A COMPACT EMBEDDING OF A SOBOLEV SPACE IS EQUIVALENT TO AN EMBEDDING INTO A BETTER SPACE

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Abstract. We prove that in the presence of the $\Delta_2$ condition the compact embedding of the Orlicz-Sobolev space is equivalent to the existence of a bounded embedding into a higher Orlicz space. We formulate results in an abstract setting of spaces of measurable functions with the property that every bounded sequence has a subsequence convergent a.e. We also provide an example showing that the theorem is not true without the $\Delta_2$ condition.

1. Introduction

If $\Omega \subset \mathbb{R}^n$ is a bounded domain and the Sobolev space $W^{1,p}(\Omega)$ is embedded into $L^q(\Omega)$, then for any $1 \leq s < q$, the embedding $W^{1,p}(\Omega) \subseteq L^s(\Omega)$ is compact, see e.g. [4, Theorem 4]. This result generalizes to the setting of Orlicz-Sobolev spaces. Let $A, \Phi, \Psi$ be Young functions. If the embedding $W^{1,A}(\Omega) \subset L^\Psi(\Omega)$ is bounded and $\Psi$ increases essentially faster than $\Phi$, $\Psi \gg \Phi$, then the embedding $W^{1,A}(\Omega) \subseteq L^\Phi(\Omega)$ is compact, see [1, Theorem 8.24]. The last statement contains the previous one since the function $t^q$ grows essentially faster than $t^s$ for $q > s$. The proof given in [4, Theorem 4] is based on the following consequence of the Rellich-Kondrachov theorem: every bounded sequence in $W^{1,p}(\Omega)$ (or $W^{1,1,\Lambda}(\Omega)$) has a subsequence that is convergent a.e. and thus it is not surprising that the results can be generalized to the setting of abstract normed spaces $W$ of measurable functions with the property that every bounded sequence has a subsequence convergent a.e., see Theorem 3.1 for a precise statement. This is nothing really new. What is new is that under the $\Delta_2$ condition the converse implication is also true: If a Young function $\Phi$ satisfies the $\Delta_2$ condition near infinity and the embedding $W \subseteq L^\Phi$ is compact, then there is a Young function $\Psi$ that grows essentially faster than $\Phi$ such that the embedding $W \subset L^\Psi$ is bounded. Hence in the presence of the $\Delta_2$ condition compact embedding is equivalent with an embedding into a better space, see Theorem 3.2. This

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result has several natural consequences. In particular it shows that the optimal embedding is never compact, of course under the $\Delta_2$ condition, see Corollary 3.3. In the last section we show an example (Theorem 4.1) that Theorem 3.2 is not true without the $\Delta_2$ condition.

2. Notation and basic definitions

In this section we recall basic definitions and facts from the theory of Orlicz spaces. For more details, see [1], [6].

We say that $\Phi : [0, \infty) \to [0, \infty)$ is a Young function if it is convex, continuous, strictly increasing, $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. If $\Phi$ and $\Psi$ are two Young functions, we say that $\Psi$ grows essentially faster near infinity than $\Phi$ if for every $k > 0$, $\Psi(t)/\Phi(kt) \to \infty$ as $t \to \infty$. We denote it by $\Psi \gg \Phi$. Finally a Young function $\Phi$ is said to satisfy the $\Delta_2$ condition near infinity if there are constants $K, t_0 > 0$ such that $\Phi(2t) \leq K \Phi(t)$ for all $t > t_0$.

Observe that if $\Phi$ satisfies the $\Delta_2$ condition near infinity, then $\Psi \gg \Phi$ if and only if $\Psi(t)/\Phi(t) \to \infty$ as $t \to \infty$.

Let $\Phi$ be a Young function and $(X, \mu)$ be a measure space. For simplicity we will always assume that $\mu(X) < \infty$. The Orlicz space $L^\Phi(X)$ consists of all measurable functions $u$ on $X$ such that

$$\int_X \Phi(\lambda |u(x)|) \, d\mu < \infty \text{ for some } \lambda > 0.$$  

It follows from the convexity of $\Phi$ that $L^\Phi(X)$ is a linear space and one can prove that this space equipped with the Luxemburg norm

$$||u||_\Phi = \inf \left\{ k > 0 : \int_X \Phi \left( \frac{|u(x)|}{k} \right) \, d\mu \leq 1 \right\}$$

is a Banach space. Note that

$$\int_X \Phi \left( \frac{|u(x)|}{||u||_\Phi} \right) \, d\mu \leq 1.$$

If $\Phi$ satisfies the $\Delta_2$ condition near infinity, then

$$L^\Phi(X) = \left\{ u : \int_X \Phi(|u(x)|) \, d\mu < \infty \right\}$$

but this claim is not true without the $\Delta_2$ condition.

