom for the velocity we obtain values of the energy spectrum in good agreement to the reference data. When we apply the Smagorinsky model in a postprocessing step we can see that more dissipation is induced and the results are more close to the $-5/3$-law. The plot also indicates that the Galerin method without any turbulence model can not predict the energy spectra from [CBC71] at the times $t = 0.87$ and $t = 2.0$.

Figure 6.4 illustrates a comparable behaviour as the explicitly uncoupled postprocessing step induces more dissipation to the system than the usual VMS approach. In this case the results of the postprocessed

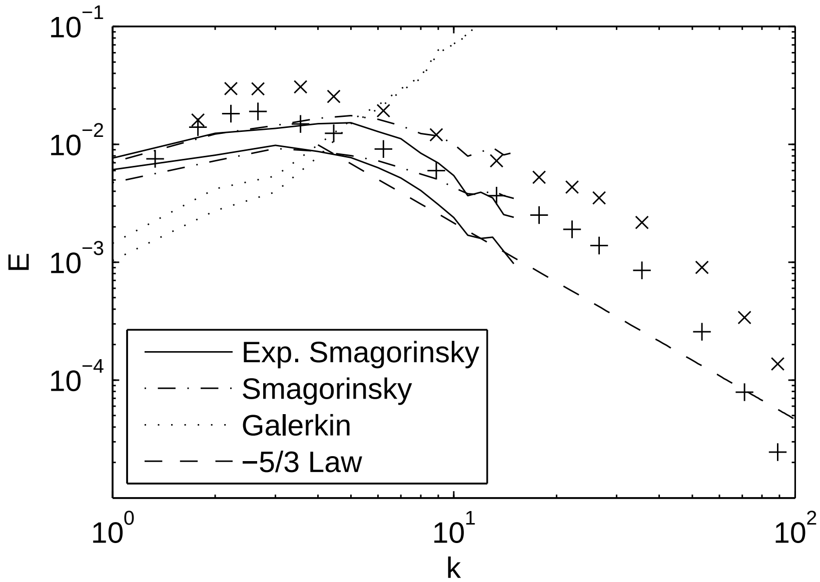

FIG. 6.3. *Energy spectra observed with the Smagorinsky model in comparison to a postprocessed Smagorinsky step, i.e. Algorithm 3.2 with $L_H = \{0\}$. 'x' and '+' denote the experimental data from Ref. [CBC71].*

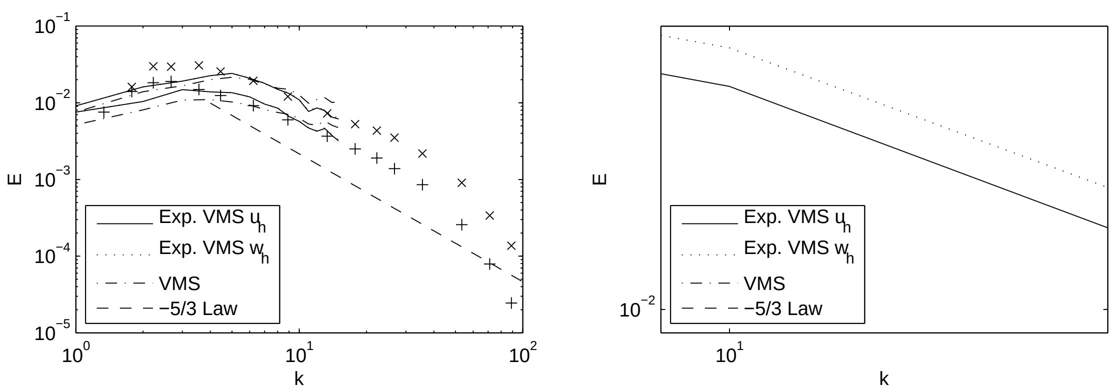

FIG. 6.4. *Energy spectra observed with the usual VMS model in comparison to a postprocessed VMS step, i.e. Algorithm 3.2 with $L_H = \mathbb{Q}_0^{disc}$, and a zoom into the plot on the right. 'x' and '+' denote the experimental data from Ref. [CBC71].*

method are even closer to the reference data than the pure VMS results. On the left side of Figure 6.4 the dotted line corresponding to the solution of Step 1 in Algorithm 3.2 is very hard to see. That is why we added a zoom into the plot on the right side, where one can see that the values for w_h are always very close to the values of u_h but always higher. This is exactly what one would expect, since the Step 2 is adding the numerical diffusion steming from the VMS method in every time step.

In Figure 6.5 we present some observations concerning Algorithm 4.4 to compute the projection in Step 2 of Algorithm 3.2. Here we see that the number of iterations decays and that very few iterations are needed, except in the first steps. This decay might be related to the stopping criteria since the L^2-norm of the solution is decaying with the energy and the stopping criterion depends on the L^2-norm. Nevertheless, the results are very satisfying and in good agreement to the theory.

