STABILITY OF PARTITIONED IMEX METHODS FOR SYSTEMS OF EVOLUTION EQUATIONS WITH SKEW-SYMMETRIC COUPLING †

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Abstract. Stability is proven for an implicit-explicit, second order, two step method for uncoupling a system of two evolution equations with exactly skew symmetric coupling. The form of the coupling studied arises in spatial discretizations of the Stokes-Darcy problem. The method proposed is an interpolation of the Crank-Nicolson Leap Frog (CNLF) combination with the BDF2-AB2 combination, being stable under the time step condition suggested by linear stability theory for the Leap-Frog scheme and BDF2-AB2.

We analyze and prove the stability of a second order IMEX method for uncoupling two evolution equations with exactly skew symmetric coupling:

$$\left\{\begin{array}{lcl} \frac{du}{dt}+A_1u+C\phi &=& f(t),\ \text{for } t>0 \text{ and } u(0)=u_0\\ \frac{d\phi}{dt}+A_2\phi-C^Tu &=& g(t),\ \text{for } t>0 \text{ and } \phi(0)=\phi_0. \end{array}\right. \tag{1}$$

This problem occurs, for example, after spatial discretization of the evolutionary Stokes-Darcy problem (e.g. [1, 4] and references therein). Here $u:[0,\infty)\to\mathbb{R}^N, \phi:[0,\infty)\to\mathbb{R}^M$, and f,g,u_0,ϕ_0 and the matrices $A_{1/2},C$ have compatible dimensions (and in particular C is $N\times M$). Note especially the exactly skew symmetric coupling linking the two equations. We assume that **the A_i are SPD**. The analysis extends to the case of A_i non-symmetric with positive definite symmetric part or even nonlinear with $\langle A(v),v\rangle \geq Const.|v|^2$. The case where $A_{1/2}$ are exactly skew symmetric, relevant to wave propagation problems with both fast and slow waves, is treated in Remark 2. In Theorem 1 we show that the *quantities of interest* for uncoupling system (1) are the following

$$\lambda_{\max}(C^TC), \qquad \min_{i=1,2}\{\lambda_{\min}(A_i)\}.$$

With superscript denoting the time step number, consider the three level θ-method [2]

$$\begin{aligned} \tfrac{(2\theta-\frac12)u^{n+1}+(-4\theta+2)u^n+(2\theta-\frac32)u^{n-1}}{\Delta t} &+ A_1\big(\theta u^{n+1}+(1-\theta)u^{n-1}\big)\\ &+C\big(2\theta\phi^n+(-2\theta+1)\phi^{n-1}\big)=f^{n+2\theta-1}, \end{aligned} \tag{2}$$

$$\begin{aligned} \tfrac{(2\theta-\frac12)\phi^{n+1}+(-4\theta+2)\phi^n+(2\theta-\frac32)\phi^{n-1}}{\Delta t} &+ A_2\big(\theta \phi^{n+1}+(1-\theta)\phi^{n-1}\big)\\ &-C^T\big(2\theta u^n+(-2\theta+1)u^{n-1}\big)=g^{n+2\theta-1}, \end{aligned} \tag{3}$$

where $\theta\in\left[\frac12,1\right]$, and u^1,ϕ^1 are computed with a second order method. When $\theta=\frac12$, one obtains the Crank-Nicolson Leap Frog (CNLF) method

$$\begin{aligned} \frac{u^{n+1}-u^{n-1}}{2\Delta t}+A_1\frac{u^{n+1}+u^{n-1}}{2}+C\phi^n=f^n,\\ \frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t}+A_2\frac{\phi^{n+1}+\phi^{n-1}}{2}-C^Tu^n=g^n, \end{aligned}$$

while $\theta=1$ gives the BDF2-AB2 method

$$\begin{aligned} \tfrac{\frac32u^{n+1}-2u^n+\frac12u^{n-1}}{\Delta t}+A_1u^{n+1}+C\big(2\phi^n-\phi^{n-1}\big)=f^{n+1},\\ \tfrac{\frac32\phi^{n+1}-2\phi^n+\frac12\phi^{n-1}}{\Delta t}+A_2\phi^{n+1}-C^T\big(2u^n-u^{n-1}\big)=g^{n+1}. \end{aligned}$$

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Remark 1. *The system* (2)-(3) *can be rewritten as CNLF in variables* u^n, ϕ^n*, corrected with a first-order scheme (Crank-Nicolson-lagged) for the errors* $e^n = u^n - u^{n-1}, \varphi^n = \phi^n - \phi^{n-1}$*:*

$$\frac{u^{n+1}-u^{n-1}}{2\Delta t}+A_1\frac{u^{n+1}+u^{n-1}}{2}+C\phi^n+(2\theta-1)\Big[\frac{e^{n+1}-e^n}{\Delta t}+A_1\frac{e^{n+1}+e^n}{2}+C\varphi^n\Big]=f^{n+2\theta-1},$$
$$\frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t}+A_2\frac{\phi^{n+1}+\phi^{n-1}}{2}-C^Tu^n+(2\theta-1)\Big[\frac{\varphi^{n+1}-\varphi^n}{\Delta t}+A_2\frac{\varphi^{n+1}+\varphi^n}{2}-C^Tu^n\Big]=g^{n+2\theta-1}.$$

This "modification" of CNLF introduces the numerical dissipation term

$$(2\theta-1)\big((2\theta-1)\min_{i=1,2}\lambda_{min}(A_i)-\Delta t\theta^2\lambda_{\max}(C^TC)\big)\sum_{n=3}^{N-2}\big(\|u^n\|^2+\|\phi^n\|^2\big).$$

Since the stability region of LF is the interval $-1 < \mathrm{Im}(z) < +1$, from the scalar case we expect a stability restriction of the form $\triangle t\sqrt{\lambda_{\max}(C^TC)} \leq 1$. (Sufficiency in the non-commutative case for CNLF was proven only recently in [4].) For vectors of the same length, we denote the usual euclidean inner product and norm by $\langle u, v\rangle = u^Tv, |\phi|^2 = \langle\phi,\phi\rangle$, and the weighted norms by $|u|^2_{A_1} = u^TA_1u, |\phi|^2_{A_2} = \phi^TA_2\phi$.