Convexity of $\Phi$ implies that for $0 < \varepsilon \leq 1$, $\Phi(x) \leq \varepsilon \Phi(x/\varepsilon)$ and hence it is easy to see that convergence $u_n \to u$ in $L^\Phi$ implies

$$(2.1) \quad \int_X \Phi(|u_n - u|) \, d\mu \to 0.$$
Convergence (2.1) is called *convergence in mean* and we note here that convergence in mean implies convergence in the Luxemburg norm only if \( \Phi \) satisfies the \( \Delta_2 \) condition near infinity.

Given an open set \( \Omega \subset \mathbb{R}^n \) and a Young function \( A \) we can define in a natural way the Orlicz-Sobolev space \( W^{1,A}(\Omega) \). If \( A(t) = t^p \), then \( W^{1,A}(\Omega) = W^{1,p}(\Omega) \). Convexity of \( A \) implies that \( A(t) \geq at \) for \( t \geq t_0 \) and hence \( W^{1,A}(\Omega) \subset W^{1,1}_{\text{loc}}(\Omega) \). Thus it follows from the Rellich-Kondrachov theorem and the standard diagonal argument that every bounded sequence in \( W^{1,A}(\Omega) \) has a subsequence that is convergent a.e.

We say that a family of functions \( F \subset L^1(X) \), is *equi-integrable* if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that

\[
\sup_{f \in F} \int_E |f| \, d\mu < \varepsilon \quad \text{whenever} \quad \mu(E) < \delta.
\]

Note that equi-integrability does not imply in general that the family \( F \) is bounded in \( L^1(X) \) even if \( \mu(X) < \infty \) (which is our standing assumption), because the measure may have atoms.

We will need the following result of de la Vallée Poussin which we state as a lemma. For a proof, see [3], [6].

**Lemma 2.1** (de la Vallée Poussin). Let \( (X, \mu) \) be a measure space with \( \mu(X) < \infty \) and let \( F \subset L^1(\mu) \) be bounded. Then \( F \) is equi-integrable if and only if there is a Young function \( \Phi \), \( \lim_{t \to \infty} \Phi(t)/t = \infty \) such that

\[
(2.2) \quad \sup_{f \in F} \int_X \Phi(|f|) \, d\mu \leq 1.
\]

In most of the statements found in the literature the condition is that the integral (2.2) is finite. Dividing \( \Phi \) by an appropriate constant we may further require that the integral is less than or equal to 1, as we do in (2.2).

### 3. Main Theorems

The following result is a common generalization of [1, Theorem 8.24], [4, Theorem 4].

**Theorem 3.1.** Let \( W(X) \) be a normed space of measurable functions on \( (X, \mu) \), \( \mu(X) < \infty \), with the property that every bounded sequence in \( W(X) \) has a subsequence that is convergent a.e. If \( \Psi \) is a Young function such that the embedding \( W(X) \subset L^\Psi(X) \) is bounded, then for every Young function \( \Phi \) such that \( \Psi \gg \gg \Phi \), the embedding \( W(X) \subset L^\Psi(X) \) is compact.

**Proof.** Since the embedding \( W \subset L^\Psi \) is bounded, there is a constant \( C > 0 \) such that \( \|f\|_\Psi \leq C\|f\|_W \) for all \( f \in W \). Let \( \{f_i\} \subset W \) be a bounded
sequence, \( \|f_i\|_W \leq M \). It suffices to prove that a subsequence of \( f_i \) is a Cauchy sequence in \( L^\Phi \). Let \( u_i = f_i / \varepsilon \). Then

\[
\|u_i - u_j\|_\Psi \leq C \|u_i - u_j\|_W \leq 2CM \varepsilon^{-1}
\]

and hence

\[
\int_X \Psi \left( \frac{|u_i - u_j|}{2CM \varepsilon^{-1}} \right) \, d\mu \leq 1 \quad \text{for all } i, j.
\]

