ON THE ESTIMATES OF DETERMINING MODES FOR NS-\(\alpha\) AND NS-\(\omega\) MODELS

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Abstract.

Key words. determining modes, Navier-Stokes equations

1. Introduction. Solutions of NSE can be represented adequately in a finite dimensional space whose basis is determining modes. Number of determining modes is an indicator of complexity of solutions to the NSE [2]. The problem of finding an optimal determining modes estimation to understand the complexity of the NSE continues to grow.

This report will give an estimate of the determining modes of NS-\(\alpha\) and NS-\(\omega\) models. We show that these models have fewer determining modes than in the case of NSE models [3], and when the radius filter goes to 0, these three models turn out to have the same number of determining modes.

We investigate the 3D equilibrium NS-\(\alpha\) regularization
\[
\begin{aligned}
-\nu \Delta u - \bar{u} \times (\nabla \times u) + \nabla p &= f \text{ in } \Omega := (0, 2\pi)^3 \\
\nabla \cdot \bar{u} &= 0 \text{ in } \Omega
\end{aligned}
\]
and NS-\(\omega\) regularization
\[
\begin{aligned}
-\nu \Delta u - \bar{u} \times (\nabla \times \bar{u}) + \nabla p &= f \text{ in } \Omega \\
\nabla \cdot u &= 0 \text{ in } \Omega
\end{aligned}
\]
with differential filter
\[
\begin{aligned}
-\delta^2 \Delta \bar{u} + \bar{u} &= u \\
\nabla \cdot \bar{u} &= 0
\end{aligned}
\]
and 2\(\pi\)-periodic boundary condition:
\[
\int_{\Omega} u(x)dx = 0
\]

Using Fourier transform, solution to both of the above models can be written
\[
u(x) = \sum_{\mathbf{k}} \hat{u}(\mathbf{k}) e^{-i\mathbf{k}x}
\]
where \(\mathbf{k} \in \mathbb{Z}^3\) and Fourier coefficient
\[
\hat{u}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int_{\Omega} u(x) e^{-i\mathbf{k}x} dx
\]

Define \(V_s = \{ u(x) = \sum_{\mathbf{k}} \hat{u}(\mathbf{k}) e^{-i\mathbf{k}x} \text{ such that } \sum_{\mathbf{k}} |\mathbf{k}|^{2s} |\hat{u}(\mathbf{k})|^2 < \infty \}. \) In \(V_s\) define the norm
\[
\| u \|_s = \| u \|_{V_s} = \left( \sum_{\mathbf{k}} |\mathbf{k}|^{2s} |u(\mathbf{k})|^2 \right)^{1/2}
\]

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We will use the following inequality in [3]

\[(u \cdot \nabla v, w) \leq c\|u\|_{s_1}\|v\|_{s_2+1}\|w\|_{s_3}\]  

(1.1)

where \(s_1 + s_2 + s_3 = \frac{3}{2}, s_1, s_2, s_3 \geq 0\)

A similar inequality can also be verified

\[(u \times (\nabla \times v), w) \leq c\|u\|_{s_1}\|v\|_{s_2+1}\|w\|_{s_3}\]  

(1.2)

The following identities will also be employed

\[(u \times (\nabla \times v), w) = -(w \times (\nabla \times v), u)\]  

(1.3)

\[(u \times (\nabla \times v), u) = 0\]  

(1.4)

\[a \times (\nabla \times b) = (\nabla \times a) \times b - a \cdot \nabla b - b \cdot \nabla a + \nabla (a \cdot b)\]  

(1.5)

2. Estimate of determining modes for NS-\(\alpha\) models. We will state and prove the following theorem

**Theorem 2.1.** Let \(k_\alpha(\delta) = \frac{1}{1+\frac{c^2\|f\|^2}{\nu^2}}\) and \(X_\alpha\) denote the finite dimensional space

\[X_\alpha := \text{span}\{e^{ikx} : |k| \leq k_\alpha(\delta)\} \cap V\]

If \(u_1\) and \(u_2\) are 2 solutions of the NS-\(\alpha\) regularization with \(P_{X_\alpha}(u_1 - u_2) = 0\) then \(u_1 \equiv u_2\).

Here \(V = \{v \in H^1(\Omega) | \int_\Omega v dx = 0, \nabla \cdot v = 0\}\), \(\|\cdot\|_*\) is the norm in \(V^*\), the dual space of \(V\), \(P_{X_\alpha}\) is the projection from \(V\) to \(X_\alpha\).