7. Conclusions. Summarizing, we conclude that the treatment of the variational multiscale method as a postprocessing step is an effective method to introduce a modern approach to turbulence in a given fluid code. It is stable and accurate. Further in our tests it has shown that it can predict the energy spectra of the decaying homogenenous isotropic turbulence very well in comparison with the usual approach. To

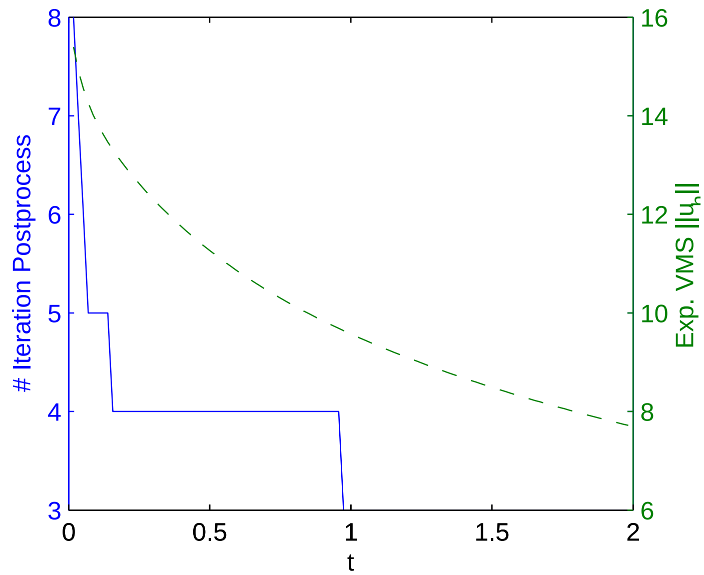

FIG. 6.5. *Number of iterations in Algorithm 4.4 with stopping criterion* $\|\underline{u}_{(j+1)} - \underline{u}_{(j)}\| < 10^{-4}$ *(solid line) and the decay of the* L^2*-norm of the approximated solution (dashed line).*

compute the projection with Algorithm 4.4 very few iterations are needed and the (preliminary) results from just lagging the troublesome term are also positive.

REFERENCES

[ALP04] M. Anitescu, W. Layton, and F. Pahlevani. Unconditional Stability for Numerical Scheme Combining Implicit Timestepping for Local Effects and Explicit Timestepping for Nonlocal Effects. *Electron. Trans. Numer. Anal.*, 18:174–187, 2004.

[AM08] T. Apel and G. Matthies. Non-conforming, anisotropic, rectangular finite elements of arbitrary order for the Stokes problem. *SIAM J. Numer. Anal.*, 46(4):1867–1891, 2008.

[BBG07] D. Barbato, L.C. Berselli, and C.R. Grisanti. Analytical and numerical results for the rational large eddy simulation model. *J. Math. Fluid Mech.*, 9:44–747, 2007.

[Ber05] L.C. Berselli. On the large eddy simulation of the taylor-green vortex. *J. Math. Fluid Mech.*, 7:164–191, 2005.

[BH82] Alexander N. Brooks and Thomas J.R. Hughes. Streamline upwind/petrov-galerkin formulations for convection dominated flows with particular emphasis on the incompressible navier-stokes equations. *Comp. Meth. Appl. Mech. Engrg.*, 32(1-3):199 – 259, 1982.

[BHK07a] W. Bangerth, R. Hartmann, and G. Kanschat. `deal.II` — a General Purpose Object Oriented Finite Element Library. *ACM Trans. Math. Software*, 33(4), 2007. article 24.

[BHK07b] W. Bangerth, R. Hartmann, and G. Kanschat. `deal.II` *Differential Equations Analysis Library, Technical Reference*, 2007. `http://www.dealii.org`.

[BIL06] L.C. Berselli, T. Iliescu, and W.J. Layton. *Mathematics of Large Eddy Simulation of Turbulent Flows.* Springer, Berlin, Heidelberg, 2006.

[BS94] S. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods.* Springer-Verlag, 1994.

[CBC71] G. Comte-Bellot and S. Corrsin. Simple Eulerian time correlation of full- and narrow-band velocity signals in grid-generated, 'isotropic' turbulence. *F. Fluid Mech.*, 48(part. 2):273–337, 1971.

[Cho68] A.J. Chorin. Numerical solution for the navier-stokes equations. *Math. Comp.*, 22:745–762, 1968.

[ELN07] V. Ervin, W.J. Layton, and M. Neda. Numerical analysis of a higher order time relaxation model of fluids. *Int. J. Numer. Anal. and Modeling*, 4:648–670, 2007.

[ELN10] V. Ervin, W.J. Layton, and M. Neda. Numerical analysis of filter based stabilization for evolution equations. *technical report, TR-MATH 10-01, http://www.mathematics.pitt.edu/research/technical-reports.php*, 2010.

[GGKW10] V. Gravemeier, M.W. Gee, M. Kronbichler, and W.A. Wall. An algebraic variational multiscale-multigrid method for large eddy simulation of turbulent flow. *Comp. Meth. Appl. Mech. Engrg.*, 199(13-16):853–864, 2010.

[Gra06] V. Gravemeier. The variational multiscale method for laminar and turbulent flow. *Arch. Comput. Meth. Engrg.*, 13:249–324, 2006.

[GS03] V. Girault and L.R. Scott. A quasi-local interpolation operator preserving the discrete divergence. *Calcolo*, 40:1–19, 2003.

[GT37] A.E. Green and G.I. Taylor. Mechanism of the production of small eddies from larger ones. *Proc. Royal Soc. A*,

