LARGE EDDY SIMULATION FOR MHD FLOWS

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Abstract. We consider the mathematical properties of a model for the simulation of the large eddies in turbulent viscous, incompressible, electrically conducting flows. We prove existence and uniqueness of weak solutions for the simplest (zeroth) closed MHD model (1.7), we prove that the solutions to the LES-MHD equations converge to the solution of the MHD equations in a weak sense as the averaging radii converge to zero, and we derive a bound on the modeling error. Furthermore, we show that the model preserve the properties of the 3D MHD equations. In particular, we prove that the kinetic energy and the magnetic helicity of the model are conserved, while the model's cross helicity is approximately conserved and converges to the cross helicity of the MHD equations as the radii δ_1, δ_2 tend to zero. Also, the model is proven to preserve the Alfvén waves, with the velocity converging to that of the MHD, as δ_1, δ_2 tend to zero.

Key words. Large eddy simulation, magnetohydrodynamics, deconvolution

1. Introduction.

Magnetically conducting fluids arise in important applications including plasma physics, geophysics and astronomy. In many of these turbulent MHD flows are typical. The difficulties of accurately modeling and simulating turbulent flows are magnified many times over in the MHD case. They are evinced by the more complex dynamics of the flow due to the coupling of Navier-Stokes and Maxwell equations via the Lorenz force and Ohm's law.

In this report we consider the problem of modeling the motion of large structures in a viscous, incompressible, electrically conducting, turbulent fluid.

The MHD (magnetohydrodynamics [2]) equations are related to engineering problems such as plasma confinement, controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD sea water propulsion.

The flow of an electrically conducting fluid is affected by Lorentz forces, induced by the interaction of electric currents and magnetic fields in the fluid. The Lorentz forces can be used to control the flow and to attain specific engineering design goals such as flow stabilization, suppression or delay of flow separation, reduction of near-wall turbulence and skin friction, drag reduction and thrust generation. There is a large body of literature dedicated to both experimental and theoretical investigations on the influence of electromagnetic force on flows (see e.g., [17, 23, 24, 16, 34, 12, 35, 18, 30, 7]). The MHD effects arising from the macroscopic interaction of liquid metals with applied currents and magnetic fields are exploited in metallurgical processes to control the flow of metallic melts: the electromagnetic stirring of molten metals [25], electromagnetic turbulence control in induction furnaces [36], electromagnetic damping of buoyancy-driven flow during solidification [26], and the electromagnetic shaping of ingots in continuous casting [27].

The mathematical description of the problem proceeds as follows. Assuming the fluid to be viscous and incompressible, the governing equations are the Navier- Stokes and pre-Maxwell equations, coupled via the Lorentz force and Ohm's law (see e.g. [29]). Let $\Omega = (0, L)^3$ be the flow domain, and $u(t, x), p(t, x), B(t, x)$ be the velocity, pressure, and the magnetic field of the flow, drived by the velocity body force f and magnetic field force $\operatorname{curl} g$. Then u, p, B satisfy the magnetohydrodynamics (MHD)

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equations:

$$
\begin{aligned}
&u_t + \nabla\cdot(uu^T) - \frac{1}{\mathrm{Re}}\Delta u + \frac{S}{2}\nabla(B^2) - S\nabla\cdot(BB^T) + \nabla p = f,\\
&B_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\, B) + \mathrm{curl}\,(B\times u) = \mathrm{curl}\, g,\\
&\nabla\cdot u = 0, \nabla\cdot B = 0,
\end{aligned}
\tag{1.1}
$$

in $Q = (0,T)\times\Omega$, with the initial data:

$$
u(0,x) = u_0(x), \quad B(0,x) = B_0(x) \quad \text{in } \Omega, \tag{1.2}
$$

and with periodic boundary conditions (with zero mean):

$$
\Phi(t, x + L e_i) = \Phi(t,x), i = 1,2,3, \quad \int_\Omega \Phi(t,x)dx = 0, \tag{1.3}
$$

for $\Phi = u, u_0, p, B, B_0, f, g$.

Here Re, $\mathrm{Re_m}$, and S are nondimensional constants that characterize the flow: the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. For derivation of (1.1), physical interpretation and mathematical analysis, see [9, 19, 28, 15] and the references therein.

If $\overline{}^{\delta_1}$, $\overline{}^{\delta_2}$ denote two local, spacing averaging operators that commute with the differentiation, then averaging (1.1) gives the following non-closed equations for $\overline{u}^{\delta_1}$, $\overline{B}^{\delta_2}$, $\overline{p}^{\delta_1}$ in $(0,T)\times\Omega$:

$$
\begin{aligned}
&\overline{u}^{\delta_1}_t + \nabla\cdot(\overline{uu^T}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta\overline{u}^{\delta_1} - S\nabla\cdot(\overline{BB^T}^{\delta_1}) + \nabla\Big(\frac{S}{2}\overline{B^2}^{\delta_1} + \overline{p}^{\delta_1}\Big) = \overline{f}^{\delta_1},\\
&\overline{B}^{\delta_2}_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\,\overline{B}^{\delta_2}) + \nabla\cdot(\overline{Bu^T}^{\delta_2}) - \nabla\cdot(\overline{uB^T}^{\delta_2}) = \mathrm{curl}\,\overline{g}^{\delta_2},\\
&\nabla\cdot\overline{u}^{\delta_2} = 0, \quad \nabla\cdot\overline{B}^{\delta_2} = 0.
\end{aligned}
\tag{1.4}
$$

The usual closure problem which we study here arises because $\overline{uu^T}^{\delta_1} \neq \overline{u}^{\delta_1}\,\overline{u}^{\delta_1}$, $\overline{BB^T}^{\delta_1} \neq \overline{B}^{\delta_1}\,\overline{B}^{\delta_1}$, $\overline{uB^T}^{\delta_2} \neq \overline{u}^{\delta_1}\,\overline{B^T}^{\delta_2}$. To isolate the turbulence closure problem from the difficult problem of wall laws for near wall turbulence, we study (1.1) hence (1.4) subject to (1.3). The closure problem is to replace the tensors $\overline{uu^T}^{\delta_1}$, $\overline{BB^T}^{\delta_1}$, $\overline{uB^T}^{\delta_2}$ with tensors $\mathscr{T}(\overline{u}^{\delta_1}, \overline{u}^{\delta_1})$, $\mathscr{T}(\overline{B}^{\delta_2}, \overline{B}^{\delta_2})$, $\mathscr{T}(\overline{u}^{\delta_1}, \overline{B}^{\delta_2})$, respectively, depending only on $\overline{u}^{\delta_1}, \overline{B}^{\delta_2}$ and not u, B. There are many closure models proposed in large eddy simulation reflecting the centrality of closure in turbulence simulation. Calling w, q, W the resulting approximations to $\overline{u}^{\delta_1}, \overline{p}^{\delta_1}, \overline{B}^{\delta_2}$, we are led to considering the following model

$$
\begin{aligned}
&w_t + \nabla\cdot\mathscr{T}(w,w) - \frac{1}{\mathrm{Re}}\Delta w - S\mathscr{T}(W,W) + \nabla q = \overline{f}^{\delta_1}\\
&W_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\, W) + \nabla\cdot\mathscr{T}(w,W) - \nabla\cdot\mathscr{T}(W,w) = \mathrm{curl}\,\overline{g}^{\delta_2},\\
&\nabla\cdot w = 0, \quad \nabla\cdot W = 0.
\end{aligned}
\tag{1.5}
$$

With any reasonable averaging operator, the true averages $\overline{u}^{\delta_1}, \overline{B}^{\delta_2}, \overline{p}^{\delta_1}$ are smoother than u, B, p. We consider the simplest, accurate closure model that is exact on constant flows (i.e., $\overline{u}^{\delta_1} = u, \overline{B}^{\delta_2} = B$) is

$$
\begin{aligned}
\overline{uu^T}^{\delta_1} &\approx \overline{\overline{u}^{\delta_1}\, \overline{u^T}^{\delta_1}}^{\delta_1} &=: \mathscr{T}(\overline{u}^{\delta_1}, \overline{u}^{\delta_1}),\\
\overline{BB^T}^{\delta_1} &\approx \overline{\overline{B}^{\delta_2}\, \overline{B^T}^{\delta_2}}^{\delta_1} &=: \mathscr{T}(\overline{B}^{\delta_2}, \overline{B}^{\delta_2}),\\
\overline{uB^T}^{\delta_2} &\approx \overline{\overline{u}^{\delta_1}\, \overline{B^T}^{\delta_2}}^{\delta_2} &=: \mathscr{T}(\overline{u}^{\delta_1}, \overline{B}^{\delta_2}),
\end{aligned}
\tag{1.6}
$$

leading to

$$
w_t + \nabla\cdot(\overline{ww^T}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta w - S\nabla\cdot(\overline{W\,W^T}^{\delta_1}) + \nabla q = \overline{f}^{\delta_1}, \tag{1.7a}
$$

$$
W_T + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\, W) + \nabla\cdot(\overline{Ww^T}^{\delta_2}) - \nabla\cdot(\overline{wW^T}^{\delta_2}) = \mathrm{curl}\,\overline{g}^{\delta_2}, \tag{1.7b}
$$

$$
\nabla\cdot w = 0, \quad \nabla\cdot W = 0, \tag{1.7c}
$$

subject to $w(x,0) = \overline{u}_0^{\delta_1}(x), W(x,0) = \overline{B}_0^{\delta_2}(x)$ and periodic boundary conditions (with zero means).

We shall show that the LES MHD model (1.7) has the mathematical properties which are expected of a model derived from the MHD equations by an averaging operation and which are important for practical computations using (1.7).