Since \( \Psi \) grows essentially faster than \( \Phi \), there is \( t_0 > 0 \) such that

\[
\Phi(t) \leq \frac{1}{4} \Psi \left( \frac{t}{2CM \varepsilon^{-1}} \right) \quad \text{for } t > t_0.
\]

On the set \( \{|u_i - u_j| < t_0\} \) we have \( \Phi(|u_i - u_j|) < \Phi(t_0) \). Let \( \delta = (4\Phi(t_0))^{-1} \). If \( E \subset X \) is such that \( \mu(X \setminus E) < \delta \), then

\[
\int_{X \setminus E} \Phi(|u_i - u_j|) \, d\mu \leq \int_{\{|u_i - u_j| > t_0\}} \Phi(|u_i - u_j|) \, d\mu + \int_X \Phi(t_0) \, d\mu
\]

\[
\leq \frac{1}{4} \int_X \Psi \left( \frac{|u_i - u_j|}{2CM \varepsilon^{-1}} \right) \, d\mu + \frac{\Phi(t_0)}{4\Phi(t_0)}
\]

\[
\leq \frac{1}{2} \quad \text{for all } i, j.
\]

By our assumptions \( u_i \) has a subsequence \( u_{ij} \) that is convergent a.e. According to the Egorov theorem there is a measurable set \( E \subset X \) such that \( \mu(X \setminus E) < \delta \) and \( u_{ij} \) converges uniformly on \( E \). Hence there is \( N \) such that

\[
|u_{ij}(x) - u_{ik}(x)| \leq \Phi^{-1} \left( \frac{1}{2\mu(X)} \right) \quad \text{for all } x \in E \text{ and } j, k \geq N.
\]

Then for \( j, k \geq N \) we have

\[
\int_X \Phi \left( \frac{|f_{ij} - f_{ik}|}{\varepsilon} \right) \, d\mu = \int_E \Phi(|u_{ij} - u_{ik}|) \, d\mu + \int_{X \setminus E} \Phi(|u_{ij} - u_{ik}|) \, d\mu
\]

\[
\leq \frac{\mu(X)}{2\mu(X)} + \frac{1}{2} = 1
\]

and hence

\[
\|f_{ij} - f_{ik}\|_\Phi \leq \varepsilon \quad \text{for all } j, k \geq N.
\]

The proof is complete.

The following theorem is the main result of the paper.

**Theorem 3.2.** Let \( W(X) \) be a normed space of measurable functions on \( (X, \mu) \), \( \mu(X) < \infty \), with the property that every bounded sequence in \( W(X) \) has a subsequence that is convergent a.e. Let \( \Phi \) be a Young function that satisfies the \( \Delta_2 \) condition near infinity. Then the following conditions are equivalent.

(a) \( W(X) \) is compactly embedded into \( L^\Phi(X) \), \( W(X) \subset L^\Phi(X) \).
(b) There is a Young function $\Psi \succ \Phi$ such that $W(X)$ is continuously embedded into $L^\Psi(X)$, $W(X) \subset L^\Psi(X)$.

Proof. The implication from (b) to (a) is contained in Theorem 3.1, so we are left with the proof of the implication from (a) to (b). Suppose that $W \subset L^\Phi$ and $\Phi$ satisfies the $\Delta_2$ condition near infinity. Let $C > 0$ be such that $\|f\|_\Phi \leq C\|f\|_W$ for $f \in W$. Consider the unit sphere in $W$

$$S = \{f \in W : \|f\|_W = 1\}.$$ 

We claim that the family

$$\mathcal{F} = \{\Phi(\frac{|f|}{C}) : f \in S\}$$

is bounded and equi-integrable in $L^1(X)$. Boundedness follows from the definition of the Luxemburg norm. Indeed, $\|f\|_\Phi \leq C$ for $f \in S$ and hence

$$\int_X \Phi(|f|/C) \, d\mu \leq 1.$$ 

Thus $\mathcal{F}$ is contained in the unit ball in $L^1(X)$. By contrary suppose that $\mathcal{F}$ is not equi-integrable. Then there is $\varepsilon > 0$ and two sequences $E_n \subset X$, $f_n \in S$ such that $\mu(E_n) < 1/n$, while

$$\int_{E_n} \Phi\left(\frac{|f_n|}{C}\right) \, d\mu \geq \varepsilon.$$ \hspace{1cm} (3.1)