158:499–521, 1937.
[Gue99] J.-L. Guermond. Stabilization of galerkin approximations of transport equations by subgrid modeling. *M2AN*, 33(6):1293–1316, nov 1999.
[Gun89] M. D. Gunzburger. *Finite Element Methods for Viscous Incompressible Flows - A Guide to Theory, Practices, and Algorithms.* Academic Press, 1989.
[HMJ00] T. Hughes, L. Mazzei, and K. Jansen. Large Eddy Simulation and the Variational Multiscale Method. *Comput. Visual. Sci.*, 3:47–59, 2000.
[HP] F. Hecht and O. Pironneau. *FreeFEM++*. `http://www.freefem.org`.
[HR90] J.G. Heywood and R. Rannacher. Finite element approximation of the nonstationary navier-stokes problem. part iv: Error analysis for second-order time discretization. *SIAM J. Numer. Anal.*, 2:353–384, 1990.
[Hug95] T.J.R. Hughes. Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles, and the origins of stabilized methods. *Comp. Meth. Appl. Mech. Engrg.*, 127:387–401, 1995.
[JK10] V. John and A. Kindl. A variational multiscale method for turbulent flow simulation with adaptive large scale space. *J. Comp. Phy.*, 229:301–312, 2010.
[JL02] V. John and W.J. Layton. Analysis of numerical errors in large eddy simulation. *SIAM J. Numer. Anal*, 40:995–1020, 2002.
[Joh04] V. John. *Large Eddy Simulation of Turbulent Incompressible Flows, Analytical and Numerical Results for a Class of LES Models*, volume 34. LNCSE Springer, Berlin, 2004.
[Lay02] W.J. Layton. A connection between subgrid scale eddy viscosity and mixed methods. *Appl. Math. Comput.*, 133:147–157, 2002.
[Lil67] D.K. Lilly. The representation of small-scale turbulence in numerical simulation experiments. *Proc. IBM Sci. Comp. Symp. on Envir. Sci.*, (320):195–210, 1967.
[LRL08] G. Lube, G. Rapin, and J. Löwe. Local projection stabilization for incompressible flows: Equal-order vs. inf-sup stable interpolation. *Electronic Transactions on Numerical Analysis*, 32:106–122, 2008.
[LRT10] W.J. Layton, L.G. Rebholz, and C. Trenchea. Modular nonlinear filter stabilization of methods for higher reynolds numbers flow. *technical report, TR-MATH 10-10, http://www.mathematics.pitt.edu/research/technical-reports.php*, 2010.
[RL10] L. Röhe and G. Lube. Analysis of a variational multiscale method for Large-Eddy simulation and its application to homogeneous isotropic turbulence. *Comp. Meth. Appl. Mech. Engrg.*, 199:2331–2342, 2010.
[RST08] H.-G. Roos, M. Stynes, and L. Tobiska. *Robust Numerical Methods for Singularly Perturbed Differential Equations.* Springer, Berlin, 2008.
[Taf96] D. Tafti. Comparison of some upwind-biased high-order formulations with a second order central-difference scheme for time integration of the incompressible navier-stokes equations. *Comput. & Fluids*, 25:647–665, 1996.
[Tay23] G.I. Taylor. On decay of vortices in a viscous fluid. *Phil. Mag.*, 46:671–674, 1923.

Appendix A. Error Analysis of the Postprocessed VMS Approximation.

In this section we prove Theorem 3.7. This proof is intricate and technical. It also exhibits the usual limitations in the final result that arise from employing the discrete Gronwall inequality (exponential error growth and the assumption that Δt is sufficiently small).

Denote $\tilde{w}_h^{n+1/2} := \frac{w_h^{n+1}+u_h^n}{2}$. To begin the analysis we rewrite Algorithm 3.2. As the spaces X_h and Q_h satisfy the usual inf-sup condition, Algorithm 3.2 is equivalent to:
For $n = 0, 1, \ldots, N_T - 1$ find w_h^{n+1}, $u_h^{n+1} \in V_h$ such that

$$(w_h^{n+1}, v_h) + \Delta t\, b^*(\tilde{w}_h^{n+1/2}, \tilde{w}_h^{n+1/2}, v_h) + \Delta t\, \nu(\nabla \tilde{w}_h^{n+1/2}, \nabla v_h) = (u_h^n, v_h) + \Delta t\, (f^{n+1/2}, v_h),\ \forall v_h \in V_h \tag{A.1}$$

$$\frac{1}{\Delta t}(w_h^{n+1} - u_h^{n+1}, v_h) = (\nu_T\, [I - P_H]\, \nabla \frac{w_h^{n+1} + u_h^{n+1}}{2}, [I - P_H]\, \nabla v_h),\ \forall v_h \in V_h. \tag{A.2}$$

To establish the optimal asymptotic error estimates for the approximation we assume true solution satisfies the regularity condition (3.3) from Section 3:

At time $t^{n+1/2} = (n + 1/2)\Delta t$ the true solution u of (2.2), (2.3) satisfies

$$\begin{aligned}(u^{n+1} - u^n, v_h) + \Delta t\, \nu(\nabla u^{n+1/2}\,, \nabla v_h) + \Delta t\, b^*(u^{n+1/2}\,, u^{n+1/2}\,, v_h) - \Delta t\, (p(t^{n+1/2}), \nabla \cdot v_h)\\ = \Delta t\, (f^{n+1/2}, v_h) + \Delta t\, Intp(u^{n+1}; v_h)\ ,\end{aligned} \tag{A.3}$$

for all $v_h \in V_h$, where $Intp(u^{n+1}; v_h)$, representing the consistency error, denotes

$$
\begin{aligned}
Intp(u^{n+1}; v_h) = & \Big((u^{n+1} - u^n)/\Delta t - u_t(t^{n+1/2}), v_h \Big) \; + \; \nu(\nabla u^{n+1/2} - \nabla u(t^{n+1/2}) \, , \, \nabla v_h) \\
& + \; b^*(u^{n+1/2} \, , u^{n+1/2} \, , \; v_h) \; - \; b^*(u(t^{n+1/2}), u(t^{n+1/2}), v_h) \\
& + \; (f(t^{n+1/2}) - f^{n+1/2}, v_h) \, .
\end{aligned} \tag{A.4}
$$