Let denote

$$\Lambda = \theta(2\theta-1)\min_{i=1,2}\{\lambda_{\min}A_i\} - \theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)}.$$

Theorem 1. *theorem Assume that the time-step* Δt *satisfies the following:*

$$\Delta t \leq \frac{1}{\sqrt{\lambda_{\max}(C^TC)}}, \qquad \text{if } \theta=\tfrac{1}{2}, \tag{4}$$
$$\Delta t\leq \min\Big\{\frac{\Lambda+\sqrt{\Lambda^2+4\theta^2(2\theta-1)\lambda_{\max}(C^TC)\nu}}{2\theta^2(2\theta-1)\lambda_{\max}(C^TC)}, \frac{(2\theta-1)}{\theta^2}\frac{\min_{i=1,2}\{\lambda_{\min}(A_i)\}}{\lambda_{\max}(C^TC)}\Big\}, \tag{5}$$

if $\theta\in(\frac{1}{2},1]$*. For the* θ*-family of IMEX methods* (2)-(3) *the following energy estimate holds*

$$\begin{aligned}
&\Big(\frac{\nu}{\Delta t}+\theta(2\theta-1)\min_{i=1,2}\{\lambda_{\min}(A_i)\}-\theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)}-\theta^2(2\theta-1)\lambda_{\max}(C^TC)\Delta t\Big)\times\\
&\qquad\times\big(\|u^{N+1}\|^2+\|u^N\|^2+\|\phi^{N+1}\|^2+\|\phi^N\|^2\big)\\
&+\big((2\theta-1)^2\min_{i=1,2}\{\lambda_{\min}(A_i)\}-\theta^2(2\theta-1)\lambda_{\max}(C^TC)\Delta t\big)\sum_{n=3}^{N-1}\big(\|u^n\|^2+\|\phi^n\|^2\big)\\
&+\theta(1-\theta)\sum_{n=1}^{N}\big(\|u^{n+1}+u^{n-1}\|^2_{A_1}+\|\phi^{n+1}+\phi^{n-1}\|^2_{A_2}\big)\\
&\leq-\frac{2\theta-1}{4\Delta t}\big(\|u^2-2u^1+u^0\|^2+\|\phi^2-2\phi^1+\phi^0\|^2\big)-\theta(2\theta-1)\big(\|u^2\|^2_{A_1}+\|\phi^2\|^2_{A_2}\big)\\
&\quad+(1-\theta)(2\theta-1)\big(\|u^1\|^2_{A_1}+\|u^0\|^2_{A_1}+\|\phi^2\|^2_{A_2}+\|\phi^0\|^2_{A_2}\big)\\
&\quad-2\theta(1-\theta)\big(\langle C\phi^0,u^1\rangle-\langle C^Tu^0,\phi^1\rangle\big)-\theta(1-2\theta)\big(\langle C\phi^0,u^2\rangle-\langle C^Tu^0,\phi^2\rangle\big)\\
&\quad+\frac{1}{\Delta t}\big(\|(u^1,u^0)\|^2_G+\|(\phi^1,\phi^0)\|^2_G\big)\\
&\quad+\sum_{n=1}^{N}\big(\langle f^{n+1},\theta u^{n+1}+(1-\theta)u^{n-1}\rangle+\langle g^{n+1},\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\rangle\big),
\end{aligned}$$

where $\nu \in \left[\frac{1}{4}, \frac{1}{2}\right]$ *is defined in* (6).

COROLLARY 0.1. *For* $\theta = \frac{1}{2}$ *(CNLF), the method is stable if*

$$\Delta t \leq \frac{1}{\sqrt{\lambda_{\max} C^T C}},$$

while for $\theta = 1$ *(BDF2-AB2), the method is stable provided*

$$\Delta t \leq \frac{\min_{i=1,2} \lambda_{\min(A_i)}}{\lambda_{\max}(C^T C)}.$$

REMARK 2. *In the case when* $A_{1,2}$ *are skew-symmetric, for* $\theta = \frac{1}{2}$*(CNLF) the method is stable. For* $\theta \in (\frac{1}{2}, 1]$*, stability is less clear. It includes dissipation terms, which, while possibly large, do not seem to control the coupling terms (see [4]).*

LEMMA 0.2 (***G*-stability**). *There exists a positive definite matrix [3]*

$$G = \begin{pmatrix} 2\theta^2 - \theta + \frac{1}{4} & -\frac{(2\theta-1)^2}{2} \\ -\frac{(2\theta-1)^2}{2} & 2\theta^2 - 3\theta + \frac{5}{4} \end{pmatrix}, \quad \theta \in \left[\tfrac{1}{2}, 1\right], \quad \textit{such that}$$

$$\begin{aligned}
&\frac{1}{\Delta t}\langle (2\theta - \tfrac{1}{2})u^{n+1} + (-4\theta + 2)u^n + (2\theta - \tfrac{3}{2})u^{n-1}, \theta u^{n+1} + (1-\theta)u^{n+1}\rangle \\
&\quad = \|(u^{n+1}, u^n)\|_G^2 - \|(u^n, u^{n-1})\|_G^2 + \tfrac{2\theta-1}{4}\|u^{n+1} - 2u^n + u^{n-1}\|^2.
\end{aligned}$$