The model considered can be developed for quite general averaging operators, see e.g. [1]. The choice of averaging operator in (1.7) is a differential filter, defined as follows. Let the $\delta > 0$ denote the averaging radius, related to the finest computationally feasible mesh. (In this report we use different lengthscales for the Navier-Stokes and Maxwell equations). Given $\phi \in L^2_0(\Omega)$, $\overline{\phi}^{\delta} \in H^2(\Omega)\cap L^2_0(\Omega)$ is the unique solution of

$$
A_\delta \overline{\phi}^{\delta} := -\delta^2\Delta\overline{\phi}^{\delta} + \overline{\phi}^{\delta} = \phi \quad \text{in } \Omega, \tag{1.8}
$$

subject to periodic boundary conditions. Under periodic boundary conditions, this averaging operator commutes with differentiation, and with this averaging operator, the model (1.6) has consistency $O(\delta^2)$, i.e.,

$$
\begin{aligned}
\overline{uu^T}^{\delta_1} &= \overline{\overline{u}^{\delta_1}\, \overline{u^T}^{\delta_1}} &+ O({\delta_1}^2),\\
\overline{BB^T}^{\delta_1} &= \overline{\overline{B}^{\delta_2}\, \overline{B^T}^{\delta_2}}^{\delta_1} &+ O({\delta_2}^2),\\
\overline{uB^T}^{\delta_2} &= \overline{\overline{u}^{\delta_1}\, \overline{B^T}^{\delta_2}}^{\delta_2} &+ O({\delta_1}^2 + {\delta_2}^2),
\end{aligned}
$$

for smooth u, B. We prove that the model (1.7) has a unique, weak solution w, W that converges in the appropriate sense $w \to u$, $W \to B$, as $\delta_1, \delta_2 \to 0$.

In Section 2 we prove the global existence and uniqueness of the solution for the closed MHD model, after giving the notations and a definition. Section 3 treats the questions of limit consistency of the model and verifiability. The conservation of the kinetic energy and helicity for the approximate deconvolution model is presented in Section 4. Section 5 shows that the model preserves the Alfén waves, with the velocity tending to the velocity of Alfvén waves in the MHD, as the radii δ_1, δ_2 tend to zero.

2. Existence and uniqueness for the MHD LES equations.

2.1. Notations and preliminaries. We shall use the standard notations for function spaces in the space periodic case (see [33]). Let $H_p^m(\Omega)$ denote the space of functions (and their vector valued counterparts also) that are locally in $H^m(\mathbb{R}^3)$, are periodic of period L and have zero mean, i.e. satisfy (1.3). We recall the solenoidal space

$$\mathscr{D}(\Omega) = \{\phi \in C^\infty(\Omega) : \phi \text{ periodic with zero mean}, \nabla \cdot \phi = 0\},$$

and the closures of $\mathscr{D}(\Omega)$ in the usual $L^2(\Omega)$ and $H^1(\Omega)$ norms :

$$\begin{aligned} H &= \{\phi \in H_2^0(\Omega), \nabla \cdot \phi = 0 \text{ in } \mathscr{D}(\Omega)'\}^2, \\ V &= \{\phi \in H_2^1(\Omega), \nabla \cdot \phi = 0 \text{ in } \mathscr{D}(\Omega)'\}^2. \end{aligned}$$

We define the operator $\mathscr{A} \in \mathscr{L}(V, V')$ by setting

$$\langle \mathscr{A}(w_1, W_1), (w_2, W_2)\rangle = \int_\Omega \left(\frac{1}{\mathrm{Re}} \nabla w_1 \cdot \nabla w_2 + \frac{S}{\mathrm{Re_m}} \mathrm{curl}\, W_1 \mathrm{curl}\, W_2 \right) dx, \tag{2.1}$$

for all $(w_i, W_i) \in V$. The operator $\mathscr{A}$ is an unbounded operator on H, with the domain $D(\mathscr{A}) = \{(w, W) \in V; (\Delta w, \Delta W) \in H\}$ and we denote again by $\mathscr{A}$ its restriction to H.
We define also a continuous tri-linear form $\mathscr{B}_0$ on $V \times V \times V$ by setting

$$\begin{aligned} \mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3)) = \int_\Omega \Bigg(\nabla \cdot (\overline{w_2 w_1^T}^{\delta_1})\, w_3 \\ - S \nabla \cdot (\overline{W_2 W_1^T}^{\delta_1})\, w_3 + \nabla \cdot (\overline{W_2 w_1^T}^{\delta_2})\, W_3 - \nabla \cdot (\overline{w_2 W_1^T}^{\delta_2})\, W_3 \Bigg) dx \end{aligned} \tag{2.2}$$

and a continuous bilinear operator $\mathscr{B}(\cdot) : V \to V$ with

$$\langle \mathscr{B}(w_1, W_1), (w_2, W_2)\rangle = \mathscr{B}_0((w_1, W_1), (w_1, W_1), (w_2, W_2))$$

for all $(w_i, W_i) \in V$.
The following properties of the trilinear form $\mathscr{B}_0$ hold (see [22, 28, 14, 11])

$$\begin{aligned} &\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} w_2, S A_{\delta_2} W_2)) = 0, \\ &\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} w_3, S A_{\delta_2} W_3)) \\ &\qquad = -\mathscr{B}_0((w_1, W_1), (w_3, W_3), (A_{\delta_1} w_2, S A_{\delta_2} W_2)), \end{aligned} \tag{2.3}$$

for all $(w_i, W_i) \in V$. Also

$$\begin{aligned} &|\mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3))| \\ &\quad \le C \|(w_1, W_1)\|_{m_1} \|(w_2, W_2)\|_{m_2+1} \|(\overline{w_3}^{\delta_1}, \overline{W_3}^{\delta_2})\|_{m_3} \end{aligned} \tag{2.4}$$

for all $(w_1, W_1) \in H^{m_1}(\Omega), (w_2, W_2) \in H^{m_2+1}(\Omega), (w_3, W_3) \in H^{m_3}(\Omega)$ and

$$\begin{aligned} m_1 + m_2 + m_3 &\ge \frac{d}{2}, \quad \text{if } m_i \ne \frac{d}{2} \text{ for all } i = 1, \ldots, d, \\ m_1 + m_2 + m_3 &> \frac{d}{2}, \quad \text{if } m_i = \frac{d}{2} \text{ for any of } i = 1, \ldots, d. \end{aligned}$$

In terms of $V, H, \mathscr{A}, \mathscr{B}(\cdot)$ we can rewrite (1.7) as

$$\begin{aligned} &\frac{d}{dt}(w,W) + \mathscr{A}(w,W)(t) + \mathscr{B}((w,W)(t)) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}^{\delta_2}), t \in (0,T), \\ &(w,W)(0) = (\overline{u}_0^{\delta_1}, \overline{B}_0^{\delta_2}), \end{aligned} \tag{2.5}$$

where $(\boldsymbol{f}, \operatorname{curl}\boldsymbol{g}) = P(f, \operatorname{curl} g)$, and $P : L^2(\Omega) \to H$ is the Hodge projection.

Definition 2.1. *Let $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in H$, $\overline{f}^{\delta_1}, curl\overline{g}^{\delta_2} \in L^2(0,T;V')$. The measurable functions $w, W : [0,T] \times \Omega \to \mathbb{R}^3$ are the weak solutions of (2.5) if $w, W \in L^2(0,T;V) \cap L^\infty(0,T;H)$, and w, W satisfy*

$$\begin{aligned} &\int_\Omega w(t)\phi dx + \int_0^t \int_\Omega \frac{1}{\mathrm{Re}} \nabla w(\tau)\nabla\phi + \overline{w(\tau)\cdot\nabla w(\tau)}^{\delta_1}\phi - S\overline{W(\tau)\cdot\nabla W(\tau)}^{\delta_1}\phi\, dx d\tau \\ &\quad = \int_\Omega \overline{u_0}^{\delta_1}\phi dx + \int_0^t \int_\Omega \overline{f(\tau)}^{\delta_1}\phi\, dx d\tau, \\ &\int_\Omega W(t)\psi dx + \int_0^t \int_\Omega \frac{1}{\mathrm{Re_m}} \nabla W(\tau)\nabla\psi + \overline{w(\tau)\cdot\nabla W(\tau)}^{\delta_2}\psi - \overline{W(\tau)\cdot\nabla w(\tau)}^{\delta_2}\psi\, dx d\tau \\ &\quad = \int_\Omega \overline{B_0}^{\delta_2}\psi dx + \int_0^t \int_\Omega curl\,\overline{g(\tau)}^{\delta_2}\psi\, dx d\tau, \end{aligned} \tag{2.6}$$

$\forall t \in [0,T), \phi, \psi \in \mathscr{D}(\Omega)$.

Also, it is easy to show that for any $u, v \in H^1(\Omega)$ with $\nabla \cdot u = \nabla \cdot v = 0$, the following identity holds

$$\nabla \times (u \times v) = v \cdot \nabla u - u \cdot \nabla v. \tag{2.7}$$

2.2. Stability and existence for the model. The first result states that the weak solution of the MHD LES model (1.7) exists globally in time, for large data and general $\mathrm{Re}, \mathrm{Re}_m > 0$ and that it satisfies an energy equality while initial data and the source terms are smooth enough.

Theorem 2.2. *Let $\delta_1, \delta_2 > 0$ be fixed. For any $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in V$ and $(\overline{f}^{\delta_1}, curl\overline{g}^{\delta_2}) \in L^2(0,T;H)$, there exists a unique weak solution w, W to (1.7). The weak solution also belongs to $L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and $w_t, W_t \in L^2((0,T)\times\Omega)$. Moreover, the following energy equality holds for $t \in [0,T]$:*

$$\mathscr{M}(t) + \int_0^t \mathscr{N}(\tau)d\tau = \mathscr{M}(0) + \int_0^t \mathscr{P}(\tau)d\tau, \tag{2.8}$$

where

$$\begin{aligned} \mathscr{M}(t) &= \frac{{\delta_1}^2}{2}\|\nabla w(t,\cdot)\|_0^2 + \frac{1}{2}\|w(t,\cdot)\|_0^2 + \frac{{\delta_2}^2 S}{2}\|\nabla W(t,\cdot)\|_0^2 + \frac{S}{2}\|W(t,\cdot)\|_0^2, \\ \mathscr{N}(t) &= \frac{{\delta_1}^2}{\mathrm{Re}}\|\Delta w(t,\cdot)\|_0^2 + \frac{1}{\mathrm{Re}}\|\nabla w(t,\cdot)\|_0^2 + \frac{{\delta_2}^2 S}{\mathrm{Re_m}}\|\Delta W(t,\cdot)\|_0^2 + \frac{S}{\mathrm{Re_m}}\|\nabla W(t,\cdot)\|_0^2, \\ \mathscr{P}(t) &= (f(t), w(t)) + S(curl\, g(t), W(t)). \end{aligned} \tag{2.9}$$

We shall use the semigroup approach proposed in [6] for the Navier-Stokes equations, based on the machinery of nonlinear differential equations of accretive type in Banach spaces.