We split the error into a Step 1 error ε_h according to (A.1), a Step 2 error e_h according to (A.2), and an approximation error Λ

$$
\begin{aligned}
u^{n+1} - w_h^{n+1} = (u^{n+1} - I_h u^{n+1}) + (I_h u^{n+1} - w_h^{n+1}) =: \Lambda^{n+1} + \varepsilon_h^{n+1}, \\
u^{n+1} - u_h^{n+1} = (u^{n+1} - I_h u^{n+1}) + (I_h u^{n+1} - u_h^{n+1}) =: \Lambda^{n+1} + e_h^{n+1},
\end{aligned} \tag{A.5}
$$

where $I_h u^{n+1} \in V_h$ will be an interpolation of u^{n+1} in V_h later in the proof but is an arbitrary element in V_h at this point. Now we subtract (A.1) from (A.3) and use $\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n) \in V_h$ as test function v_h to obtain

$$
\begin{aligned}
\frac{1}{2} \left(\|\varepsilon_h^{n+1}\|^2 \; - \; \|e_h^n\|^2 \right) \; + \; \Delta t \, \nu \|\nabla \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 = & \\
- \, (\Lambda^{n+1} - \Lambda^n, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \; - \; \Delta t \, \nu(\nabla \Lambda^{n+1/2}, \nabla \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) & \\
- \; \Delta t \, b^*(u^{n+1/2} \, , u^{n+1/2} \, , \; \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \; + \; \Delta t \, b^*(\tilde{w}_h^{n+1/2}, \tilde{w}_h^{n+1/2}, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) & \\
+ \; \Delta t (p(t^{n+1/2}) - q_h^{n+1}, \nabla \cdot \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \; + \; \Delta t \, Intp(u^{n+1}; \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)). &
\end{aligned} \tag{A.6}
$$

The key to this equation is that $\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)$ is discretely divergence free and hence a possible test function v_h. Next we estimate the terms on the RHS of (A.6) and get

$$
\begin{aligned}
(\Lambda^{n+1} - \Lambda^n, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) & \leq \frac{1}{2} \Delta t \| \frac{\Lambda^{n+1} - \Lambda^n}{\Delta t} \|^2 + \frac{1}{2} \Delta t \| \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \\
& = \; \frac{1}{2} \Delta t \int_\Omega \left(\frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \Lambda_t \, dt \right)^2 d\Omega \; + \frac{1}{2} \Delta t \| \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \\
& \leq \; \frac{1}{2} \Delta t \int_\Omega \left(\frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} |\Lambda_t|^2 \, dt \right) d\Omega \; + \frac{1}{2} \Delta t \| \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \\
& \leq \; \frac{1}{2} \int_{t^n}^{t^{n+1}} \|\Lambda_t\|^2 \, dt \; + \frac{1}{4} \Delta t \left(\|\varepsilon_h^{n+1}\|^2 \; + \; \|e_h^n\|^2 \right)
\end{aligned} \tag{A.7}
$$

$$
\nu(\nabla \Lambda^{n+1/2}, \nabla \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \; \leq \; \frac{\nu}{10} \|\nabla \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \; + \; C\nu \, \|\nabla \Lambda^{n+1/2}\|^2 \, . \tag{A.8}
$$

We rewrite $b^*(u^{n+1/2} \, , u^{n+1/2} \, , \; \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \; - \; b^*(\tilde{w}_h^{n+1/2}, \tilde{w}_h^{n+1/2}, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n))$ as

$$
\begin{aligned}
& b^*(u^{n+1/2} \, , u^{n+1/2} \, , \; \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \; - \; b^*(\tilde{w}_h^{n+1/2}, \tilde{w}_h^{n+1/2}, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \\
= \; & b^*(u^{n+1/2} \, , u^{n+1/2} \, , \; \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \; - \; b^*(\tilde{w}_h^{n+1/2}, u^{n+1/2}, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \\
& + \; b^*(\tilde{w}_h^{n+1/2}, u^{n+1/2}, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \; - \; b^*(\tilde{w}_h^{n+1/2}, \tilde{w}_h^{n+1/2}, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n))
\end{aligned}
$$

$$
\begin{aligned}
&= b^*(\frac{1}{2}((u^{n+1} - w_h^{n+1}) + (u^n - u_h^n))\,, u^{n+1/2}\,,\, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \\
&\quad + \, b^*(\tilde{w}_h^{n+1/2}, \frac{1}{2}((u^{n+1} - w_h^{n+1}) + (u^n - u_h^n)), \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \\
&= b^*(\Lambda^{n+1/2} + \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\,, u^{n+1/2}\,,\, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \\
&\quad + b^*(\tilde{w}_h^{n+1/2}, \Lambda^{n+1/2} + \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\,, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \\
&= b^*(\Lambda^{n+1/2}, u^{n+1/2}\,,\, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \, + b^*(\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\,, u^{n+1/2}\,,\, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \\
&\quad + b^*(\tilde{w}_h^{n+1/2}, \Lambda^{n+1/2}\,, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n))\,,
\end{aligned}
\tag{A.9}
$$

where we used the skew symmetry of b^*. Using $b^*(u,v,w) \le C(\Omega)\sqrt{\|u\|\,\|\nabla u\|}\,\|\nabla v\|\,\|\nabla w\|$, for $u,\ v,\ w\ \in X$, and Young's inequality, we bound the terms on the RHS of (A.9) as follows.