LEMMA 0.3. $\|(u^{n+1}, u^n)\|_G^2 = \nu\big(\|u^{n+1}\|^2 + \|u^n\|^2\big) + \|au^{n+1} + bu^n\|^2$, *where*

$$\begin{aligned}
&\nu = \frac{1}{16(2\theta^2 - 3\theta + \frac{5}{4})} \in \left[\tfrac{1}{4}, \tfrac{1}{2}\right], \qquad (6) \\
&a = -\sqrt{\tfrac{2\theta-1}{2}\big(\sqrt{1 + (2\theta-1)^2} + 1\big)}, \; b = \sqrt{\tfrac{2\theta-1}{2}\big(\sqrt{1 + (2\theta-1)^2} - 1\big)}.
\end{aligned}$$

Proof of Theorem 1. The proof is based on energy type estimates. First, multiply (2) by $\theta u^{n+1} + (1-\theta)u^{n-1}$ and (3) by $\theta\phi^{n+1} + (1-\theta)\phi^{n-1}$ to obtain

$$\begin{aligned}
&\frac{1}{\Delta t}\Big(\|(u^{n+1}, u^n)\|_G^2 - \|(u^n, u^{n-1})\|_G^2\Big) + \frac{2\theta - 1}{4\Delta t}\|u^{n+1} - 2u^n + u^{n-1}\|^2 \qquad (7) \\
&+ \frac{1}{\Delta t}\Big(\|(\phi^{n+1}, \phi^n)\|_G^2 - \|(\phi^n, \phi^{n-1})\|_G^2\Big) + \frac{2\theta - 1}{4\Delta t}\|\phi^{n+1} - 2\phi^n + \phi^{n-1}\|^2 \\
&+ \|\theta u^{n+1} + (1-\theta)u^{n-1}\|_{A_1}^2 + \|\theta\phi^{n+1} + (1-\theta)\phi^{n-1}\|_{A_2}^2 \\
&+ \langle C\big(2\theta\phi^n + (-2\theta+1)\phi^{n-1}\big), \theta u^{n+1} + (1-\theta)u^{n-1}\rangle \\
&- \langle C^T\big(2\theta u^n + (-2\theta+1)u^{n-1}\big), \theta\phi^{n+1} + (1-\theta)\phi^{n-1}\rangle \\
&= \langle f^{n+1}, \theta u^{n+1} + (1-\theta)u^{n-1}\rangle + \langle g^{n+1}, \theta\phi^{n+1} + (1-\theta)\phi^{n-1}\rangle,
\end{aligned}$$

which by summation from $n = 1$ to N yields

$$\begin{aligned}
&\frac{1}{\Delta t}\big(\|(u^{N+1}, u^N)\|_G^2 + \|(\phi^{N+1}, \phi^N)\|_G^2\big) \qquad (8) \\
&+ \frac{2\theta-1}{4\Delta t}\sum_{n=1}^{N}\big(\|u^{n+1} - 2u^n + u^{n-1}\|^2 + \|\phi^{n+1} - 2\phi^n + \phi^{n-1}\|^2\big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{N}\big(\|\theta u^{n+1}+(1-\theta)u^{n-1}\|_{A_1}^2+\|\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\|_{A_2}^2\big)\\
&+\sum_{n=1}^{N}\Big(\langle C\big(2\theta\phi^{n}+(1-2\theta)\phi^{n-1}\big),\theta u^{n+1}+(1-\theta)u^{n-1}\rangle\\
&\quad-\langle C^T\big(2\theta u^{n}+(1-2\theta)u^{n-1}\big),\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\rangle\Big)\\
&=\frac{1}{\Delta t}\big(\|(u^1,u^0)\|_G^2+\|(\phi^1,\phi^0)\|_G^2\big)\\
&+\sum_{n=1}^{N}\big(\langle f^{n+1},\theta u^{n+1}+(1-\theta)u^{n-1}\rangle+\langle g^{n+1},\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\rangle\big).
\end{aligned}
$$

LEMMA 0.4. *The contribution of the dissipative terms to the energy equation is*

$$
\begin{aligned}
&\sum_{n=1}^{N}\big(\|\theta u^{n+1}+(1-\theta)u^{n-1}\|_{A_1}^2+\|\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\|_{A_2}^2\big)\\
&=\theta(2\theta-1)\big(\|u^{N+1}\|_{A_1}^2+\|\phi^{N+1}\|_{A_2}^2+\|u^{N}\|_{A_1}^2+\|\phi^{N}\|_{A_2}^2\big)\\
&\quad+(2\theta-1)^2\sum_{n=2}^{N-1}\big(\|u^n\|_{A_1}^2+\|\phi^n\|_{A_2}^2\big)\\
&\quad-(1-\theta)(2\theta-1)\big(\|u^1\|_{A_1}^2+\|u^0\|_{A_1}^2+\|\phi^1\|_{A_2}^2+\|\phi^0\|_{A_2}^2\big)\\
&\quad+\theta(1-\theta)\sum_{n=1}^{N}\big(\|u^{n+1}+u^{n-1}\|_{A_1}^2+\|\phi^{n+1}+\phi^{n-1}\|_{A_2}^2\big).
\end{aligned}
\tag{9}
$$