Let us define the modified nonlinearity $\mathscr{B}_N(\cdot) : V \to V$ by setting

$$\mathscr{B}_N(w,W) = \begin{cases} \mathscr{B}(w,W) & \text{if } \|(w,W)\|_1 \le N, \\ \left(\frac{N}{\|(w,W)\|_1}\right)^2 \mathscr{B}(w,W) & \text{if } \|(w,W)\|_1 > N. \end{cases} \tag{2.10}$$

By (2.4) we have for the case of $\|(w_1,W_1)\|_1, \|(w_2,W_2)\|_1 \le N$

$$\begin{aligned} &|\langle \mathscr{B}_N(w_1,W_1) - \mathscr{B}_N(w_2,W_2), (w_1-w_2, W_1-W_2)\rangle| \\ &= |\mathscr{B}_0((w_1-w_2,W_1-W_2),(w_1,W_1),(w_1-w_2,W_1-W_2)| \\ &\quad + \mathscr{B}_0((w_2,W_2),(w_1-w_2,W_1-W_2),(w_1-w_2,W_1-W_2)| \\ &\le C\|(w_1-w_2,W_1-W_2)\|_{1/2}\|(w_1,W_1)\|_1\|(\overline{w_1-w_2}^{\delta_1}, \overline{W_1-W_2}^{\delta_2})\|_1 \\ &\le \frac{\nu}{2}\|(w_1-w_2,W_1-W_2)\|_1^2 + C_N\|(w_1-w_2,W_1-W_2)\|_0^2, \end{aligned}$$

where $\nu = \inf\{1/\mathrm{Re}, S/\mathrm{Re}_m\}$.
In the case of $\|(w_i,W_i)\|_1 > N$ we have

$$\begin{aligned} &|\langle \mathscr{B}_N(w_1,W_1) - \mathscr{B}_N(w_2,W_2), (w_1-w_2, W_1-W_2)\rangle| \\ &= \frac{N^2}{\|(w_1,W_1)\|_1^2}\mathscr{B}_0((w_1-w_2,W_1-W_2),(w_1,W_1),(w_1-w_2,W_1-W_2)) \\ &\quad + \left(\frac{N^2}{\|(w_1,W_1)\|_1^2} - \frac{N^2}{\|(w_2,W_2)\|_1^2}\right)\mathscr{B}_0((w_2,W_2),(w_2,W_2),(w_1-w_2,W_1-W_2)) \\ &\le CN\|(w_1-w_2,W_1-W_2)\|_1^{3/2}\|(w_1-w_2,W_1-W_2)\|_0^{1/2} \\ &\quad + CN\|(w_1-w_2,W_1-W_2)\|_1^2 \\ &\le \frac{\nu}{2}\|(w_1-w_2,W_1-W_2)\|_1^2 + C_N\|(w_1-w_2,W_1-W_2)\|_0^2. \end{aligned}$$

For the case of $\|(w_1,W_1)\|_1 > N, \|(w_2,W_2)\|_1 \le N$ (similar estimates are obtained when $\|(w_1,W_1)\|_1 \le N, \|(w_2,W_2)\|_1 > N$) we have

$$\begin{aligned} &|\langle \mathscr{B}_N(w_1,W_1) - \mathscr{B}_N(w_2,W_2), (w_1-w_2, W_1-W_2)\rangle| \\ &= \frac{N^2}{\|(w_1,W_1)\|_1^2}\mathscr{B}_0((w_1-w_2,W_1-W_2),(w_1,W_1),(w_1-w_2,W_1-W_2)) \\ &\quad - \left(1 - \frac{N^2}{\|(w_1,W_1)\|_1^2}\right)\mathscr{B}_0((w_2,W_2),(w_2,W_2),(w_1-w_2,W_1-W_2)) \\ &\le CN\|(w_1-w_2,W_1-W_2)\|_1^{3/2}\|(w_1-w_2,W_1-W_2)\|_0^{1/2} \\ &\quad + CN\|(w_1-w_2,W_1-W_2)\|_1\|(w_1-w_2,W_1-W_2)\|_{1/2} \\ &\le \frac{\nu}{2}\|(w_1-w_2,W_1-W_2)\|_1^2 + C_N\|(w_1-w_2,W_1-W_2)\|_0^2. \end{aligned}$$

Combining all the cases above we conclude that

$$\begin{aligned} &|\langle \mathscr{B}_N(w_1,W_1) - \mathscr{B}_N(w_2,W_2), (w_1-w_2, W_1-W_2)\rangle| \\ &\quad \le \frac{\nu}{2}\|(w_1-w_2,W_1-W_2)\|_1^2 + C_N\|(w_1-w_2,W_1-W_2)\|_0^2. \end{aligned} \tag{2.11}$$

The operator $\mathscr{B}_N$ is continuous from V to V'. Indeed, as above we have (using (2.4) with $m_1 = 1, m_2 = 0, m_3 = 1$)

$$
\begin{aligned}
&|\langle \mathscr{B}_N(w_1, W_1) - \mathscr{B}_N(w_2, W_2), (w_3, W_3)\rangle| \\
&\le |\mathscr{B}_0\left((w_1 - w_2, W_1 - W_2), (w_1, W_1), (w_3, W_3)\right)| \\
&\quad + |\mathscr{B}_0\left((w_2, W_2), (w_1 - w_2, W_1 - W_2), (w_3, W_3)\right)| \\
&\le C_N \|(w_1 - w_2, W_1 - W_2)\|_1 \|(w_3, W_3)\|_1.
\end{aligned}
\tag{2.12}
$$

Now consider the operator $\Gamma_N : D(\Gamma_N) \to H$ defined by

$$
\Gamma_N = \mathscr{A} + \mathscr{B}_N, \quad D(\Gamma_N) = D(\mathscr{A}).
$$

Here we used (2.4) with $m_1 = 1, m_2 = 1/2, m_3 = 0$ and interpolation results (see e.g. [13, 32, 11]) to show that

$$
\|\mathscr{B}_N(w, W)\|_0 \le C \|(w, W)\|_1^{3/2} \|\mathscr{A}(w, W)\|_0^{1/2} \le C_N \|\mathscr{A}(w, W)\|_0^{1/2}. \tag{2.13}
$$

LEMMA 2.3. *There exists $\alpha_N > 0$ such that $\Gamma_N + \alpha_N I$ is m-accretive (maximal monotone) in $H \times H$.*

Proof. By (2.11) we have that

$$
\begin{aligned}
&((\Gamma_N + \lambda)(w_1, W_1) - (\Gamma_N + \lambda)(w_2, W_2), (w_1 - w_2, W_1 - W_2)) \\
&\ge \frac{\nu}{2} \|(w_1 - w_2, W_1 - W_2)\|_1^2, \text{ for all } (w_i, W_i) \in D(\Gamma_N),
\end{aligned}
\tag{2.14}
$$

for $\lambda \ge C_N$. Next we consider the operator

$$
\mathscr{F}_N(w, W) = \mathscr{A}(w, W) + \mathscr{B}_N(w, W) + \alpha_N(w, W), \quad \text{for all } (w, W) \in D(\mathscr{F}_N),
$$

with

$$
D(\mathscr{F}_N) = \{(w, W) \in V; \mathscr{A}(w, W) + \mathscr{B}_N(w, W) \in H\}.
$$

By (2.12) and (2.14) we see that $\mathscr{F}_N$ is monotone, coercive and continuous from V to V'. We infer that $\mathscr{F}_N$ is maximal monotone from V to V' and the restriction to H is maximal monotone on H with the domain $D(\mathscr{F}_N) \supseteq D(\mathscr{A})$ (see e.g. [8, 4]). Moreover, we have $D(\mathscr{F}_N) = D(\mathscr{A})$. For this we use the perturbation theorem for nonlinear m-accretive operators and split $\mathscr{F}_N$ into a continuous and a ω-m-accretive operator on H

$$
\begin{aligned}
&\mathscr{F}_N^1 = (1 - \frac{\varepsilon}{2})\mathscr{A}, \quad D(\mathscr{F}_N^1) = D(\mathscr{A}), \\
&\mathscr{F}_N^2 = \frac{\varepsilon}{2}\mathscr{A} + \mathscr{B}_N(\cdot) + \alpha_N I, \quad D(\mathscr{F}_N^2) = \{(w, W) \in V, \mathscr{F}_N^2(w, W) \in H\}.
\end{aligned}
$$

As seen above by (2.13) we have

$$
\begin{aligned}
&\left\|\mathscr{F}_N^2(w, W)\right\|_0 \le \frac{\varepsilon}{2} \|\mathscr{A}(w, W)\|_0 + \|\mathscr{B}_N(w, W)\|_0 + \alpha_N \|(w, W)\|_0 \\
&\le \varepsilon \|\mathscr{A}(w, W)\|_0 + \alpha_N \|(w, W)\|_0 + \frac{C_N^2}{2\varepsilon}, \quad \text{for all } (w, W) \in D(\mathscr{F}_N^1) = D(\mathscr{A}),
\end{aligned}
$$

where $0 < \varepsilon < 1$.
Since $\mathscr{F}_N^1 + \mathscr{F}_N^2 = \Gamma_N + \alpha_N I$ we infer that $\Gamma_N + \alpha_N I$ with domain $D(\mathscr{A})$ is m-accretive in H as claimed. ☐

Proof. [Proof of Theorem 2.2] As a consequence of Lemma 2.3 (see, e.g., [4, 5]) we have that for $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in D(\mathscr{A})$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \in W^{1,1}([0,T], H)$ the equation

$$\begin{aligned} &\frac{d}{dt}(w, W) + \mathscr{A}(w, W)(t) + \mathscr{B}_N((w, W)(t)) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}),\ t \in (0, T), \\ &(w, W)(0) = (\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}), \end{aligned} \tag{2.15}$$

has a unique strong solution $(w_N, W_N) \in W^{1,\infty}([0,T]; H) \cap L^\infty(0, T; D(\mathscr{A}))$.
By a density argument (see, e.g., [5, 22]) it can be shown that if $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in H$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \in L^2(0, T, V')$ then there exist absolute continuous functions $(w_N, W_N) : [0, T] \to V'$ that satisfy $(w_N, W_N) \in C([0,T]; H) \cap L^2(0, T : V) \cap W^{1,2}([0,T], V')$ and (2.15) a.e. in $(0, T)$, where d/dt is considered in the strong topology of V'.