$$
\begin{aligned}
b^*(\Lambda^{n+1/2}\,, u^{n+1/2}\,,\, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) &\le C\,\sqrt{\|\Lambda^{n+1/2}\|\,\|\nabla\Lambda^{n+1/2}\|}\,\|\nabla u^{n+1/2}\|\,\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\| \\
&\le \frac{\nu}{10}\,\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \ + \ C\,\nu^{-1}\,\|\Lambda^{n+1/2}\|\,\|\nabla\Lambda^{n+1/2}\|\,\|\nabla u^{n+1/2}\|^2
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
b^*(\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\,, u^{n+1/2}\,,\, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) &\le C\,\|\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^{1/2}\,\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^{3/2}\,\|\nabla u^{n+1/2}\| \\
&\le \frac{\nu}{10}\,\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \ + \ C\,\nu^{-3}\,\|\nabla u^{n+1/2}\|^4\,\|\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \\
&\le \frac{\nu}{10}\,\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \ + \ C\,\nu^{-3}\,\|\nabla u^{n+1/2}\|^4\,\left(\|\varepsilon_h^{n+1}\|^2 \ + \ \|e_h^n\|^2\right)
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
b^*(\tilde{w}_h^{n+1/2}\,, \Lambda^{n+1/2}\,,\, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) &\le C\,\|\nabla\tilde{w}_h^{n+1/2}\|\,\|\nabla\Lambda^{n+1/2}\|\,\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\| \\
&\le \frac{\nu}{10}\,\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \ + \ C\,\nu^{-1}\,\|\nabla\tilde{w}_h^{n+1/2}\|^2\,\|\nabla\Lambda^{n+1/2}\|^2
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
(p(t^{n+1/2}) - q_h^{n+1}, \nabla\cdot\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) &\le \|p(t^{n+1/2}) - q_h^{n+1}\|\,\|\nabla\cdot\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\| \\
&\le \frac{\nu}{10}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \ + \ C\,\nu^{-1}\,\|p(t^{n+1/2}) - q_h^{n+1}\|^2\,.
\end{aligned}
\tag{A.13}
$$

The *consistency error term* $\Delta t|Intp(u^{n+1};\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n))|$ in (A.6) can be bounded as follows.

LEMMA A.1. *Under the regularity assumption* (3.3) *from Section 3 there holds*

$$
\begin{aligned}
\Delta t|Intp(u^{n+1};\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n))| &\le \frac{\Delta t}{2}\left(\|\varepsilon_h^{n+1}\|^2 + \|e_h^n\|^2\right) + \frac{\nu\Delta t}{4}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \\
+\frac{C(\Delta t)^5}{\nu}\left(\|\nabla u^{n+1/2}\|^4 + \|\nabla u(t^{n+1/2})\|^4\right) &+ C(\Delta t)^4\int_{t^n}^{t^{n+1}}\left(\|u_{ttt}\|^2 + \nu\|\nabla u_{tt}\|^2 + \frac{1}{\nu}\|\nabla u_{tt}\|^4 + \|f_{tt}\|^2\right)dt.
\end{aligned}
$$

Proof. We want to estimate every term in the definition of $Intp(u^{n+1};v_h))$ from (A.4) and obtain

$$
\begin{aligned}
((u^{n+1} - u^n)/\Delta t - u_t(t^{n+1/2}), \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) &\le \frac{1}{2}\|\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 + \frac{1}{2}\|(u^{n+1} - u^n)/\Delta t - u_t(t^{n+1/2})\|^2 \\
&\le \frac{1}{4}\|\varepsilon_h^{n+1}\|^2 + \frac{1}{4}\|e_h^n\|^2 + \frac{1}{2}\frac{(\Delta t)^3}{1280}\int_{t^n}^{t^{n+1}}\|u_{ttt}\|^2 dt
\end{aligned}
$$

$$(f(t^{n+1/2}) - f^{n+1/2}, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \le \frac{1}{2}\|\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 + \frac{1}{2}\|f(t^{n+1/2}) - f^{n+1/2}\|^2$$
$$\le \frac{1}{4}\|\varepsilon_h^{n+1}\|^2 + \frac{1}{4}\|e_h^n\|^2 + \frac{1}{2}\frac{(\Delta t)^3}{48}\int_{t^n}^{t^{n+1}} \|f_{tt}\|^2 dt$$

$$\nu(\nabla u^{n+1/2} - \nabla u(t^{n+1/2}), \nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \le \frac{\nu}{8}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 + 2\nu\|\nabla u^{n+1/2} - \nabla u(t^{n+1/2})\|^2$$
$$\le \frac{\nu}{8}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 + 2\nu\frac{(\Delta t)^3}{48}\int_{t^n}^{t^{n+1}} \|\nabla u_{tt}\|^2 dt,$$

where we used inequalities of Section 6 in [ELN07]. Also with these inequalities we get an estimate of the terms of the nonlinearity