Proof. Using the polarized identity, the dissipative terms can be rewritten as

$$
\begin{aligned}
&\|\theta u^{n+1}+(1-\theta)u^{n-1}\|_{A_1}^2+\|\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\|_{A_2}^2\\
&=\theta(2\theta-1)\big(\|u^{n+1}\|_{A_1}^2+\|\phi^{n+1}\|_{A_2}^2\big)+\theta(1-\theta)\big(\|u^{n+1}+u^{n-1}\|_{A_1}^2+\|\phi^{n+1}+\phi^{n-1}\|_{A_2}^2\big)\\
&\quad-(1-\theta)(2\theta-1)\big(\|u^{n-1}\|_{A_1}^2+\|\phi^{n-1}\|_{A_2}^2\big)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\sum_{n=1}^{N}\big(\|\theta u^{n+1}+(1-\theta)u^{n-1}\|_{A_1}^2+\|\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\|_{A_2}^2\big)\\
&=\theta(2\theta-1)\big(\|u^{N+1}\|_{A_1}^2+\|\phi^{N+1}\|_{A_2}^2+\|u^{N}\|_{A_1}^2+\|\phi^{N}\|_{A_2}^2\big)\\
&\quad+(2\theta-1)^2\sum_{n=2}^{N-1}\big(\|u^n\|_{A_1}^2+\|\phi^n\|_{A_2}^2\big)\\
&\quad-(1-\theta)(2\theta-1)\big(\|u^1\|_{A_1}^2+\|u^0\|_{A_1}^2+\|\phi^1\|_{A_2}^2+\|\phi^0\|_{A_2}^2\big)\\
&\quad+\theta(1-\theta)\sum_{n=1}^{N}\big(\|u^{n+1}+u^{n-1}\|_{A_1}^2+\|\phi^{n+1}+\phi^{n-1}\|_{A_2}^2\big)
\end{aligned}
$$

which completes the proof. □

LEMMA 0.5. *The contribution of the coupling terms to the energy equation is*

$$
\sum_{n=1}^{N}\Big(\langle C(2\theta\phi^{n}+(1-2\theta)\phi^{n-1}),\theta u^{n+1}+(1-\theta)u^{n-1}\rangle
\tag{10}
$$

$$
\begin{aligned}
&-\langle C^T(2\theta u^n+(1-2\theta)u^{n-1}),\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\rangle\Big)\\
&\geq \big(-\theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)}-\theta^2(2\theta-1)\Delta t\lambda_{\max}(C^TC)\big)\times\\
&\quad\times\Big(\|u^{N+1}\|^2+\|\phi^{N+1}\|^2+\|u^N\|^2+\|\phi^N\|^2\Big)\\
&\quad-\frac{2\theta-1}{4\Delta t}\sum_{n=2}^{N}\big(\|\phi^{n+1}-2\phi^n+\phi^{n-1}\|^2+\|u^{n+1}-2u^n+u^{n-1}\|^2\big)\\
&\quad-\theta^2(2\theta-1)\Delta t\lambda_{\max}(C^TC)\sum_{n=3}^{N-1}\big(\|u^n\|^2+\|\phi^n\|^2\big)\\
&\quad+2\theta(1-\theta)(\langle C\phi^0,u^1\rangle-\langle C^Tu^0,\phi^1\rangle)+\theta(1-2\theta)(\langle C\phi^0,u^2\rangle-\langle C^Tu^0,\phi^2\rangle).
\end{aligned}
$$

Proof. We note that the coupling terms can be written as

$$
\begin{aligned}
&\langle C(2\theta\phi^n+(-2\theta+1)\phi^{n-1}),\theta u^{n+1}+(1-\theta)u^{n-1}\rangle\\
&\quad-\langle C^T(2\theta u^n+(-2\theta+1)u^{n-1}),\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\rangle\\
&=2\theta^2(\langle C\phi^n,u^{n+1}\rangle-\langle C^Tu^n,\phi^{n+1}\rangle)+2\theta(1-\theta)(\langle C\phi^n,u^{n-1}\rangle-\langle C^Tu^n,\phi^{n-1}\rangle)\\
&\quad+\theta(1-2\theta)(\langle C\phi^{n-1},u^{n+1}\rangle-\langle C^Tu^{n-1},\phi^{n+1}\rangle)\\
&\quad+(1-2\theta)(1-\theta)\underbrace{(\langle C\phi^{n-1},u^{n-1}\rangle-\langle C^Tu^{n-1},\phi^{n-1}\rangle)}_{=0}.
\end{aligned}
$$

Sum the coupling terms from $n=1$ to N and rearrange to obtain after some calculations