First, we show that $D(\mathscr{A})$ is dense in H. Indeed, if $(w, W) \in H$ we set $(w_\varepsilon, W\varepsilon) = (I + \varepsilon\Gamma_N)^{-1}(w, W)$, where I is the unity operator in H. Multiplying the equation

$$(w_\varepsilon, W_\varepsilon) + \varepsilon\Gamma_N(w_\varepsilon, W_\varepsilon) = (w, W)$$

by $(w_\varepsilon, W_\varepsilon)$ it follows by (2.3), (2.11) that

$$\|(w_\varepsilon, W_\varepsilon)\|_0^2 + 2\varepsilon\nu\|(w_\varepsilon, W_\varepsilon)\|_1^2 \le \|(w, W)\|_0^2$$

and by (2.10)

$$\|(w_\varepsilon - w, W_\varepsilon - W)\|_{-1} = \varepsilon\|\Gamma_\varepsilon(w_\varepsilon, W_\varepsilon)\|_{-1} \le \varepsilon N\|(w_\varepsilon, W_\varepsilon)\|_0^{1/2}\|(w_\varepsilon, W_\varepsilon)\|_1^{1/2}.$$

Hence, $\{(w_\varepsilon, W_\varepsilon)\}$ is bounded in H and $(w_\varepsilon, W_\varepsilon) \to (w, W)$ in V' as $\varepsilon \to 0$. Therefore, $(w_\varepsilon, W_\varepsilon) \rightharpoonup (w, W)$ in H as $\varepsilon \to 0$, which implies that $D(\Gamma_N)$ is dense in H.

Secondly, let $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in H$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \in L^2(0, T, V')$. Then there are sequences $\{(\overline{u_0}_n^{\delta_1}, \overline{B_0}_n^{\delta_2})\} \subset D(\Gamma_N)$, $\{(\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2})\} \subset W^{1,1}([0,T]; H)$ such that

$$\begin{aligned} &(\overline{u_0}_n^{\delta_1}, \overline{B_0}_n^{\delta_2}) \to (\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \quad \text{in } H, \\ &(\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2}) \to (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \quad \text{in } L^2(0, T; V'), \end{aligned}$$

as $n \to \infty$. Let $(w_N^n, W_N^n) \in W^{1,\infty}([0,T]; H)$ be the solution to problem (2.15) where $(w, W)(0) = (\overline{u_0}_n^{\delta_1}, \overline{B_0}_n^{\delta_2})$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) = (\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2})$. By (2.14) we have

$$\begin{aligned} &\frac{d}{dt}\|(w_N^n - w_N^m, W_N^n - W_N^m)\|_0^2 + \frac{\nu}{2}\|(w_N^n - w_N^m, W_N^n - W_N^m)\|_1^2 \\ &\quad \le 2C_N\|(w_N^n - w_N^m, W_N^n - W_N^m)\|_0^2 + \frac{2}{\nu}\|(\overline{\boldsymbol{f}}_n^{\delta_1} - \overline{\boldsymbol{f}}_m^{\delta_1}, \operatorname{curl}(\overline{\boldsymbol{g}}_n^{\delta_2} - \overline{\boldsymbol{g}}_m^{\delta_2}))\|_{-1}^2, \end{aligned}$$

for a.e. $t \in (0, T)$. By the Gronwall inequality we obtain

$$\begin{aligned} &\|(w_N^n - w_N^m, W_N^n - W_N^m)(t)\|_0^2 \le e^{2C_N t}\|(\overline{u_0}_n^{\delta_1} - \overline{u_0}_m^{\delta_1}, \overline{B_0}_n^{\delta_2} - \overline{B_0}_m^{\delta_2})\|_0^2 \\ &\quad + \frac{2e^{2C_N t}}{\nu}\int_0^t \|(\overline{\boldsymbol{f}}_n^{\delta_1} - \overline{\boldsymbol{f}}_m^{\delta_1}, \operatorname{curl}(\overline{\boldsymbol{g}}_n^{\delta_2} - \overline{\boldsymbol{g}}_m^{\delta_2}))(\tau)\|_{-1}^2 d\tau. \end{aligned}$$

Hence

$$(w_N(t), W_N(t)) = \lim_{n\to\infty}(w_N^n(t), W_N^n(t))$$

exists in H uniformly in t on $[0,T]$. Similarly we obtain

$$\begin{aligned}&\|w_N^n(t)\|_0^2+\|W_N^n(t)\|_0^2+\int_0^t\left(\frac{1}{\mathrm{Re}}(\|\nabla w_N^n(s)\|_0^2+\frac{S}{\mathrm{Re_m}}(\|\mathrm{curl}\,W_N^n(s)\|_0^2\right)ds\\&\quad\le C_N\left[\|\overline{u_0}_n^{\delta_1}\|_0^2+\|\overline{B_0}_n^{\delta_2}\|_0^2+\int_0^t\left(\|\overline{\boldsymbol{f}}_n^{\delta_1}(s)\|_{-1}^2+\|\mathrm{curl}\,\overline{\boldsymbol{g}}_n^{\delta_2}(s)\|_{-1}^2\right)ds\right],\end{aligned}$$

and

$$\begin{aligned}&\int_0^T\left\|\frac{d}{dt}(w_N^n,W_N^n)(t)\right\|_{-1}^2dt\\&\quad\le C_N\left[\|\overline{u_0}_n^{\delta_1}\|_0^2+\|\overline{B_0}_n^{\delta_2}\|_0^2+\int_0^t\left(\|\overline{\boldsymbol{f}}_n^{\delta_1}(s)\|_{-1}^2+\|\mathrm{curl}\,\overline{\boldsymbol{g}}_n^{\delta_2}(s)\|_{-1}^2\right)ds\right].\end{aligned}$$

Hence on a sequence we have

$$\begin{aligned}&(w_N^n,W_N^n)\to(w_N,W_N)\qquad\text{weakly in }L^2(0,T;V),\\&\frac{d}{dt}(w_N^n,W_N^n)\to\frac{d}{dt}(w_N,W_N)\qquad\text{weakly in }L^2(0,T;V'),\end{aligned}$$

where $d(w_N,W_N)/dt$ is considered in the sense of V'-valued distributions on $(0,T)$. We proved that $(w_N,W_N)\in C([0,T];H)\cap L^2(0,T;V)\cap W^{1,2}([0,T];V')$.

It remains to prove that (w_N,W_N) satisfies the equation (2.15) a.e. on $(0,T)$. Let $(w,W)\in V$ be arbitrary but fixed. We multiply the equation

$$\frac{d}{dt}(w_N^n,W_N^n)+\Gamma_N(w_N^n,W_N^n)=(\overline{\boldsymbol{f}}_n^{\delta_1},\mathrm{curl}\,\overline{\boldsymbol{g}}_n^{\delta_2}),\quad\text{a.e. }t\in(0,T),$$

by (w_N^n-w,W_N^n-W), integrate on (s,t) and get

$$\begin{aligned}&\frac{1}{2}\left(\|(w_N^n(t),W_N^n(t))-(w,W)\|_0^2-\|(w_N^n(s),W_N^n(s))-(w,W)\|_0^2\right)\\&\quad\le\int_s^t\langle(\overline{\boldsymbol{f}}_n^{\delta_1}(\tau),\mathrm{curl}\,\overline{\boldsymbol{g}}_n^{\delta_2}(\tau))-\Gamma_N(w,W),(w_N^n(\tau),W_N^n(\tau))-(w,W)\rangle d\tau.\end{aligned}$$

After we let $n\to\infty$ we get

$$\begin{aligned}&\left\langle\frac{(w_N(t),W_N(t))-(w_N(s),W_N(s))}{t-s},(w_N(s),W_N(s))-(w,W)\right\rangle\\&\quad\le\frac{1}{t-s}\int_s^t\langle(\overline{\boldsymbol{f}}^{\delta_1}(\tau),\mathrm{curl}\,\overline{\boldsymbol{g}}^{\delta_2}(\tau))-\Gamma_N(w,W),(w_N(\tau),W_N(\tau))-(w,W)\rangle d\tau.\end{aligned}\tag{2.16}$$

Let t_0 denote a point at which (w_N,W_N) is differentiable and

$$(\overline{\boldsymbol{f}}^{\delta_1}(t_0),\mathrm{curl}\,\overline{\boldsymbol{g}}^{\delta_2}(t_0))=\lim_{h\to0}\frac{1}{h}\int_{t_0}^{t_0+h}(\overline{\boldsymbol{f}}^{\delta_1}(h),\mathrm{curl}\,\overline{\boldsymbol{g}}^{\delta_2}(h))dh.$$

Then by (2.16) we have

$$\left\langle\frac{d(w_N,W_N)}{dt}(t_0)-(\overline{\boldsymbol{f}}^{\delta_1},\mathrm{curl}\,\overline{\boldsymbol{g}}^{\delta_2})(t_0)+\Gamma_N(w,W),(w_N,W_N)(t_0)-(w,W)\right\rangle\le0.$$

Since (w, W) is arbitrary in V and Γ_N is maximal monotone in $V \times V'$ we conclude that

$$\frac{d(w_N, W_N)}{dt}(t_0) + \Gamma_N(w_N, W_N)(t_0) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2})(t_0).$$

If we multiply (2.15) by $(A_{\delta_1} w_N, S A_{\delta_2} W_N)$, use (2.3) and integrate in time we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|w_N(t)\|_0^2 + S\|W_N(t)\|_0^2\right) + \frac{{\delta_1}^2}{2}\|\nabla w_N(t)\|_0^2 + \frac{{\delta_2}^2 S}{2}\|\operatorname{curl} W_N(t)\|_0^2 \\
&\quad + \int_0^t \left(\frac{1}{\mathrm{Re}}(\|\nabla w_N(s)\|_0^2 + {\delta_1}^2\|\Delta w_N(s)\|_0^2) \right. \\
&\quad \left. + \frac{S}{\mathrm{Re_m}}(\|\operatorname{curl} W_N(s)\|_0^2 + {\delta_2}^2\|\operatorname{curl}\operatorname{curl} W_N(s)\|_0^2) \right) ds \\
&= \frac{1}{2}\left(\|\overline{u_0}^{\delta_1}\|_0^2 + S\|\overline{B_0}^{\delta_2}\|_0^2\right) + \frac{{\delta_1}^2}{2}\|\nabla \overline{u_0}^{\delta_1}\|_0^2 + \frac{{\delta_2}^2 S}{2}\|\operatorname{curl} \overline{B_0}^{\delta_2}\|_0^2 \\
&\quad + \int_0^t \left(\|\overline{f}^{\delta_1}(s)\|_{-1}\|w_N(s)\|_1 + S\|\operatorname{curl} \overline{g}^{\delta_2}(s)\|_{-1}\|W_N(s)\|_1 \right) ds.
\end{aligned}$$

Using the Cauchy-Schwarz and Gronwall inequalities this implies

$$\|(w_N, W_N)(t)\|_1 \le C_{\delta_1,\delta_2} \quad \text{for all } t \in (0, T),$$

where C_{δ_1,δ_2} is independent of N. In particular, for N sufficiently large it follows from (2.10) that $\mathscr{B}_N = \mathscr{B}$ and $(w_N, W_N) = (w, W)$ is a solution to (1.7).