$$\begin{aligned}
&b^*(u^{n+1/2}, u^{n+1/2}, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) - b^*(u(t^{n+1/2}), u(t^{n+1/2}), \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \\
&= b^*(u^{n+1/2} - u(t^{n+1/2}), u^{n+1/2}, \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) + b^*(u(t^{n+1/2}), u^{n+1/2} - u(t^{n+1/2}), \frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)) \\
&\le C\|\nabla(u^{n+1/2} - u(t^{n+1/2}))\|\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\| \left(\|\nabla u^{n+1/2}\| + \|\nabla u(t^{n+1/2})\|\right) \\
&\le \frac{\nu}{8}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 + C\nu^{-1}\|\nabla(u^{n+1/2} - u(t^{n+1/2}))\|^2 \left(\|\nabla u^{n+1/2}\|^2 + \|\nabla u(t^{n+1/2})\|^2\right) \\
&\le \frac{\nu}{8}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 + C\nu^{-1}\frac{(\Delta t)^3}{48}\left(\|\nabla u^{n+1/2}\|^2 + \|\nabla u(t^{n+1/2})\|^2\right)\int_{t^n}^{t^{n+1}} \|\nabla u_{tt}\|^2 dt \\
&\le \frac{\nu}{8}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 + C\nu^{-1}\frac{(\Delta t)^3}{48}\int_{t^n}^{t^{n+1}} \|\nabla u_{tt}\|^2 \left(\|\nabla u^{n+1/2}\|^2 + \|\nabla u(t^{n+1/2})\|^2\right) dt \\
&\le \frac{\nu}{8}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 + C\nu^{-1}\frac{(\Delta t)^3}{48}\left(\int_{t^n}^{t^{n+1}} \|\nabla u_{tt}\|^4 dt + \int_{t^n}^{t^{n+1}} \left(\|\nabla u^{n+1/2}\|^4 + \|\nabla u(t^{n+1/2})\|^4\right) dt\right) \\
&\le \frac{\nu}{8}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 + C\frac{(\Delta t)^3}{48\nu}\left(\Delta t\left(\|\nabla u^{n+1/2}\|^4 + \|\nabla u(t^{n+1/2})\|^4\right) + \int_{t^n}^{t^{n+1}} \|\nabla u_{tt}\|^4 dt\right).
\end{aligned}$$

Combining all estimates yields the lemma. ◻

The application of Lemma A.1 to (A.6) together with the estimates (A.7)–(A.13) gives

$$\begin{aligned}
&\frac{1}{2}\left(\|\varepsilon_h^{n+1}\|^2 - \|e_h^n\|^2\right) + \Delta t\,\frac{\nu}{4}\|\nabla\frac{1}{2}(\varepsilon_h^{n+1} + e_h^n)\|^2 \\
&\qquad \le C\Delta t\,(1 + \nu^{-3}\|\nabla u^{n+1/2}\|^4)\left(\|\varepsilon_h^{n+1}\|^2 + \|e_h^n\|^2\right) + C\nu\Delta t\|\nabla\Lambda^{n+1/2}\|^2 \\
&\qquad\quad + \frac{C\Delta t}{\nu}\|\nabla\tilde{w}_h^{n+1/2}\|^2\|\nabla\Lambda^{n+1/2}\|^2 + \frac{C\Delta t}{\nu}\|\nabla u^{n+1/2}\|^2\|\Lambda^{n+1/2}\|\|\nabla\Lambda^{n+1/2}\| \\
&\qquad\quad + \frac{C\Delta t}{\nu}\|p(t^{n+1/2}) - q_h^{n+1}\|^2 + \frac{C(\Delta t)^5}{\nu}\left(\|\nabla u^{n+1/2}\|^4 + \|\nabla u(t^{n+1/2})\|^4\right) \\
&\qquad\quad + C\int_{t^n}^{t^{n+1}} \|\Lambda_t\|^2 dt + C(\Delta t)^4\int_{t^n}^{t^{n+1}} \left(\|u_{ttt}\|^2 + \nu\|\nabla u_{tt}\|^2 + \frac{1}{\nu}\|\nabla u_{tt}\|^4 + \|f_{tt}\|^2\right) dt.
\end{aligned} \tag{A.14}$$

As u_h^{n+1} and w_h^{n+1} are connected through the *variational multiscale* projection in Step 2, we next use that equation to obtain a relationship between $\|\varepsilon_h^n\|$ and $\|e_h^n\|$.

Lemma A.2. *There holds*

$$\|\varepsilon_h^{n+1}\|^2 = \|e_h^{n+1}\|^2 + \frac{1}{2}\Delta t\|\sqrt{\nu_T}[I - P_H]\nabla(\varepsilon_h^{n+1} + e_h^{n+1})\|^2$$
$$+ \Delta t(\nu_T[I - P_H]\nabla(\Lambda^{n+1} - u^{n+1}), [I - P_H]\nabla(\varepsilon_h^{n+1} + e_h^{n+1})).$$

Proof. From (A.2) we have

$$\left(\frac{w_h^{n+1}-u_h^{n+1}}{\Delta t}, v_h\right) = \left(\nu_T[I-P_H]\nabla\frac{w_h^{n+1}+u_h^{n+1}}{2}, [I-P_H]\nabla v_h\right)$$

and set $v_h = (w_h^{n+1} - I_h u^{n+1}) + (u_h^{n+1} - I_h u^{n+1}) = -(\varepsilon_h^{n+1} + e_h^{n+1})$. We obtain

$$\left(\frac{-(\varepsilon_h^{n+1}-e_h^{n+1})}{\Delta t}, -(\varepsilon_h^{n+1}+e_h^{n+1})\right) = \\ \left(\nu_T[I-P_H]\nabla\frac{-(\varepsilon_h^{n+1}+e_h^{n+1})+2I_h u^{n+1}}{2}, [I-P_H]\nabla(-(\varepsilon_h^{n+1}+e_h^{n+1}))\right).$$

Hence

$$\frac{1}{\Delta t}(\|\varepsilon_h^{n+1}\|^2 - \|e_h^{n+1}\|^2) = \frac{1}{2}\left\|\sqrt{\nu_T}[I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1})\right\|^2 \\ -\ (\nu_T[I-P_H]\nabla I_h u^{n+1}, [I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1}))$$

and with $I_h u^{n+1} = u^{n+1} - \Lambda^{n+1}$ from (A.5) we conclude the proof. □