$$
\begin{aligned}
&2\theta^2(\langle C\phi^N,u^{N+1}\rangle-\langle C^Tu^N,\phi^{N+1}\rangle)+2\theta(2\theta-1)\sum_{n=2}^{N-1}(\langle C\phi^n,u^{n+1}\rangle-\langle C^Tu^n,\phi^{n+1}\rangle)\\
&\quad+2\theta(1-\theta)(\langle C\phi^0,u^1\rangle-\langle C^Tu^0,\phi^1\rangle)+\theta(1-2\theta)\sum_{n=1}^{N}(\langle C\phi^{n-1},u^{n+1}\rangle-\langle C^Tu^{n-1},\phi^{n+1}\rangle)\\
&=2\theta^2(\langle C\phi^N,u^{N+1}\rangle-\langle C^Tu^N,\phi^{N+1}\rangle)+\theta(1-2\theta)(\langle C\phi^{N-1},u^{N+1}\rangle-\langle C^Tu^{N-1},\phi^{N+1}\rangle)\\
&\quad+2\theta(2\theta-1)\sum_{n=2}^{N-1}(\langle C\phi^n,u^{n+1}\rangle-\langle C^Tu^n,\phi^{n+1}\rangle)\\
&\quad+2\theta(1-\theta)(\langle C\phi^0,u^1\rangle-\langle C^Tu^0,\phi^1\rangle)+\theta(1-2\theta)\sum_{n=1}^{N-1}(\langle C\phi^{n-1},u^{n+1}\rangle-\langle C^Tu^{n-1},\phi^{n+1}\rangle)\\
&=2\theta^2(\langle C\phi^N,u^{N+1}\rangle-\langle C^Tu^N,\phi^{N+1}\rangle)+\theta(1-2\theta)(\langle C\phi^{N-1},u^{N+1}\rangle-\langle C^Tu^{N-1},\phi^{N+1}\rangle)\\
&\quad+2\theta(2\theta-1)\sum_{n=2}^{N-1}(\langle C\phi^n,u^{n+1}\rangle-\langle C^Tu^n,\phi^{n+1}\rangle)\\
&\quad+2\theta(1-\theta)(\langle C\phi^0,u^1\rangle-\langle C^Tu^0,\phi^1\rangle)+\theta(1-2\theta)\sum_{n=1}^{N-1}(\langle C\phi^{n-1},u^{n+1}\rangle-\langle C^Tu^{n-1},\phi^{n+1}\rangle)\\
&=2\theta^2(\langle C\phi^N,u^{N+1}\rangle-\langle C^Tu^N,\phi^{N+1}\rangle)+\theta(1-2\theta)(\langle C\phi^{N-1},u^{N+1}\rangle-\langle C^Tu^{N-1},\phi^{N+1}\rangle)\\
&\quad+\theta(2\theta-1)\sum_{n=2}^{N-1}(\langle C(2\phi^n-\phi^{n-1}),u^{n+1}\rangle-\langle C^T(2u^n-u^{n-1}),\phi^{n+1}\rangle)
\end{aligned}
$$

$$
\begin{aligned}
&+2\theta(1-\theta)(\langle C\phi^0, u^1\rangle - \langle C^T u^0, \phi^1\rangle) + \theta(1-2\theta)(\langle C\phi^0, u^2\rangle - \langle C^T u^0, \phi^2\rangle) \\
&= 2\theta^2(\langle C\phi^N, u^{N+1}\rangle - \langle C^T u^N, \phi^{N+1}\rangle) + \theta(1-2\theta)(\langle C\phi^{N-1}, u^{N+1}\rangle - \langle C^T u^{N-1}, \phi^{N+1}\rangle) \\
&\quad +\theta(2\theta-1)\sum_{n=2}^{N-1}(\langle C(-\phi^{n+1}+2\phi^n-\phi^{n-1}), u^{n+1}\rangle - \langle C^T(-u^{n+1}+2u^n-u^{n-1}), \phi^{n+1}\rangle) \\
&\quad +2\theta(1-\theta)(\langle C\phi^0, u^1\rangle - \langle C^T u^0, \phi^1\rangle) + \theta(1-2\theta)(\langle C\phi^0, u^2\rangle - \langle C^T u^0, \phi^2\rangle) \\
&= \langle C \underbrace{(2\theta^2\phi^N + \theta(1-2\theta)\phi^{N-1})}_{=2\theta(1-\theta)\phi^N - 2\theta(1-2\theta)\phi^N + \theta(1-2\theta)\phi^{N-1}}, u^{N+1}\rangle - \langle C^T \underbrace{(2\theta^2 u^N + \theta(1-2\theta)u^{N-1})}_{=2\theta(1-\theta)u^N - 2\theta(1-2\theta)u^N + \theta(1-2\theta)u^{N-1}}, \phi^{N+1}\rangle \\
&\quad +\theta(2\theta-1)\sum_{n=2}^{N-1}(\langle C(-\phi^{n+1}+2\phi^n-\phi^{n-1}), u^{n+1}\rangle - \langle C^T(-u^{n+1}+2u^n-u^{n-1}), \phi^{n+1}\rangle) \\
&\quad +2\theta(1-\theta)(\langle C\phi^0, u^1\rangle - \langle C^T u^0, \phi^1\rangle) + \theta(1-2\theta)(\langle C\phi^0, u^2\rangle - \langle C^T u^0, \phi^2\rangle). \\
&= \langle C(2\theta(1-\theta)\phi^N - 2\theta(1-2\theta)\phi^N + \theta(1-2\theta)\phi^{N-1}), u^{N+1}\rangle \\
&\quad -\langle C^T(2\theta(1-\theta)u^N - 2\theta(1-2\theta)u^N + \theta(1-2\theta)u^{N-1}), \phi^{N+1}\rangle \\
&\quad +\theta(2\theta-1)\sum_{n=2}^{N-1}(\langle C(-\phi^{n+1}+2\phi^n-\phi^{n-1}), u^{n+1}\rangle - \langle C^T(-u^{n+1}+2u^n-u^{n-1}), \phi^{n+1}\rangle) \\
&\quad +2\theta(1-\theta)(\langle C\phi^0, u^1\rangle - \langle C^T u^0, \phi^1\rangle) + \theta(1-2\theta)(\langle C\phi^0, u^2\rangle - \langle C^T u^0, \phi^2\rangle) \\
&= \theta(1-2\theta)\langle C(-2\phi^N + \phi^{N-1}), u^{N+1}\rangle - \theta(1-2\theta)\langle C^T(-2u^N + u^{N-1}), \phi^{N+1}\rangle \\
&\quad +2\theta(1-\theta)\langle C\phi^N, u^{N+1}\rangle - 2\theta(1-\theta)\langle C^T u^N, \phi^{N+1}\rangle \\
&\quad +\theta(2\theta-1)\sum_{n=2}^{N-1}(\langle C(-\phi^{n+1}+2\phi^n-\phi^{n-1}), u^{n+1}\rangle - \langle C^T(-u^{n+1}+2u^n-u^{n-1}), \phi^{n+1}\rangle) \\
&\quad +2\theta(1-\theta)(\langle C\phi^0, u^1\rangle - \langle C^T u^0, \phi^1\rangle) + \theta(1-2\theta)(\langle C\phi^0, u^2\rangle - \langle C^T u^0, \phi^2\rangle) \\
&= 2\theta(1-\theta)\big(\langle C\phi^N, u^{N+1}\rangle - \langle C^T u^N, \phi^{N+1}\rangle\big) \\
&\quad + \theta(2\theta-1)\sum_{n=2}^{N}(\langle C(-\phi^{n+1}+2\phi^n-\phi^{n-1}), u^{n+1}\rangle - \langle C^T(-u^{n+1}+2u^n-u^{n-1}), \phi^{n+1}\rangle) \\
&\quad + 2\theta(1-\theta)(\langle C\phi^0, u^1\rangle - \langle C^T u^0, \phi^1\rangle) + \theta(1-2\theta)(\langle C\phi^0, u^2\rangle - \langle C^T u^0, \phi^2\rangle)
\end{aligned}
$$