In the following we prove the uniqueness of the weak solution. Let (w_1, W_1) and (w_2, W_2) be two solutions of the system (2.5) and set $\varphi = w_1 - w_2$, $\Phi = B_1 - B_2$. Thus (φ, Φ) is a solution to the problem

$$\begin{aligned}
&\frac{d}{dt}(\varphi, \Phi) + \mathscr{A}(\varphi, \Phi)(t) = -\mathscr{B}((w_1, W_1)(t)) + \mathscr{B}((w_2, W_2)(t)), \quad t \in (0, T), \\
&(\varphi, \Phi)(0) = (0, 0).
\end{aligned}$$

We take $(A_{\delta_1}\varphi, S A_{\delta_2}\Phi)$ as test function, integrate in space, use the incompressibility condition (2.3) and the estimate (2.4) to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\varphi\|_0^2 + {\delta_1}^2\|\nabla\varphi\|_0^2 + S\|\Phi\|_0^2 + S{\delta_2}^2\|\nabla\Phi\|_0^2\right) \\
&\qquad + \frac{1}{\mathrm{Re}}\left(\|\nabla\varphi\|_0^2 + \delta_1^2\|\Delta\varphi\|_0^2\right) + \frac{S}{\mathrm{Re_m}}\left(\|\nabla\Phi\|_0^2 + \delta_2^2\|\Delta\Phi\|_0^2\right) \\
&= \mathscr{B}_0((\varphi, \Phi), (w_1, W_1), (A_{\delta_1}\varphi, S A_{\delta_2}\Phi)) \\
&\le C\|(w_1, W_1)\|_0 \|(\varphi, \Phi)\|_0^{1/2} \|(\nabla\varphi, \nabla\Phi)\|_0^{3/2} \\
&\le C_{\delta_1,\delta_2}\|(w_1, W_1)\|_0 \left(\|\varphi\|_0^2 + {\delta_1}^2\|\nabla\varphi\|_0^2 + S\|\Phi\|_0^2 + S{\delta_2}^2\|\nabla\Phi\|_0^2\right).
\end{aligned}$$

Applying the Gronwall's lemma we deduce that (φ, Φ) vanishes for all $t \in [0, T]$, and hence the uniqueness of the solution. □

REMARK 2.1. *The pressure is recovered from the weak solution via the classical DeRham theorem (see [21]).*

2.3. Regularity.

THEOREM 2.4. *Let $m \in \mathbb{N}$, $(u_0, B_0) \in V \cap H^{m-1}(\Omega)$ and $(f, curl g) \in L^2(0,T; H^{m-1}(\Omega))$. Then there exists a unique solution w, W, q to the equation (1.7) such that*

$$(w, W) \in L^\infty(0,T; H^{m+1}(\Omega)) \cap L^2(0,T; H^{m+2}(\Omega)),$$
$$q \in L^2(0,T; H^m(\Omega)).$$

Proof. The result is already proved when $m = 0$ in Theorem 2.2. For any $m \in \mathbb{N}^*$, we assume that

$$(w, W) \in L^\infty(0,T; H^m(\Omega)) \cap L^2(0,T; H^{m+1}(\Omega)) \tag{2.17}$$

so it remains to prove

$$(D^m w, D^m W) \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),$$

where D^m denotes any partial derivative of total order m. We take the m^{th} derivative of (1.7) and have

$$(D^m w)_t - \frac{1}{\text{Re}}\Delta(D^m w) + \overline{D^m(w\cdot\nabla w)}^{\delta_1} - S\overline{D^m(W\cdot\nabla W)}^{\delta_1} = \overline{D^m f}^{\delta_1},$$
$$(D^m W)_t + \frac{1}{\text{Re}_\text{m}}\nabla\times\nabla\times(D^m W) + \overline{D^m(w\cdot\nabla W)}^{\delta_2} - \overline{D^m(W\cdot\nabla w)}^{\delta_2} = \nabla\times\overline{D^m g}^{\delta_2},$$
$$\nabla\cdot(D^m w) = 0, \nabla\cdot(D^m W) = 0,$$
$$D^m w(0,\cdot) = D^m\overline{u_0}^{\delta_1}, D^m W(0,\cdot) = D^m\overline{B_0}^{\delta_2},$$

with periodic boundary conditions and zero mean, and the initial conditions with zero divergence and mean. Taking $A_{\delta_1} D^m w, A_{\delta_1} D^m W$ as test functions we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|D^m w\|_0^2 + {\delta_1}^2\|\nabla D^m w\|_0^2 + S\|D^m W\|_0^2 + S{\delta_2}^2\|\nabla D^m W\|_0^2\right) \\
&\quad + \frac{1}{\text{Re}}\left(\|\nabla D^m w\|_0^2 + \delta_1^2\|\Delta D^m w\|_0^2\right) + \frac{1}{\text{Re}_\text{m}}\left(\|\nabla D^m W\|_0^2 + \delta_2^2\|\Delta D^m W\|_0^2\right) \\
&= \int_\Omega (D^m f D^m w + \nabla\times g D^m W)\,dx - \mathscr{X},
\end{aligned} \tag{2.18}$$

where

$$\mathscr{X} = \int_\Omega (D^m(w\cdot\nabla w) - SD^m(W\cdot\nabla W) + D^m(w\cdot\nabla W) - D^m(W\cdot\nabla w))\,dx.$$

Now we apply (2.4) and use the induction assumption (2.17)

$$\begin{aligned}
\mathscr{X} &= \sum_{|\alpha|\le m}\binom{m}{\alpha}\sum_{i,j=1}^{3}\int_\Omega D^\alpha w_i D^{m-\alpha}D_i w_j D^m w_j - SD^\alpha W_i D^{m-\alpha}D_i W_j D^m w_j \\
&\qquad - D^\alpha w_i D^{m-\alpha}D_i W_j D^m W_j - D^\alpha W_i D^{m-\alpha}D_i w_j D^m W_j \\
&\le \|w\|_{m+1}^{3/2}\|w\|_{m+2}^{1/2}\|w\|_m + \|W\|_{m+1}^{3/2}\|W\|_{m+2}^{1/2}\|w\|_m \\
&\qquad + \|w\|_{m+1}\|W\|_{m+1}^{1/2}\|W\|_{m+2}^{1/2}\|W\|_m + \|W\|_{m+1}^{3/2}\|W\|_{m+2}^{1/2}\|W\|_m.
\end{aligned}$$

Integrating (2.18) on $(0,T)$, using the Cauchy-Schwarz and Hölder inequalities, and the assumption (2.17) we obtain the desired result for w, W. We conclude the proof mentioning that the regularity of the pressure term q is obtained via classical methods, see e.g. [31, 3]. ◻

3. Accuracy of the model.

We will address first the question of consistency error, i.e., we show in Theorem 3.1 that the solution of the closed model (1.7) converges to a weak solution of the MHD equations (1.1) when δ_1, δ_2 go to zero. This proves that the model is consistent as $\delta_1, \delta_2 \to 0$.

Let $\tau_u, \tau_B, \tau_{Bu}$ denote the model's consistency errors

$$\tau_u = \overline{u}^{\delta_1}\overline{u}^{\delta_1} - uu, \quad \tau_B = \overline{B}^{\delta_2}\overline{B}^{\delta_2} - BB, \quad \tau_{Bu} = \overline{B}^{\delta_2}\overline{u}^{\delta_1} - Bu, \tag{3.1}$$

where u, B is a solution of the MHD equations obtained as a limit of a subsequence of the sequence $w_{\delta_1}, W_{\delta_2}$.

We will also prove in Theorem 3.2 that $\|\overline{u}^{\delta_1} - w\|_{L^\infty(0,T;L^2(Q))}$, $\|\overline{B}^{\delta_2} - W\|_{L^\infty(0,T;L^2(Q))}$ are bounded by $\|\tau_u\|_{L^2(Q_T)}$, $\|\tau_B\|_{L^2(Q_T)}$, $\|\tau_{Bu}\|_{L^2(Q_T)}$.

3.1. Limit consistency of the model.

THEOREM 3.1. *There exist two sequences $\delta_1^n, \delta_2^n \to 0$ as $n \to 0$ such that*

$$(w_{\delta_1^n}, W_{\delta_2^n}, q_{\delta_1^n}) \to (u, B, p) \quad \text{as } \delta_1^n, \delta_2^n \to 0,$$

where $(u, B, p) \in L^\infty(0,T;H) \cap L^2(0,t;V) \times L^{\frac{4}{3}}(0,T;L^2(\Omega))$ is a weak solution of the MHD equations (1.1). The sequences $\{w_{\delta_1^n}\}_{n\in\mathbb{N}}, \{W_{\delta_2^n}\}_{n\in\mathbb{N}}$ converge strongly to u, B in $L^{\frac{4}{3}}(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;H^1(\Omega))$, respectively, while $\{q_{\delta_1^n}\}_{n\in\mathbb{N}}$ converges weakly to p in $L^{\frac{4}{3}}(0,T;L^2(\Omega))$.

Proof. The proof follows that of Theorem 3.1 in [20], and is an easy consequence of Theorem 3.2 and Proposition 3.4; we will sketch it for the reader's convenience. □

3.2. Verifiability of the model.

THEOREM 3.2. *Suppose that the true solution of (1.1) satisfies the regularity condition $(u, B) \in L^4(0,T;V)$. Then $e = \overline{u}^{\delta_1} - w$, $E = \overline{B}^{\delta_2} - W$ satisfy*

$$\begin{aligned} &\|e(t)\|_0^2 + S\|E(t)\|_0^2 + \int_0^t \left(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2 + \frac{S}{\mathrm{Re_m}}\|curl\, E(s)\|_0^2\right) ds \\ &\quad \le C\Phi(t)\int_0^t \left(\mathrm{Re}\|\tau_u(s) + S\tau_B(s)\|_0^2 + \mathrm{Re_m}\|\tau_{Bu}(s) - {\tau_{Bu}}^T(s)\|_0^2\right) ds, \end{aligned} \tag{3.2}$$

where $\Phi(t) = \exp\left\{\mathrm{Re}^3 \int_0^t \|\nabla u\|_0^4 ds, {\mathrm{Re_m}}^3 \int_0^t \|\nabla u\|_0^4 ds + \mathrm{Re_m}\mathrm{Re}^2 \int_0^t \|\nabla B\|_0^4\right\}$.