**Proof.** Let \(u_1\) and \(u_2\) be 2 solutions to the equilibrium problem and \(\phi = u_1 - u_2\)

\[\begin{cases} 
-\nu \Delta u_1 - \bar{u}_1 \times (\nabla \times u_1) + \nabla p_1 = f \\
-\nu \Delta u_2 - \bar{u}_2 \times (\nabla \times u_2) + \nabla p_2 = f 
\end{cases}\]

It turns out

\[-\nu \Delta (u_1 - u_2) + \nabla (\bar{u}_1 \times (\nabla \times u_1) - \bar{u}_2 \times (\nabla \times u_2)) + \nabla (p_1 - p_2) = 0\]

\[\Rightarrow -\nu \Delta \phi - (\phi \times (\nabla \times u_1) + \bar{u}_2 \times (\nabla \times \phi)) + \nabla (p_1 - p_2) = 0\]

\[\Rightarrow -\nu \Delta \phi = \bar{\phi} \times (\nabla \times u_1) + \bar{u}_2 \times (\nabla \times \phi) - \nabla (p_1 - p_2)\]

Multiplying both sides by \(\bar{\phi}\) and \(\int_\Omega\) yields

**LHS:**

\[-\nu (\Delta \phi, \bar{\phi}) = \nu (\nabla \phi, \nabla \bar{\phi}) = \nu (\nabla \bar{\phi}, \nabla \phi) - \nu \delta^2 (\nabla (\Delta \bar{\phi}), \nabla \phi)\]

\[=\nu \|\nabla \bar{\phi}\|^2 + \nu \delta^2 \|\Delta \bar{\phi}\|^2\]  

(2.1)

**RHS:**

\[(\bar{\phi} \times (\nabla \times u_1), \bar{\phi}) + (\bar{u}_2 \times (\nabla \times \phi), \bar{\phi}) - (\nabla (p_1 - p_2), \bar{\phi}) = (\bar{u}_2 \times (\nabla \times \phi), \bar{\phi})\]

Applying identity (1.5) we have

\[(\bar{u}_2 \times (\nabla \times \phi), \bar{\phi})\]

\[= - (\phi \times (\nabla \times \bar{u}_2), \bar{\phi}) - (\bar{u}_2 \cdot \nabla \phi, \bar{\phi}) - (\phi \cdot \nabla \bar{u}_2, \bar{\phi}) + (\nabla (\bar{u}_2 \cdot \phi), \bar{\phi})\]

\[= - (\bar{\phi} \times (\nabla \times \bar{u}_2), \bar{\phi}) + \delta^2 (\Delta \bar{\phi} \times (\nabla \times \bar{u}_2), \bar{\phi}) + (\bar{u}_2 \cdot \nabla \phi, \bar{\phi}) - \delta^2 (\bar{u}_2 \cdot \nabla \phi, \Delta \bar{\phi}) - (\bar{\phi} \cdot \nabla \bar{u}_2, \bar{\phi}) + \delta^2 (\Delta \bar{\phi} \cdot \nabla \bar{u}_2, \bar{\phi})\]

\[= \delta^2 (\Delta \bar{\phi} \times (\nabla \times \bar{u}_2), \bar{\phi}) - \delta^2 (\bar{u}_2 \cdot \nabla \phi, \Delta \bar{\phi}) - (\bar{\phi} \cdot \nabla \bar{u}_2, \bar{\phi}) + \delta^2 (\Delta \bar{\phi} \cdot \nabla \bar{u}_2, \bar{\phi})\]
By (1.1) and (1.2) we get
\[ \delta^2 (\Delta \tilde{\phi} \times (\nabla \times \tilde{u}_2) + \tilde{\phi}) \leq c \delta^2 \| \Delta \tilde{\phi} \| \| \nabla \tilde{u}_2 \| \| \tilde{\phi} \|_{3/2} \]
\[ - \delta^2 (\tilde{u}_2 \cdot \nabla \tilde{\phi}, \Delta \tilde{\phi}) \leq c \delta^2 \| \nabla \tilde{u}_2 \| \| \tilde{\phi} \|_{3/2} \| \Delta \tilde{\phi} \| \]
\[ \delta^2 (\Delta \tilde{\phi} \cdot \nabla \tilde{u}_2, \tilde{\phi}) \leq c \delta^2 \| \Delta \tilde{\phi} \| \| \nabla \tilde{u}_2 \| \| \tilde{\phi} \|_{3/2} \]
\[ - (\tilde{\phi} \cdot \nabla \tilde{u}_2), \tilde{\phi} \leq c \| \tilde{\phi} \|_{1/2} \| \nabla \tilde{u}_2 \| \| \nabla \tilde{\phi} \| \]