Now, use

$$
\begin{aligned}
&2\theta(1-\theta)\big(\langle C\phi^{N+1}, u^N\rangle - \langle C\phi^N, u^{N+1}\rangle\big) \\
&\qquad \le \theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)}\Big(\|u^{N+1}\|^2 + \|\phi^{N+1}\|^2 + \|u^N\|^2 + \|\phi^N\|^2\Big)
\end{aligned}
$$

and the Cauchy-Schwarz inequality

$$
\begin{aligned}
&\theta(2\theta-1)\sum_{n=2}^{N}(\langle C(-\phi^{n+1}+2\phi^n-\phi^{n-1}), u^{n+1}\rangle - \langle C^T(-u^{n+1}+2u^n-u^{n-1}), \phi^{n+1}\rangle) \\
&\le \frac{2\theta-1}{4\Delta t}\sum_{n=2}^{N}\big(\|\phi^{n+1}-2\phi^n+\phi^{n-1}\|^2 + \|u^{n+1}-2u^n+u^{n-1}\|^2\big) \\
&\qquad + \theta^2(2\theta-1)\Delta t\sum_{n=2}^{N}\big(\|C^Tu^{n+1}\|^2 + \|C\phi^{n+1}\|^2\big)
\end{aligned}
$$

to obtain

$$
\begin{aligned}
&\sum_{n=1}^{N}\Big(\langle C(2\theta\phi^n+(1-2\theta)\phi^{n-1}),\theta u^{n+1}+(1-\theta)u^{n-1}\rangle\\
&\qquad-\langle C^T(2\theta u^n+(1-2\theta)u^{n-1}),\theta\phi^{n+1}+(1-\theta)\phi^{n-1}\rangle\Big)\\
&=2\theta(1-\theta)\big(\langle C\phi^N,u^{N+1}\rangle-\langle C^Tu^N,\phi^{N+1}\rangle\big)\\
&\qquad+\theta(2\theta-1)\sum_{n=2}^{N}(\langle C(-\phi^{n+1}+2\phi^n-\phi^{n-1}),u^{n+1}\rangle-\langle C^T(-u^{n+1}+2u^n-u^{n-1}),\phi^{n+1}\rangle)\\
&\qquad+2\theta(1-\theta)(\langle C\phi^0,u^1\rangle-\langle C^Tu^0,\phi^1\rangle)+\theta(1-2\theta)(\langle C\phi^0,u^2\rangle-\langle C^Tu^0,\phi^2\rangle)\\
&\ge-\theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)}\Big(\|u^{N+1}\|^2+\|\phi^{N+1}\|^2+\|u^N\|^2+\|\phi^N\|^2\Big)\\
&-\frac{2\theta-1}{4\Delta t}\sum_{n=2}^{N}\big(\|\phi^{n+1}-2\phi^n+\phi^{n-1}\|^2+\|u^{n+1}-2u^n+u^{n-1}\|^2\big)\\
&\qquad-\theta^2(2\theta-1)\Delta t\sum_{n=2}^{N}\big(\|C^Tu^{n+1}\|^2+\|C\phi^{n+1}\|^2\big)\\
&\qquad+2\theta(1-\theta)(\langle C\phi^0,u^1\rangle-\langle C^Tu^0,\phi^1\rangle)+\theta(1-2\theta)(\langle C\phi^0,u^2\rangle-\langle C^Tu^0,\phi^2\rangle)\\
&\ge\big(-\theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)}-\theta^2(2\theta-1)\Delta t\lambda_{\max}(C^TC)\big)\times\\
&\qquad\times\Big(\|u^{N+1}\|^2+\|\phi^{N+1}\|^2+\|u^N\|^2+\|\phi^N\|^2\Big)\\
&\quad-\frac{2\theta-1}{4\Delta t}\sum_{n=2}^{N}\big(\|\phi^{n+1}-2\phi^n+\phi^{n-1}\|^2+\|u^{n+1}-2u^n+u^{n-1}\|^2\big)\\
&\quad-\theta^2(2\theta-1)\Delta t\sum_{n=2}^{N-2}\big(\|C^Tu^{n+1}\|^2+\|C\phi^{n+1}\|^2\big)\\
&\quad+2\theta(1-\theta)(\langle C\phi^0,u^1\rangle-\langle C^Tu^0,\phi^1\rangle)+\theta(1-2\theta)(\langle C\phi^0,u^2\rangle-\langle C^Tu^0,\phi^2\rangle)\\
&=\big(-\theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)}-\theta^2(2\theta-1)\Delta t\lambda_{\max}(C^TC)\big)\times\\
&\qquad\times\Big(\|u^{N+1}\|^2+\|\phi^{N+1}\|^2+\|u^N\|^2+\|\phi^N\|^2\Big)\\
&\quad-\frac{2\theta-1}{4\Delta t}\sum_{n=2}^{N}\big(\|\phi^{n+1}-2\phi^n+\phi^{n-1}\|^2+\|u^{n+1}-2u^n+u^{n-1}\|^2\big)\\
&\quad-\theta^2(2\theta-1)\Delta t\sum_{n=3}^{N-1}\big(\|C^Tu^n\|^2+\|C\phi^n\|^2\big)\\
&\quad+2\theta(1-\theta)(\langle C\phi^0,u^1\rangle-\langle C^Tu^0,\phi^1\rangle)+\theta(1-2\theta)(\langle C\phi^0,u^2\rangle-\langle C^Tu^0,\phi^2\rangle).
\end{aligned}
$$