Proof. The errors $e = \overline{u}^{\delta_1} - w, E = \overline{B}^{\delta_2} - W$ satisfy in variational sense

$$\begin{aligned} &e_t + \nabla\cdot(\overline{\overline{u}^{\delta_1}\overline{u}^{\delta_1} - ww}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta e + S\nabla\cdot(\overline{\overline{B}^{\delta_2}\overline{B}^{\delta_2} - WW}^{\delta_1}) + \nabla(\overline{p}^{\delta_1} - q) \\ &\quad = \nabla\cdot(\overline{\tau}_u^{\delta_1} + S\overline{\tau}_B^{\delta_1}), \\ &E_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl\,curl}\, E + \nabla\cdot(\overline{\overline{B}^{\delta_2}\overline{u}^{\delta_1} - Ww}^{\delta_2}) - \nabla\cdot(\overline{\overline{u}^{\delta_1}\overline{B}^{\delta_2} - wW}^{\delta_2}) \\ &\quad = \nabla\cdot(\overline{\tau}_{Bu}^{\delta_2} - \overline{\tau}_{Bu}^{\delta_2\,T}), \end{aligned}$$

and $\nabla\cdot e=\nabla\cdot E=0$, $e(0)=E(0)=0$. Taking the inner product with $(A_{\delta_1}e, SA_{\delta_2}E)$ we get as in (2.8) the energy estimate

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|e\|_0^2+S\|E\|_0^2+\delta_1^2\|\nabla e\|_0^2+\delta_2^2S\|\mathrm{curl}\,E\|_0^2\right)\\
&\qquad+\frac{1}{\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\ \mathrm{E}\|_0^2+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2+\frac{\delta_2^2S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathrm{curl}\,E\|_0^2\\
&\qquad+\int_\Omega\Big(\nabla\cdot(\overline{u}^{\delta_1}\overline{u}^{\delta_1}-ww)e+S\nabla\cdot(\overline{B}^{\delta_2}\overline{B}^{\delta_2}-WW)e\\
&\qquad\qquad+S\nabla\cdot(\overline{B}^{\delta_2}\overline{u}^{\delta_1}-Ww)E-S\nabla\cdot(\overline{u}^{\delta_1}\overline{B}^{\delta_2}-wW)E\Big)dx\\
&=-\int_\Omega\Big((\tau_u+S\tau_B)\cdot\nabla e+S(\tau_{Bu}-{\tau_{Bu}}^T)\cdot\nabla E\Big)dx\\
&\le\frac{1}{2\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{2\mathrm{Re_m}}\|\mathrm{curl}\,E\|_0^2+\frac{\mathrm{Re}}{2}\|\tau_u+S\tau_B\|_0^2+\frac{\mathrm{Re_m}}{2S}\|\tau_{Bu}-{\tau_{Bu}}^T\|_0^2.
\end{aligned}
$$

Since $\overline{u}^{\delta_1}\overline{u}^{\delta_1}-ww=e\overline{u}^{\delta_1}+we$, $\overline{B}^{\delta_2}\overline{B}^{\delta_2}-WW=E\overline{B}^{\delta_2}+WE$, $\overline{B}^{\delta_2}\overline{u}^{\delta_1}-Ww=E\overline{u}^{\delta_1}+We$, $\overline{u}^{\delta_1}\overline{B}^{\delta_2}-wW=e\overline{B}^{\delta_1}+wE$, and $\int_\Omega\nabla\cdot(we)edx=\int_\Omega\nabla\cdot(WE)Edx=0$ we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|e\|_0^2+S\|E\|_0^2+\delta_1^2\|\nabla e\|_0^2+S\delta_2^2\|\mathrm{curl}\,E\|_0^2\right)\\
&\quad+\frac{1}{\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\ \mathrm{E}\|_0^2+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2+\frac{\delta_2^2S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathrm{curl}\,E\|_0^2\\
&\le\int_\Omega\Big(-e\cdot\nabla\overline{u}^{\delta_1}e-S\nabla\cdot(E\overline{B}^{\delta_2})e-S\nabla\cdot(E\overline{u}^{\delta_1})E+Se\cdot\nabla\overline{B}^{\delta_2}E\Big)dx\\
&\quad+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-{\tau_{Bu}}^T\|_0^2\\
&\le C\Big(\|\nabla e\|_0^{3/2}\|e\|_0^{1/2}\|\nabla\overline{u}^{\delta_1}\|_0+2S\|E\|_0^{1/2}\|\nabla E\|_0^{1/2}\|\nabla\overline{B}^{\delta_2}\|_0\|\nabla e\|_0\\
&\qquad+S\|E\|_0^{1/2}\|\nabla E\|_0^{3/2}\|\nabla\overline{u}^{\delta_1}\|_0\Big)+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-{\tau_{Bu}}^T\|_0^2.
\end{aligned}
$$

Using $ab\le\varepsilon a^{4/3}+C\varepsilon^{-3}b^4$ we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|e\|_0^2+S\|E\|_0^2+\delta_1^2\|\nabla e\|_0^2+S\delta_2^2\|\mathrm{curl}\,E\|_0^2\right)\\
&\quad+\frac{1}{\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\ \mathrm{E}\|_0^2+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2+\frac{\delta_2^2S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathrm{curl}\,E\|_0^2\\
&\le C\Big(\mathrm{Re}^3\|e\|_0^2\|\nabla\overline{u}^{\delta_1}\|_0^4+\mathrm{Re_m}\mathrm{Re}^2\|E\|_0^2\|\nabla\overline{B}^{\delta_2}\|_0^4+{\mathrm{Re_m}}^3\|E\|_0^2\|\nabla\overline{u}^{\delta_1}\|_0^4\Big)\\
&\quad+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-{\tau_{Bu}}^T\|_0^2
\end{aligned}
$$

and by the Gronwall inequality we deduce

$$
\begin{aligned}
&\|e(t)\|_0^2+S\|E(t)\|_0^2+\int_0^t\left(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\,E(s)\|_0^2\right)ds\\
&\le C\Psi(t)\int_0^t\left(\mathrm{Re}\|\tau_u(s)+S\tau_B(s)\|_0^2+\mathrm{Re_m}\|\tau_{Bu}(s)-{\tau_{Bu}}^T(s)\|_0^2\right)ds,
\end{aligned}
$$

where

$$
\Psi(t)=\exp\left\{\mathrm{Re}^3\int_0^t\|\nabla\overline{u}^{\delta_1}\|_0^4ds,{\mathrm{Re_m}}^3\int_0^t\|\nabla\overline{u}^{\delta_1}\|_0^4ds+\mathrm{Re_m}\mathrm{Re}^2\int_0^t\|\nabla\overline{B}^{\delta_2}\|_0^4ds\right\}.
$$

Using the stability bounds $\|\nabla \overline{u}^{\delta_1}\|_0 \leq \|\nabla u\|_0$, $\|\nabla \overline{B}^{\delta_2}\|_0 \leq \|\nabla B\|_0$ we conclude the proof. ▯

3.3. Consistency error estimate. Here we shall give bounds on the consistency errors (3.1) as $\delta_1, \delta_2 \to 0$ in $L^1((0,T) \times \Omega)$ and $L^2((0,T) \times \Omega)$.

PROPOSITION 3.3. *Let us assume that $(f, curl g) \in L^2(0,T;V')$. Then the following holds*

$$
\begin{aligned}
\|\tau_u\|_{L^1(0,T;L^1(\Omega))} &\leq 2^{3/2}\delta_1 T^{1/2}\mathrm{Re}^{1/2}\mathscr{E}(T),\\
\|\tau_B\|_{L^1(0,T;L^1(\Omega))} &\leq 2^{3/2}\delta_2 T^{1/2}\frac{\mathrm{Re_m}^{1/2}}{S}\mathscr{E}(T),\\
\|\tau_{Bu}\|_{L^1(0,T;L^1(\Omega))} &\leq 2^{1/2}T^{1/2}\frac{1}{S}(\delta_1\mathrm{Re}^{1/2} + \delta_2\mathrm{Re_m}^{1/2})\mathscr{E}(T),
\end{aligned} \tag{3.3}
$$

where

$$
\mathscr{E}(T) = \left(\|u_0\|_0^2 + S\|B_0\|_0^2 + \mathrm{Re}\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \frac{\mathrm{Re_m}}{S}\|\mathrm{curl}\, g\|^2_{L^2(0,T;H^{-1}(\Omega))}\right).
$$

Proof. Using the stability bounds we have

$$
\begin{aligned}
\|\tau_u\|_{L^1(0,T;L^1(\Omega))} &\leq \|u + \overline{u}^{\delta_1}\|_{L^2(0,T;L^2(\Omega))}\|\overline{u}^{\delta_1} - u\|_{L^2(0,T;L^2(\Omega))}\\
&\leq 2\|u\|_{L^2(0,T;L^2(\Omega))}\sqrt{2}\delta_1\|\nabla u\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\|\tau_B\|_{L^1(0,T;L^1(\Omega))} &\leq \|B + \overline{B}^{\delta_2}\|_{L^2(0,T;L^2(\Omega))}\|\overline{B}^{\delta_2} - B\|_{L^2(0,T;L^2(\Omega))}\\
&\leq 2\|B\|_{L^2(0,T;L^2(\Omega))}\sqrt{2}\delta_2\|\nabla B\|_{L^2(0,T;L^2(\Omega))},\\
\|\tau_{Bu}\|_{L^1(0,T;L^1(\Omega))} &\leq \|\overline{B}^{\delta_2} - B\|_{L^2(Q)}\|\overline{u}^{\delta_1}\|_{L^2(Q)} + \|B\|_{L^2(Q)}\|\overline{u}^{\delta_1} - u\|_{L^2(Q)}\\
&\leq \sqrt{2}\delta_2\|\nabla B\|_{L^2(Q)}\|u\|_{L^2(Q)} + \sqrt{2}\delta_1\|\nabla u\|_{L^2(Q)}\|B\|_{L^2(Q)}.
\end{aligned}
$$

The classical energy estimates for the MHD system (1.1) will yield now (3.3). ▯

Assuming more regularity on (u, B) leads to the sharper bounds on the consistency errors.