Substituting Lemma A.2 into (A.14), we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|e_h^{n+1}\|^2 - \|e_h^n\|^2\right) + \frac{\Delta t}{4}\left(\nu\|\nabla\frac{1}{2}(\varepsilon_h^{n+1}+e_h^n)\|^2 + \|\sqrt{\nu_T}[I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1})\|^2\right)\\
&\le C\Delta t(1+\nu^{-3}\|\nabla u^{n+1/2}\|^4)\left(\|e_h^{n+1}\|^2 + \|e_h^n\|^2\right) + C\nu\Delta t\|\nabla\Lambda^{n+1/2}\|^2\\
&\quad + C(\Delta t)^2(1+\nu^{-3}\|\nabla u^{n+1/2}\|^4)\Big(\frac{1}{2}\|\sqrt{\nu_T}[I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1})\|^2\\
&\quad + (\nu_T[I-P_H]\nabla(\Lambda^{n+1}-u^{n+1}), [I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1}))\Big)\\
&\quad + \frac{\Delta t}{2}(\nu_T[I-P_H]\nabla(u^{n+1}-\Lambda^{n+1}), [I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1}))\\
&\quad + \frac{C\Delta t}{\nu}\|\nabla\tilde{w}_h^{n+1/2}\|^2\|\nabla\Lambda^{n+1/2}\|^2 + \frac{C\Delta t}{\nu}\|\nabla u^{n+1/2}\|^2\|\Lambda^{n+1/2}\|\,\|\nabla\Lambda^{n+1/2}\|\\
&\quad + \frac{C\Delta t}{\nu}\|p(t^{n+1/2}) - q_h^{n+1}\|^2 + \frac{C(\Delta t)^5}{\nu}\left(\|\nabla u^{n+1/2}\|^4 + \|\nabla u(t^{n+1/2})\|^4\right)\\
&\quad + C\int_{t^n}^{t^{n+1}}\|\Lambda_t\|^2 dt + C(\Delta t)^4\int_{t^n}^{t^{n+1}}\left(\|u_{ttt}\|^2 + \nu\|\nabla u_{tt}\|^2 + \frac{1}{\nu}\|\nabla u_{tt}\|^4 + \|f_{tt}\|^2\right)dt.
\end{aligned} \tag{A.15}$$

Since we can estimate

$$|(\nu_T[I-P_H]\nabla(\Lambda^{n+1}-u^{n+1}), [I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1}))| \\ \le \frac{1}{8}\|\sqrt{\nu_T}[I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1})\|^2 + C\|\sqrt{\nu_T}[I-P_H]\nabla(\Lambda^{n+1}-u^{n+1})\|^2,$$

it is possible to choose Δt sufficiently small, i.e., $C\Delta t < \frac{1}{16}(1+\nu^{-3}\|\nabla u^{n+1/2}\|^4)^{-1}$ such that the terms steming from the VMS method are hidden and after summing this up from $n=0$ to $n=N-1$ equation

(A.15) results in

$$
\begin{aligned}
&\frac{1}{2}\|e_h^N\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\left(\nu\|\nabla\frac{1}{2}(\varepsilon_h^{n+1}+e_h^n)\|^2 + \frac{1}{2}\|\sqrt{\nu_T}[I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1})\|^2\right)\\
&\qquad \le \sum_{n=0}^{N-1}\Bigg\{C\Delta t(1+\nu^{-3}\|\nabla u^{n+1/2}\|^4)(\|e_h^{n+1}\|^2+\|e_h^n\|^2)\\
&\qquad + C\nu\Delta t\|\nabla\Lambda^{n+1/2}\|^2 + C\Delta t\left(\|\sqrt{\nu_T}[I-P_H]\nabla\Lambda^{n+1}\|^2 + \|\sqrt{\nu_T}[I-P_H]\nabla u^{n+1}\|^2\right)\\
&\qquad + \frac{C\Delta t}{\nu}\|\nabla\tilde{w}_h^{n+1/2}\|^2\,\|\nabla\Lambda^{n+1/2}\|^2 + \frac{C\Delta t}{\nu}\|\nabla u^{n+1/2}\|^2\|\Lambda^{n+1/2}\|\,\|\nabla\Lambda^{n+1/2}\|\\
&\qquad + \frac{C\Delta t}{\nu}\|p(t^{n+1/2})-q_h^{n+1}\|^2 + \frac{C(\Delta t)^5}{\nu}\left(\|\nabla u^{n+1/2}\|^4 + \|\nabla u(t^{n+1/2})\|^4\right)\\
&\qquad + C\int_{t^n}^{t^{n+1}}\|\Lambda_t\|^2dt + C(\Delta t)^4\int_{t^n}^{t^{n+1}}\left(\|u_{ttt}\|^2+\nu\|\nabla u_{tt}\|^2+\frac{1}{\nu}\|\nabla u_{tt}\|^4+\|f_{tt}\|^2\right)dt\Bigg\}.
\end{aligned}
$$