The sequence $2f_n/C$ is bounded in $W$ and since the embedding $W \subset L^\Phi$ is compact, the sequence has a subsequence (still denoted by $2f_n/C$) convergent in $L^\Phi$ to some function $g \in L^\Phi$. The convergence in mean (2.1) gives

$$\int_X \Phi\left(\left|\frac{2f_n}{C} - g\right|\right) \, d\mu < \varepsilon \quad \text{for} \quad n \geq n_1.$$ 

Since $g \in L^\Phi$ and $\Phi$ satisfies the $\Delta_2$ condition near infinity, $\int_X \Phi(|g|) \, d\mu < \infty$ and hence there is $n_2$ such that

$$\int_{E_n} \Phi(|g|) \, d\mu < \varepsilon \quad \text{for} \quad n \geq n_2$$

by absolute continuity of the integral. For $n > \max\{n_1, n_2\}$ convexity of $\Phi$ gives

$$\int_{E_n} \Phi\left(\frac{|f_n|}{C}\right) \, d\mu \leq \int_{E_n} \Phi\left(\frac{1}{2} \left|\frac{2f_n}{C} - g\right| + \frac{1}{2} |g|\right) \, d\mu \leq \frac{1}{2} \int_{E_n} \Phi\left(\frac{2f_n}{C} - g\right) \, d\mu + \frac{1}{2} \int_{E_n} \Phi(|g|) \, d\mu < \varepsilon$$

which contradicts (3.1). We proved that the family $\mathcal{F}$ satisfies assumptions of the de la Vallée Poussin theorem and hence there is a Young function $\eta$
such that $\eta(t)/t \to \infty$ as $t \to \infty$ and

$$\sup_{f \in S} \int_X \eta \left( \Phi \left( \frac{|f|}{C} \right) \right) \, d\mu \leq 1.$$ 

Hence for all $0 \neq f \in W$ and $\Psi = \eta \circ \Phi$

$$\int_X \Psi \left( \frac{|f|}{C\|f\|_W} \right) \, d\mu \leq 1$$

which proves boundedness of the embedding $W \subset L^\Psi$ with the same constant $\|f\|_\Psi \leq C\|f\|_W$. It remains to observe that $\Psi \gg \Phi$. Indeed, for any $k > 0$

$$\lim_{t \to \infty} \frac{\Psi(t)}{\Phi(kt)} = \lim_{t \to \infty} \frac{\eta(\Phi(t))}{\Phi(t)} = \infty$$

since $\Phi(t)/\Phi(kt)$ is bounded away from 0 by the $\Delta_2$ condition. \hfill $\square$

**Corollary 3.3.** If $\Phi$ satisfies the $\Delta_2$ condition near infinity and a bounded embedding $W(X) \subset L^\Phi(X)$ is optimal in the category of Orlicz spaces, then it is not compact.

All of the above theorems apply to Sobolev and Orlicz-Sobolev spaces since in the statements we can take $W = W^{1,A}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set of finite measure.

Nečas [5, Théorème 1.4] proved that if $\Omega \subset \mathbb{R}^n$ is a bounded domain with continuous boundary (i.e. the boundary is locally a graph of a continuous function), then the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compact. As an immediate consequence of this result and Theorem 3.2 we obtain

**Corollary 3.4.** If $\Omega \subset \mathbb{R}^n$ is a bounded domain with continuous boundary, then there is a Young function $\Phi$ that grows essentially faster at infinity than $t^2$, such that the embedding $W^{1,2}(\Omega) \subset L^\Phi(\Omega)$ is bounded.

A more precise description of the function $\Phi$ can be obtained from the information about the modulus of continuity of the functions used to represent the boundary as a graph, but it is interesting to observe that our argument implies the existence of $\Phi$ without any careful investigation of the structure of the boundary.