REMARK 3.1. *Let $(u, B) \in L^2(0,T;H^2(\Omega))$. Then*

$$
\begin{aligned}
\|\tau_u\|_{L^1(0,T;L^1(\Omega))} &\leq C\delta_1^2,\\
\|\tau_B\|_{L^1(0,T;L^1(\Omega))} &\leq C\delta_2^2,\\
\|\tau_{Bu}\|_{L^1(0,T;L^1(\Omega))} &\leq C(\delta_1^2 + \delta_2^2),
\end{aligned}
$$

where $C = C(T, Re, Re_m, \|(u,B)\|_{L^2(0,T;L^2(\Omega))}, \|(u,B)\|_{L^2(0,T;H^2(\Omega))})$.

Proof. The result is obtained by following the proof of Proposition 3.3 and using the bounds

$$
\begin{aligned}
\|\overline{u}^{\delta_1} - u\|_{L^2(0,T;L^2(\Omega))} &\leq \delta_1^2\|\Delta u\|_{L^2(0,T;L^2(\Omega))},\\
\|\overline{B}^{\delta_2} - B\|_{L^2(0,T;L^2(\Omega))} &\leq \delta_2^2\|\Delta B\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
$$

▯

Next we estimate the L^2-norms of the consistency errors $\tau_u, \tau_B, \tau_{Bu}$, which were used in Theorem 3.2 to estimate the filtering errors e, E.

PROPOSITION 3.4. *Let u, B be a solution of the MHD equations (1.1) and assume that*

$$(u, B) \in L^4((0,T) \times \Omega) \cap L^2(0,T; H^2(\Omega)).$$

Then we have

$$\begin{aligned}
&\|\tau_u\|_{L^2(Q)} \le C\delta_1, \\
&\|\tau_B\|_{L^2(Q)} \le C\delta_2, \\
&\|\tau_{Bu}\|_{L^2(Q)} \le C(\delta_1 + \delta_2),
\end{aligned}$$

where $C = C(\|(u,B)\|_{L^4((0,T)\times\Omega)}, \|(u,B)\|_{L^2(0,T;H^2(\Omega))})$.

Proof. As in the proof of Proposition 3.3, using the stability bounds we have

$$\begin{aligned}
\|\tau_u\|_{L^2(Q)} &\le 2\|u\|_{L^4(Q)}\|\overline{u}^{\delta_1} - u\|_{L^4(Q)} \\
&\le 2^{3/2}\|u\|_{L^4(Q)}\left(\int_0^T \|\overline{u}^{\delta_1} - u\|_{L^2(\Omega)}\|\nabla(\overline{u}^{\delta_1} - u)\|^3_{L^2(\Omega)}dt\right)^{1/4} \\
&\le 2^{3/2}\|u\|_{L^4(Q)}\left(\int_0^T 4\delta_1^4\|\nabla u\|_{L^2(\Omega)}\|\Delta u\|^3_{L^2(\Omega)}dt\right)^{1/4} \\
&\le 4\delta_1\|u\|_{L^4(Q)}\|u\|_{L^2(0,T;H^1(\Omega))}\|u\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

Similarly we deduce

$$\|\tau_B\|_{L^2(Q)} \le 4\delta_2\|B\|_{L^4(Q)}\|B\|_{L^2(0,T;H^1(\Omega))}\|B\|_{L^2(0,T;H^2(\Omega))},$$

and

$$\begin{aligned}
\|\tau_{Bu}\|_{L^2(Q)} &\le \|u\|_{L^4(Q)}\|\overline{B}^{\delta_2} - B\|_{L^4(Q)} + \|B\|_{L^4(Q)}\|\overline{u}^{\delta_2} - u\|_{L^4(Q)} \\
&\le 2\delta_2\|u\|_{L^4(Q)}\|B\|_{L^2(0,T;H^1(\Omega))}\|B\|_{L^2(0,T;H^2(\Omega))} \\
&\quad + 2\delta_1\|B\|_{L^4(Q)}\|u\|_{L^2(0,T;H^1(\Omega))}\|u\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

□

As in Remark 3.1, assuming extra regularity on (u, B) leads to the sharper bounds.

REMARK 3.2. *Let*

$$(u, B) \in L^4((0,T) \times \Omega) \cap L^4(0,T; H^2(\Omega)).$$

Then

$$\begin{aligned}
&\|\tau_u\|_{L^2(Q)} \le C\delta_1^2, \\
&\|\tau_B\|_{L^2(Q)} \le C\delta_2^2, \\
&\|\tau_{Bu}\|_{L^2(Q)} \le C(\delta_1^2 + \delta_2^2),
\end{aligned}$$

where $C = C(\|(u,B)\|_{L^4((0,T)\times\Omega)}, \|(u,B)\|_{L^4(0,T;H^2(\Omega))})$.

The proof repeats the one of Remark 3.1.

4. Conservation laws.

As our model is some sort of a regularizing numerical scheme, we would like to make sure that the model inherits some of the original properties of the 3D MHD equations.

It is well known that kinetic energy and helicity are critical in the organization of the flow.

The energy $E = \frac{1}{2}\int_\Omega (u(x)\cdot u(x) + SB(x)\cdot B(x))dx$, the cross helicity $H_C = \frac{1}{2}\int_\Omega (u(x)\cdot B(x))dx$ and the magnetic helicity $H_M = \frac{1}{2}\int_\Omega (\mathbb{A}(x)\cdot B(x))dx$ (where $\mathbb{A}$ is the vector potential, $B = \nabla\times\mathbb{A}$) are the three invariants of the MHD equations (1.1) in the absence of kinematic viscosity and magnetic diffusivity ($\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$).

Introduce the characteristic quantities of the model

$$E_{LES} = \frac{1}{2}[(A_{\delta_1}w, w) + S(A_{\delta_2}W, W)],$$

$$H_{C,LES} = \frac{1}{2}(A_{\delta_1}w, A_{\delta_2}W),$$

and

$$H_{M,LES} = \frac{1}{2}(A_{\delta_2}W, \overline{\mathbb{A}}^{\delta_2}),\ \text{where } \overline{\mathbb{A}}^{\delta_2} = A_{\delta_2}^{-1}\mathbb{A}.$$

This section is devoted to proving that these quantities are conserved by (1.7) with the periodic boundary conditions and $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$. Also, note that

$$E_{LES}\to E,\ H_{C,LES}\to H_C,\ H_{M,LES}\to H_M, \qquad \text{as } \delta_{1,2}\to 0.$$

THEOREM 4.1 (Conservation Laws). *The following conservation laws hold,* $\forall T > 0$

$$E_{LES}(T) = E_{LES}(0), \tag{4.1}$$

$$H_{C,LES}(T) = H_{C,LES}(0) + C(T)\max_{i=1,2}\delta_i^2, \tag{4.2}$$

and

$$H_{M,LES}(T) = H_{M,LES}(0). \tag{4.3}$$

Note that the cross helicity $H_{C,LES}$ of the model is not conserved exactly, but it possesses two important properties:

$$H_{C,LES}\to H_C \text{ as } \delta_{1,2}\to 0,$$

and

$$H_{C,LES}(T)\to H_{C,LES}(0) \text{ as } N \text{ increases.}$$

Proof. Start by proving (4.1). Consider (1.7) with $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$. Multiply (1.7a) by $A_{\delta_1}w$, and multiply (1.7b) by $SA_{\delta_2}W$. Integrating both equations over Ω gives

$$\frac{1}{2}\frac{d}{dt}(A_{\delta_1}w, w) = S((\nabla\times W)\times W, w), \tag{4.4}$$

$$\frac{1}{2}S\frac{d}{dt}(A_{\delta_2}W, W) - S(W\cdot\nabla w, W) = 0. \tag{4.5}$$

Use the identity

$$((\nabla\times v)\times u, w) = (u\cdot\nabla v, w) - (w\cdot\nabla v, u). \tag{4.6}$$

Add (4.4) and (4.5). Using (4.6) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[(A_{\delta_1}w, w) + S(A_{\delta_2}W, W)\right]\\ &\quad = S(W\cdot\nabla W, w) - S(w\cdot\nabla W, W) + S(W\cdot\nabla w, W).\end{aligned}$$

Hence

$$\frac{1}{2}\frac{d}{dt}[(A_{\delta_1}w, w) + S(A_{\delta_2}W, W)] = 0, \tag{4.7}$$

which proves (4.1).

To prove (4.2), multiply (1.7a) by $A_{\delta_1}W$, and multiply (1.7b) by $A_{\delta_2}w$. Integrating both equations over Ω gives

$$(\frac{\partial A_{\delta_1}w}{\partial t}, W) + (w\cdot\nabla w, W) = 0, \tag{4.8}$$

$$(\frac{\partial A_{\delta_2}W}{\partial t}, w) + (w\cdot\nabla W, w) = 0. \tag{4.9}$$

Add (4.8) and (4.9); the identity $(u\cdot\nabla v, w) = -(u\cdot\nabla w, v)$ implies

$$(\frac{\partial A_{\delta_1}w}{\partial t}, W) + (\frac{\partial A_{\delta_2}W}{\partial t}, w) = 0. \tag{4.10}$$

It follows from (1.8) that

$$\begin{aligned}w &= A_{\delta_1}w + \delta_1^2\Delta w,\\ W &= A_{\delta_2}W + \delta_2^2\Delta W.\end{aligned} \tag{4.11}$$

Then (4.10) gives

$$\begin{aligned}&(\frac{\partial A_{\delta_1}w}{\partial t}, A_{\delta_2}W) + (\frac{\partial A_{\delta_2}W}{\partial t}, A_{\delta_1}w)\\ &= (\frac{\partial A_{\delta_1}w}{\partial t}, \delta_2^2\Delta W) + (\frac{\partial A_{\delta_2}W}{\partial t}, \delta_1^2\Delta w).\end{aligned} \tag{4.12}$$

Hence,

$$\begin{aligned}\frac{d}{dt}(A_{\delta_1}w, A_{\delta_2}W) = \delta_2^2(&\frac{\partial A_{\delta_1}w}{\partial t}, \Delta W)\\ &+ \delta_1^2(\frac{\partial A_{\delta_2}W}{\partial t}, \Delta w),\end{aligned} \tag{4.13}$$

which proves (4.2).