Now we choose the interpolation operator in V_h, constructed in [GS03, AM08], and a usual interpolation operator for the pressure, which leads us to

$$\|u - I_h u\|_r \le Ch^{k+1-r}|u|_{k+1},$$

where $r \le k$ and k is the polynomial degree of the corresponding FE space. Since P_H also fulfills the interpolation property, due to the regularity assumptions and Theorem 3.5 this gives

$$
\begin{aligned}
&\frac{1}{2}\|e_h^N\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\left(\nu\|\nabla\frac{1}{2}(\varepsilon_h^{n+1}+e_h^n)\|^2 + \frac{1}{2}\|\sqrt{\nu_T}[I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1})\|^2\right)\\
&\le \sum_{n=0}^{N} C\Delta t\left(1+\nu^{-3}|||\nabla u|||_{\infty,0}^4\right)\|e_h^n\|^2\\
&+ C\nu h^{2k}|||u|||_{2,k+1}^2 + C\nu_T h^{2k}|||u|||_{2,k+1}^2 + C\nu_T H^{2k}|||u|||_{2,k+1}^2\\
&+ C\frac{h^{2k}}{\nu^2}|||u|||_{\infty,k+1}^2 + C\frac{h^{2k+1}}{\nu}\left(|||u|||_{4,k+1}^4 + |||\nabla u|||_{4,0}^4\right) + C\frac{h^{2s+2}}{\nu}|||p_{1/2}|||_{2,s+1}^2\\
&+ Ch^{2k+2}|||u_t|||_{2,k+1}^2 + C(\Delta t)^4\Big(\frac{1}{\nu}|||\nabla u|||_{4,0}^4 + \frac{1}{\nu}|||\nabla u_{1/2}|||_{4,0}^4\\
&+ |||u_{ttt}|||_{2,0}^2 + \nu|||\nabla u_{tt}|||_{2,0}^2 + \frac{1}{\nu}|||\nabla u_{tt}|||_{4,0}^4 + |||f_{tt}|||_{2,0}^2\Big)
\end{aligned}
$$

The next step will be the application of Lemma A.3, the discrete Gronwall inequality, e.g., [HR90].

LEMMA A.3. *Let $D \ge 0$ and κ_n, A_n, B_n, $C_n \ge 0$ for any integer $n \ge 0$ and satisfy*

$$A_N + \Delta t\sum_{n=0}^{N} B_n \le \Delta t\sum_{n=0}^{N}\kappa_n A_n + \Delta t\sum_{n=0}^{N} C_n + D \text{ for } N \ge 0.$$

Suppose that for all n, $\Delta t\kappa_n < 1$, and set $g_n = (1-\Delta t\kappa_n)^{-1}$. Then,

$$A_N + \Delta t\sum_{n=0}^{N} B_n \le exp\left(\Delta t\sum_{n=0}^{N} g_n\kappa_n\right)\left[\Delta t\sum_{n=0}^{N} C_n + D\right] \text{ for } N \ge 0.$$

Let Δt be sufficiently small, i.e., $C\Delta t < (1+\nu^{-3}|||\nabla u|||_{\infty,0}^4)^{-1}$, it is allowed to apply the lemma and we obtain

$$
\begin{aligned}
&\frac{1}{2}\|e_h^N\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\left(\nu\|\nabla\frac{1}{2}(\varepsilon_h^{n+1}+e_h^n)\|^2 + \frac{1}{2}\|\sqrt{\nu_T}[I-P_H]\nabla(\varepsilon_h^{n+1}+e_h^{n+1})\|^2\right)\\
&\leq C\nu h^{2k}|||u|||_{2,k+1}^2 + C\nu_T h^{2k}|||u|||_{2,k+1}^2 + C\nu_T H^{2k}|||u|||_{2,k+1}^2\\
&+C\frac{h^{2k}}{\nu^2}|||u|||_{\infty,k+1}^2 + C\frac{h^{2k+1}}{\nu}\left(|||u|||_{4,k+1}^4 + |||\nabla u|||_{4,0}^4\right) + C\frac{h^{2s+2}}{\nu}|||p_{1/2}|||_{2,s+1}^2\\
&+Ch^{2k+2}|||u_t|||_{2,k+1}^2 + C(\Delta t)^4\Big(\frac{1}{\nu}|||\nabla u|||_{4,0}^4 + \frac{1}{\nu}|||\nabla u_{1/2}|||_{4,0}^4\\
&+|||u_{ttt}|||_{2,0}^2 + \nu|||\nabla u_{tt}|||_{2,0}^2 + \frac{1}{\nu}|||\nabla u_{tt}|||_{4,0}^4 + |||f_{tt}|||_{2,0}^2\Big)
\end{aligned}
$$

Now we have an estimate for the model error e_h and it is left to find an error estimate for the whole error. We obtain

$$
\begin{aligned}
&\frac{1}{2}\|u^N - u_h^N\|^2 +\\
&\quad\frac{\Delta t}{4}\sum_{n=0}^{N-1}\left(\nu\|\nabla(u(t^{n+1/2}) - (w_h^{n+1}+u_h^n)/2)\|^2 + \|\sqrt{\nu_T}[I-P_H]\nabla(u(t^{n+1}) - (w_h^{n+1}+u_h^{n+1})/2)\|^2\right)\\
&\quad\leq \|\Lambda^N\|^2 + \|e_h^N\|^2 + C\nu\Delta t\sum_{n=0}^{N-1}(\|\nabla(u^{n+1/2} - u(t^{n+1/2}))\|^2 + \|\nabla\Lambda^{n+1/2}\|^2 + \|\nabla\frac{1}{2}(\varepsilon_h^{n+1}+e_h^n)\|^2)\\
&\qquad + C\Delta t\sum_{n=0}^{N-1}(\|\sqrt{\nu_T}[I-P_H]\nabla\Lambda^{n+1}\|^2 + \|\sqrt{\nu_T}[I-P_H]\nabla\frac{1}{2}(\varepsilon_h^{n+1}+e_h^{n+1})\|^2),
\end{aligned}
$$

where the upcoming new terms are either already contained in the RHS of the model error, or easy to handle like e.g. with Lemma A.1. Combining all estimates from above we get Theorem 3.7 and (in the particular case) Corollary 3.8.