4. Example

The following example shows that we cannot avoid the $\Delta_2$ condition in Theorem 3.2. In particular the example shows that if we do not assume the $\Delta_2$ condition, the optimal embedding can be compact, differently than in the case of Corollary 3.3. We do not know if there is a similar example in the setting of Sobolev spaces.
Theorem 4.1. There is a Banach space $W$ of measurable functions on $[0, 1]$ with the following properties:

(a) Every bounded sequence in $W$ has a subsequence convergent a.e.
(b) $W \subset L^\Phi([0, 1])$ for $\Phi(t) = \frac{2}{\pi} (e^t - 1)$.
(c) There is no Young function $\Psi \succ \succ \Phi$ such that $W \subset L^\Psi([0, 1])$.

Remark 4.2. A we do not require in (c) that the embedding $W \subset L^\Psi([0, 1])$ has to be bounded. We only assume that every function in $W$ belongs to $L^\Psi([0, 1])$.

Proof. First we will define auxiliary functions that will be used to construct the space $W$. Let

$$f(x) = -\log(x + x \log^2 x), \quad x \in (0, 1].$$

Note that $f$ is strictly decreasing from $\infty$ to 0. We have

$$\int_0^1 \Phi(|f(x)|) \, dx = \frac{2}{\pi} \int_0^1 \left(e^{f(x)} - 1\right) \, dx = \frac{2}{\pi} \left(\frac{\pi}{2} - 1\right) < 1$$

since the antiderivative of $e^{f(x)} = (x + x \log^2 x)^{-1}$ is $\arctan \log x$. It is easy to see that for any $0 < k < 1$

$$\int_0^1 \Phi\left(\frac{|f(x)|}{k}\right) \, dx = \infty$$

and hence $\|f\|_\Phi = 1$. For $n \geq 2$ we define

$$g_n = c_n \chi_{[0, \frac{1}{n}]}, \quad \text{where} \quad c_n = -\log\left(\frac{\pi}{2} + \arctan \log \frac{1}{n}\right).$$

Observe that $c_n > 0$ for $n \geq 2$. Finally let $f_n = f + g_n$. We have

$$\int_0^1 \Phi(|f_n(x)|) \, dx = \frac{2}{\pi} \left(\int_0^{1/n} \left(e^{f(x)} e^{c_n} - 1\right) \, dx + \int_{1/n}^1 \left(e^{f(x)} - 1\right) \, dx\right) \leq \frac{2}{\pi} \left(1 - \frac{1}{n} + \frac{\pi}{2} - 1\right) < 1$$

and for $0 < k < 1$

$$\int_0^1 \Phi\left(\frac{|f_n(x)|}{k}\right) \, dx > \int_0^1 \Phi\left(\frac{|f(x)|}{k}\right) \, dx = \infty,$$

so $\|f_n\|_\Phi = 1$. Note also that $f_n \to f$ in $L^\Phi$ as $n \to \infty$. Indeed, for every $\varepsilon > 0$

$$\lim_{n \to \infty} \int_0^1 \Phi\left(\frac{|f_n - f|}{\varepsilon}\right) \, dx = \lim_{n \to \infty} \frac{2}{\pi} \frac{1}{n} \left(e^{c_n/\varepsilon} - 1\right) = 0$$

by a simple application of the l’Hospital rule.
Now we define a Banach space $W$ of measurable functions on $[0,1]$ as

$$W = \left\{ h = \sum_{i=1}^{\infty} a_i f_i : (a_i)_{i=1}^{\infty} \in \ell^1 \right\}$$

with the norm

$$\|h\|_W = \left\| \sum_{i=2}^{\infty} a_i f_i \right\|_W := \sum_{i=2}^{\infty} |a_i|.$$

Since for every $x \in (0,1]$, $f_i(x) = f(x)$ for all sufficiently large $i$, the series $\sum_{i=2}^{\infty} a_i f_i(x)$ converges at every $x \in (0,1]$ and hence it defines a measurable function. Considering intervals $\left(\frac{1}{n+1}, \frac{1}{n}\right]$, $n = 1, 2, 3, \ldots$ one can easily check by induction that if $\sum_{i=2}^{\infty} a_i f_i = 0$ a.e., then $a_i = 0$ for all $i$, so the coefficients $a_i$ are uniquely determined and hence $\| \cdot \|_W$ is a well defined norm. Now it is obvious that $W$ is isometric to $\ell^1$ and hence $W$ is a Banach space.