Next, we prove (4.3) by multiplying (1.7b) by $A_{\delta_2}\overline{\mathbb{A}}^{\delta_2}$, and integrating over Ω. This gives

$$\frac{1}{2}\frac{d}{dt}(\nabla \times A_{\delta_2}\overline{\mathbb{A}}^{\delta_2}, \overline{\mathbb{A}}^{\delta_2}) + (w \cdot \nabla W, \overline{\mathbb{A}}^{\delta_2}) - (W \cdot \nabla w, \overline{\mathbb{A}}^{\delta_2}) = 0. \tag{4.14}$$

Since the cross-product of two vectors is orthogonal to each of them,

$$((\nabla \times \overline{\mathbb{A}}^{\delta_2}) \times w, \nabla \times \overline{\mathbb{A}}^{\delta_2}) = 0.$$

It follows from (4.15) and (4.6) that

$$(w \cdot \nabla \overline{\mathbb{A}}^{\delta_2}, \nabla \times \overline{\mathbb{A}}^{\delta_2}) = ((\nabla \times \overline{\mathbb{A}}^{\delta_2}) \cdot \nabla \overline{\mathbb{A}}^{\delta_2}, w). \tag{4.15}$$

Since $W = \nabla \times \overline{\mathbb{A}}^{\delta_2}$, we obtain from (4.14) and (4.15) that (4.3) holds. ☐

5. Alfvén waves. In this section we prove that our model possesses a very important property of the MHD: the ability of the magnetic field to transmit transverse inertial waves - Alfvén waves. We follow the argument typically used to prove the existence of Alfvén waves in MHD, see, e.g., [10].

Using the density ρ and permeability μ, we write the equations of the model (1.7) in the form

$$w_t + \nabla \cdot (\overline{ww^T}^{\delta_1}) + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho\mu}\overline{(\nabla \times W) \times W}^{\delta_1} - \nu \nabla \times (\nabla \times w), \tag{5.1a}$$

$$\frac{\partial W}{\partial t} = \overline{\nabla \times (w \times W)}^{\delta_2} - \eta \nabla \times (\nabla \times W), \tag{5.1b}$$

$$\nabla \cdot w = 0, \quad \nabla \cdot W = 0, \tag{5.1c}$$

where $\nu = \frac{1}{\mathrm{Re}}$, $\eta = \frac{1}{\mathrm{Re_m}}$.

Proposition 5.1. *The magnetic field in (5.1) transmits the Alfvén waves with group velocity*

$$\tilde{v}_a = v_a + O(\delta_1^2 + \delta_2^2),$$

where v_a is the velocity of Alfvén waves in MHD.

Proof. Assume a uniform, steady magnetic field W_0, perturbed by a small velocity field w. We denote the perturbations in current density and magnetic field by j_{model} and W_p, with

$$\nabla \times W_p = \mu j_{model}. \tag{5.2}$$

Also, the vorticity of the model is

$$\omega_{model} = \nabla \times w. \tag{5.3}$$

Since $w \cdot \nabla w$ is quadratic in the small quantity w, it can be neglected in the Navier-Stokes equation (5.1a), and therefore

$$\frac{\partial w}{\partial t} + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho\mu}\overline{(\nabla \times W_p) \times W_0}^{\delta_1} - \nu \nabla \times (\nabla \times w). \tag{5.4}$$

The leading order terms in the induction equation (5.1b) are

$$\frac{\partial W_p}{\partial t} = \overline{\nabla \times (w \times W_0)}^{\delta_2} - \eta \nabla \times (\nabla \times W_p). \tag{5.5}$$

Using (5.2), we rewrite (5.4) as

$$\frac{\partial w}{\partial t} + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho} \overline{j_{model} \times W_0}^{\delta_1} + \nu \Delta w. \tag{5.6}$$

Take *curl* of (5.6) and use the identity (2.7). Since $\nabla W_0 = 0$, we obtain from (5.3) that

$$\frac{\partial \omega_{model}}{\partial t} = \frac{1}{\rho} \overline{W_0 \cdot \nabla j_{model}}^{\delta_1} + \nu \Delta \omega_{model}. \tag{5.7}$$

Taking *curl* of (5.5) and using (5.2),(5.3) yields

$$\mu \frac{\partial j_{model}}{\partial t} = \overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2} + \eta \mu \Delta j_{model}. \tag{5.8}$$

Divide (5.8) by μ to obtain

$$\frac{\partial j_{model}}{\partial t} = \frac{1}{\mu} \overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2} + \eta \Delta j_{model}. \tag{5.9}$$

We now eliminate j_{model} from (5.7) by taking the time derivative of (5.7) and substituting for $\frac{\partial j_{model}}{\partial t}$ using (5.9). This yields

$$\frac{\partial^2 \omega_{model}}{\partial t^2} = \frac{1}{\rho} \overline{W_0 \cdot \nabla \Big(\frac{1}{\mu} \overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2} + \eta \Delta j_{model} \Big)}^{\delta_1} + \nu \Delta \frac{\partial \omega_{model}}{\partial t}. \tag{5.10}$$

The linearity of $A_{\delta_1}^{-1}$ implies

$$\begin{aligned} \frac{\partial^2 \omega_{model}}{\partial t^2} &= \frac{1}{\rho \mu} \overline{W_0 \cdot \nabla (\overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2})}^{\delta_1} \\ &+ \frac{\eta}{\rho} \overline{W_0 \cdot \nabla (\Delta j_{model})}^{\delta_1} + \nu \Delta \frac{\partial \omega_{model}}{\partial t}. \end{aligned} \tag{5.11}$$

In order to eliminate the term containing Δj_{model} from (5.11), we take the Laplacian of (5.7):

$$\Delta \frac{\partial \omega_{model}}{\partial t} = \frac{1}{\rho} \overline{W_0 \cdot \nabla (\Delta j_{model})}^{\delta_1} + \nu \Delta^2 \omega_{model}. \tag{5.12}$$

It follows from (5.11)-(5.12) that

$$\begin{aligned} \frac{\partial^2 \omega_{model}}{\partial t^2} &= \frac{1}{\rho \mu} \overline{W_0 \cdot \nabla (\overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2})}^{\delta_1} \\ &+ (\eta + \nu) \Delta \frac{\partial \omega_{model}}{\partial t} - \eta \nu \Delta^2 \omega_{model}. \end{aligned} \tag{5.13}$$

Next we look for the plane-wave solutions of the form

$$\omega_{model} \sim \omega_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \theta t)}, \tag{5.14}$$

where $\mathbf{k}$ is the wavenumber. It immediately follows from (5.14) that

$$\begin{aligned}\frac{\partial \omega_{model}}{\partial t} &= -i\theta\omega_{model}, \\ \frac{\partial^2 \omega_{model}}{\partial t^2} &= -\theta^2\omega_{model}, \\ \Delta\frac{\partial \omega_{model}}{\partial t} &= i\theta k^2\omega_{model}, \\ \Delta^2(\omega_{model}) &= k^4\omega_{model}.\end{aligned} \tag{5.15}$$

Substitute (5.14) into the wave equation (5.13). Using (5.15) gives

$$\begin{aligned}-\theta^2\omega_{model} = \frac{1}{\rho\mu}&\overline{W_0\cdot\nabla(\overline{W_0\cdot\nabla\omega_{model}}^{\delta_2})}^{\delta_1} \\ &+(\eta+\nu)i\theta k^2\omega_{model} - \eta\nu k^4\omega_{model}.\end{aligned} \tag{5.16}$$

It follows from (1.8) that

$$\begin{aligned}\overline{W_0\cdot\nabla\omega_{model}}^{\delta_2} &= W_0\cdot\nabla\omega_{model} + O(\delta_2^2), \\ \overline{W_0\cdot\nabla(\overline{W_0\cdot\nabla\omega_{model}}^{\delta_2})}^{\delta_1} &= (W_0\cdot\nabla)^2\omega_{model} + O(\delta_1^2) + O(\delta_2^2).\end{aligned} \tag{5.17}$$

Thus we obtain from (5.16),(5.17) that

$$\begin{aligned}-\theta^2\omega_{model} = \frac{1}{\rho\mu}&(W_0\cdot\nabla)^2\omega_{model} + (\eta+\nu)i\theta k^2\omega_{model} \\ &- \eta\nu k^4\omega_{model} + O(\delta_1^2+\delta_2^2).\end{aligned} \tag{5.18}$$

It follows from (5.14) that

$$(W_0\cdot\nabla)^2\omega_{model} = -W_0^2k_{||}^2\omega_{model}, \tag{5.19}$$

where $k_{||}$ is the component of $\mathbf{k}$ parallel to W_0. Hence, (5.18),(5.19) imply

$$\begin{aligned}-\theta^2\omega_{model} = -\frac{W_0^2k_{||}^2}{\rho\mu}&\omega_{model} + (\eta+\nu)i\theta k^2\omega_{model} \\ &- \eta\nu k^4\omega_{model} + O(\delta_1^2+\delta_2^2).\end{aligned} \tag{5.20}$$

This gives

$$-\theta^2 = -\frac{W_0^2k_{||}^2}{\rho\mu} + (\eta+\nu)i\theta k^2 - \eta\nu k^4 + O(\delta_1^2+\delta_2^2). \tag{5.21}$$

Solving this quadratic equation for θ gives the dispersion relationship

$$\theta = -\frac{(\eta+\nu)k^2}{2}i \pm \Big(\sqrt{\frac{W_0^2k_{||}^2}{\rho\mu} - \frac{(\nu-\eta)^2k^4}{4}} + O(\delta_1^2+\delta_2^2)\Big). \tag{5.22}$$

Hence, for a perfect fluid ($\nu=\eta=0$) we obtain

$$\begin{aligned}\theta &= \pm\tilde{v}_a k_{||}, \\ \tilde{v}_a &= v_a + O(\delta_1^2+\delta_2^2),\end{aligned}$$

where v_a is the Alfvén velocity $W_0/\sqrt{\rho\mu}$.

When $\nu = 0$ and η is small (i.e. for high Re_m) we have

$$\theta = \pm\tilde{v}_a k_{||} - \frac{\eta k^2}{2} i,$$

which represents a transverse wave with a group velocity equal to $\pm v_a + O(\delta_1^2 + \delta_2^2)$. □

We conclude that our model (1.7) preserves the Alfvén waves and the group velocity of the waves $\tilde{v}_a$ tends to the true Alfvén velocity v_a as the radii tend to zero.

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