□

Using (8) and (9) in (10) we have

$$
\begin{aligned}
&\frac{1}{\Delta t}\big(\|(u^{N+1},u^N)\|_G^2+\|(\phi^{N+1},\phi^N)\|_G^2\big)+\frac{2\theta-1}{4\Delta t}\big(\|u^2-2u^1+u^0\|^2+\|\phi^2-2\phi^1+\phi^0\|^2\big)\\
&+\big(\theta(2\theta-1)\min_{i=1,2}\{\lambda_{\min}(A_i)\}-\theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)}-\theta^2(2\theta-1)\lambda_{\max}(C^TC)\Delta t\big)\times
\end{aligned}
$$

$$
\begin{aligned}
&\qquad \times \big(\|u^{N+1}\|^2 + \|u^N\|^2 + \|\phi^{N+1}\|^2 + \|\phi^N\|^2\big) \\
&+ \big((2\theta-1)^2 \min_{i=1,2}\{\lambda_{\min}(A_i)\} - \theta^2(2\theta-1)\lambda_{\max}(C^TC)\Delta t\big) \sum_{n=3}^{N-1} \big(\|u^n\|^2 + \|\phi^n\|^2\big) \\
&\qquad + \theta(2\theta-1)\big(\|u^2\|_{A_1}^2 + \|\phi^2\|_{A_2}^2\big) \\
&- (1-\theta)(2\theta-1)\big(\|u^1\|_{A_1}^2 + \|u^0\|_{A_1}^2 + \|\phi^2\|_{A_2}^2 + \|\phi^0\|_{A_2}^2\big) \\
&+ \theta(1-\theta) \sum_{n=1}^{N} \big(\|u^{n+1} + u^{n-1}\|_{A_1}^2 + \|\phi^{n+1} + \phi^{n-1}\|_{A_2}^2\big) \\
&+ 2\theta(1-\theta)\big(\langle C\phi^0, u^1\rangle - \langle C^T u^0, \phi^1\rangle\big) + \theta(1-2\theta)\big(\langle C\phi^0, u^2\rangle - \langle C^T u^0, \phi^2\rangle\big) \\
&= \frac{1}{\Delta t}\big(\|(u^1, u^0)\|_G^2 + \|(\phi^1, \phi^0)\|_G^2\big) \\
&\quad + \sum_{n=1}^{N}\big(\langle f^{n+1}, \theta u^{n+1} + (1-\theta)u^{n-1}\rangle + \langle g^{n+1}, \theta\phi^{n+1} + (1-\theta)\phi^{n-1}\rangle\big).
\end{aligned}
$$

Equivalently

$$
\begin{aligned}
&\frac{1}{\Delta t}\big(\|(u^{N+1}, u^N)\|_G^2 + \|(\phi^{N+1}, \phi^N)\|_G^2\big) \\
&+ \big(\theta(2\theta-1) \min_{i=1,2}\{\lambda_{\min}(A_i)\} - \theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)} - \theta^2(2\theta-1)\lambda_{\max}(C^TC)\Delta t\big) \times \\
&\qquad \times \big(\|u^{N+1}\|^2 + \|u^N\|^2 + \|\phi^{N+1}\|^2 + \|\phi^N\|^2\big) \\
&+ \big((2\theta-1)^2 \min_{i=1,2}\{\lambda_{\min}(A_i)\} - \theta^2(2\theta-1)\lambda_{\max}(C^TC)\Delta t\big) \sum_{n=3}^{N-1} \big(\|u^n\|^2 + \|\phi^n\|^2\big) \\
&+ \theta(1-\theta) \sum_{n=1}^{N} \big(\|u^{n+1} + u^{n-1}\|_{A_1}^2 + \|\phi^{n+1} + \phi^{n-1}\|_{A_2}^2\big) \\
&\le -\frac{2\theta-1}{4\Delta t}\big(\|u^2 - 2u^1 + u^0\|^2 + \|\phi^2 - 2\phi^1 + \phi^0\|^2\big) - \theta(2\theta-1)\big(\|u^2\|_{A_1}^2 + \|\phi^2\|_{A_2}^2\big) \\
&\quad + (1-\theta)(2\theta-1)\big(\|u^1\|_{A_1}^2 + \|u^0\|_{A_1}^2 + \|\phi^2\|_{A_2}^2 + \|\phi^0\|_{A_2}^2\big) \\
&\quad - 2\theta(1-\theta)\big(\langle C\phi^0, u^1\rangle - \langle C^T u^0, \phi^1\rangle\big) - \theta(1-2\theta)\big(\langle C\phi^0, u^2\rangle - \langle C^T u^0, \phi^2\rangle\big) \\
&\quad + \frac{1}{\Delta t}\big(\|(u^1, u^0)\|_G^2 + \|(\phi^1, \phi^0)\|_G^2\big) \\
&\qquad + \sum_{n=1}^{N}\big(\langle f^{n+1}, \theta u^{n+1} + (1-\theta)u^{n-1}\rangle + \langle g^{n+1}, \theta\phi^{n+1} + (1-\theta)\phi^{n-1}\rangle\big).
\end{aligned}
$$