The partial sums of the series series $\sum_{i=2}^{\infty} a_i f_i$ form a Cauchy sequence in $L^\Phi$ because

$$\left\| \sum_{i=k}^{\ell} a_i f_i \right\|_\Phi \leq \sum_{i=k}^{\ell} |a_i| \| f_i \|_\Phi = \sum_{i=k}^{\ell} |a_i|$$

and hence the series converges in the Banach space $L^\Phi$. This also shows that $W$ is continuously embedded into $L^\Phi$.

$$\|h\|_\Phi = \left\| \sum_{i=2}^{\infty} a_i f_i \right\|_\Phi \leq \sum_{i=2}^{\infty} |a_i| \| f_i \|_\Phi = \sum_{i=2}^{\infty} |a_i| = \|h\|_W,$$

but what is more interesting, the embedding is compact, $W \subseteq L^\Phi([0,1])$. Before we prove this fact observe that compactness of the embedding implies that every bounded sequence in $W$ has a subsequence that is convergent a.e. which is the property (a).

Recall that $f_n \to f$ in $L^\Phi$ as $n \to \infty$ and hence the set

$$F = \{ f_i \}_{i=1}^{\infty} \subset L^\Phi, \text{ where } f_1 = f$$

is compact. Then also family of functions

$$K = \{ x \mapsto tf_i(x) : t \in [-M, M], \ i \geq 1 \} \subset L^\Phi$$

is compact. Indeed, $K$ is the image of a continuous mapping defined on a compact set

$$\lambda : [-M, M] \times F \to L^\Phi, \ \lambda(t, f_i) = tf_i, \ \lambda([-M, M] \times F) = K.$$ 

According to Mazur’s theorem [2, Theorem 4.8], the convex hull $\text{co}(K)$ is relatively compact in $L^\Phi$. With this introduction we can complete the proof of (b) as follows.
Let \( h_n \in W \) be a bounded sequence and let \( \tilde{h}_n = \sum_{i=2}^{k(n)} a_i^n f_i \) be such that \( \|h_n - \tilde{h}_n\|_W < 1/n \). The sequence \( \tilde{h}_n \) is bounded, say
\[
\|\tilde{h}_n\|_W = \sum_{i=2}^{k(n)} |a_i^n| \leq M.
\]
Then
\[
\tilde{h}_n = \sum_{i=2}^{k(n)} \frac{|a_i^n|}{\|h_n\|_W} \left( \text{sgn} \left( a_i^n \right) \|\tilde{h}_n\|_W f_i \right) \in \text{co}(K)
\]
and hence \( \tilde{h}_n \) has a subsequence convergent in \( L^\Phi \). This also implies that \( h_n \) has a subsequence convergent in \( L^\Phi \) to the same limit.

We are left with the proof of (c). Suppose that there is \( \Psi \gg \Phi \) such that \( W \subset L^\Psi \). It follows from the closed graph theorem that the embedding is bounded. Indeed, if \( h_n \rightarrow h \) in \( W \) and \( h_n \rightarrow g \) in \( L^\Psi \), then from boundedness of the embedding into \( L^\Phi \), \( h_n \rightarrow h \) in \( L^\Phi \) and hence \( g = h \).

Since \( \|f_n\|_W = 1 \), the sequence \( f_n \) is bounded in \( L^\Psi \), say \( \|f\|_\Psi \leq C \), so
\begin{equation}
\int_0^1 \Psi \left( \frac{|f_n(x)|}{C} \right) \, dx \leq 1.
\end{equation}

Note that
\[
\inf_{x \in [0,1/n]} f_n(x) \geq f(1/n) \rightarrow \infty \quad \text{as } n \rightarrow \infty
\]
and therefore the condition \( \Psi \gg \Phi \) implies
\[
A_n = \inf_{x \in [0,1/n]} \frac{\Psi \left( \frac{|f_n(x)|}{C} \right)}{\Phi(|f_n(x)|)} \rightarrow \infty \quad \text{as } n \rightarrow \infty.
\]
Thus
\[
\int_0^1 \Psi \left( \frac{|f_n(x)|}{C} \right) \, dx \geq A_n \int_0^{1/n} \Phi(|f_n(x)|) \, dx
\]
\[
= \frac{2}{\pi} A_n \int_0^{1/n} \left( e^{f_n(x)} - 1 \right) \, dx
\]
\[
= \frac{2}{\pi} A_n \left( 1 - \frac{1}{n} \right) \rightarrow \infty
\]
which contradicts (1.1). The proof is complete.

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