Therefore, using the G-stability property (6) we obtain

$$
\begin{aligned}
&\big(\frac{\nu}{\Delta t} + \theta(2\theta-1) \min_{i=1,2}\{\lambda_{\min}(A_i)\} - \theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)} - \theta^2(2\theta-1)\lambda_{\max}(C^TC)\Delta t\big) \times \\
&\qquad \times \big(\|u^{N+1}\|^2 + \|u^N\|^2 + \|\phi^{N+1}\|^2 + \|\phi^N\|^2\big) \\
&+ \big((2\theta-1)^2 \min_{i=1,2}\{\lambda_{\min}(A_i)\} - \theta^2(2\theta-1)\lambda_{\max}(C^TC)\Delta t\big) \sum_{n=3}^{N-1} \big(\|u^n\|^2 + \|\phi^n\|^2\big) \\
&+ \theta(1-\theta) \sum_{n=1}^{N} \big(\|u^{n+1} + u^{n-1}\|_{A_1}^2 + \|\phi^{n+1} + \phi^{n-1}\|_{A_2}^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq -\frac{2\theta-1}{4\Delta t}\big(\|u^2-2u^1+u^0\|^2+\|\phi^2-2\phi^1+\phi^0\|^2\big) - \theta(2\theta-1)\big(\|u^2\|^2_{A_1}+\|\phi^2\|^2_{A_2}\big)\\
&\quad + (1-\theta)(2\theta-1)\big(\|u^1\|^2_{A_1} + \|u^0\|^2_{A_1}+\|\phi^2\|^2_{A_2} + \|\phi^0\|^2_{A_2}\big)\\
&\quad - 2\theta(1-\theta)\big(\langle C\phi^0, u^1\rangle - \langle C^T u^0, \phi^1\rangle\big) - \theta(1-2\theta)\big(\langle C\phi^0, u^2\rangle - \langle C^T u^0, \phi^2\rangle\big)\\
&\quad + \frac{1}{\Delta t}\big(\|(u^1,u^0)\|^2_G + \|(\phi^1,\phi^0)\|^2_G\big)\\
&\qquad + \sum_{n=1}^{N}\big(\langle f^{n+1}, \theta u^{n+1}+(1-\theta)u^{n-1}\rangle + \langle g^{n+1}, \theta\phi^{n+1}+(1-\theta)\phi^{n-1}\rangle\big)
\end{aligned}
$$

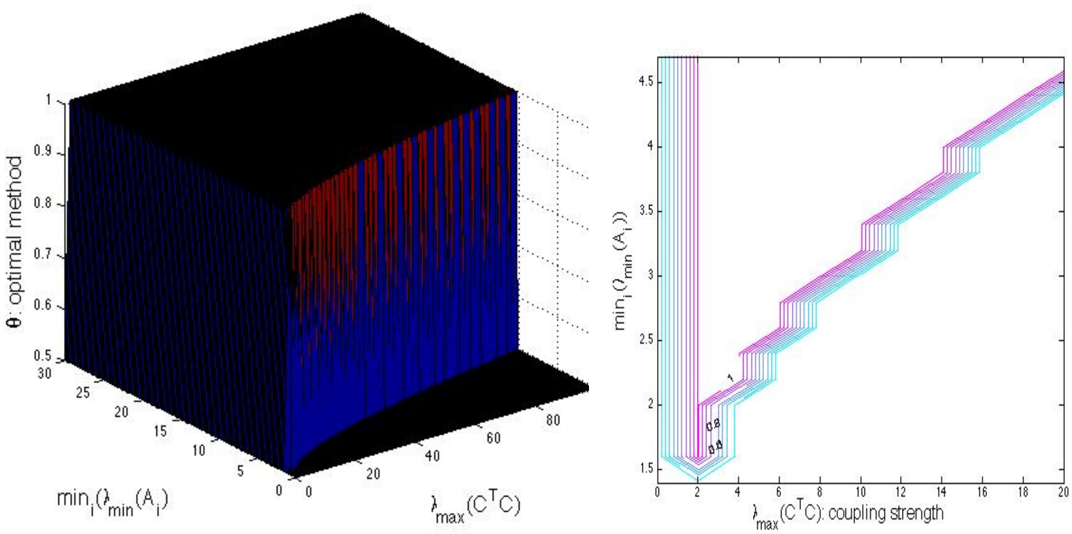
and (conditional) stability under the following time-step restrictions

$$
\begin{aligned}
0 \leq \nu + \Big[\theta(2\theta-1)\min_{i=1,2}\{\lambda_{\min}(A_i)\} - \theta(1-\theta)\sqrt{\lambda_{\max}(C^TC)}\Big]\Delta t\\
- \theta^2(2\theta-1)\lambda_{\max}(C^TC)\Delta t^2,
\end{aligned}
$$

and

$$
(2\theta-1)\Delta t \leq \frac{(2\theta-1)^2}{\theta^2}\frac{\min_{i=1,2}\{\lambda_{\min}(A_i)\}}{\lambda_{\max}(C^TC)},
$$

which yield (4) and (5).



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