Therefore
\[ (\tilde{u}_2 \times (\nabla \times \phi), \tilde{\phi}) \leq c \| \phi \|_{1/2} \| \nabla \tilde{u}_2 \| \| \nabla \tilde{\phi} \| + c \delta^2 \| \Delta \tilde{\phi} \| \| \nabla \tilde{u}_2 \| \| \tilde{\phi} \|_{3/2} \]
\[ \leq \frac{c^2}{2\nu} \| \tilde{\phi} \|^2 \| \nabla \tilde{u}_2 \|^2 + \frac{\nu}{2} \| \nabla \tilde{\phi} \|^2 + \frac{c^2 \delta^2}{2\nu} \| \tilde{\phi} \|^2 \| \nabla \tilde{u}_2 \|^2 + \frac{\nu \delta^2}{2} \| \Delta \tilde{\phi} \|^2 \]  \hspace{1cm} (2.2)

From (2.1) and (2.2)
\[ \frac{\nu}{2} \| \nabla \tilde{\phi} \|^2 + \frac{\nu \delta^2}{2} \| \Delta \tilde{\phi} \|^2 \leq \frac{c^2}{2\nu} \| \tilde{\phi} \|^2 \| \nabla \tilde{u}_2 \|^2 + \frac{c^2 \delta^2}{2\nu} \| \tilde{\phi} \|^2 \| \nabla \tilde{u}_2 \|^2 \]
\[ \| \nabla \tilde{\phi} \|^2 + \delta^2 \| \Delta \tilde{\phi} \|^2 \leq \frac{c^2}{\nu^2} \| \nabla \tilde{u}_2 \|^2 (\| \tilde{\phi} \|^2 \| \nabla \tilde{u}_2 \|^2 + \delta^2 \| \tilde{\phi} \|^2 \| \nabla \tilde{u}_2 \|^2) \]

Now by Fourier transform
\[ \text{LHS} = \sum_{k} \frac{|k|^2}{(1 + \delta^2 |k|^2)^2} |\hat{\phi}(k)|^2 + \sum_{k} \frac{\delta^2 |k|^4}{(1 + \delta^2 |k|^2)^2} |\hat{\phi}(k)|^2 \]
\[ = \sum_{k} \frac{|k|^2}{1 + \delta^2 |k|^2} |\hat{\phi}(k)|^2 \]  \hspace{1cm} (2.3)

RHS: Since
\[ - \nu \Delta \tilde{u}_2 - \tilde{u}_2 \times (\nabla \times \tilde{u}_2) + \nabla p_2 = f \]
\[ \Rightarrow - \nu (\Delta \tilde{u}_2, \tilde{u}_2) = (f, \tilde{u}_2) \]
\[ \Rightarrow \nu (\nabla \tilde{u}_2, \nabla \tilde{u}_2) = (f, \tilde{u}_2) \]
\[ \Rightarrow \nu (\nabla \tilde{u}_2, \nabla \tilde{u}_2) - \nu \delta^2 (\nabla (\Delta \tilde{u}_2), \nabla \tilde{u}_2) = (f, \tilde{u}_2) \]
\[ \Rightarrow \nu \| \nabla \tilde{u}_2 \|^2 + \nu \delta^2 \| \Delta \tilde{u}_2 \|^2 = (f, \tilde{u}_2) \leq \| f \| \| \nabla \tilde{u}_2 \| \leq \frac{\| f \|^2}{2\nu} + \frac{\nu}{2} \| \nabla \tilde{u}_2 \|^2 \]
\[ \Rightarrow \| \nabla \tilde{u}_2 \|^2 + \delta^2 \| \Delta \tilde{u}_2 \|^2 \leq \frac{\| f \|^2}{\nu^2} \]
\[ \Rightarrow \sum_{k} \left( \frac{|k|^2}{(1 + \delta^2 |k|^2)^2} + \frac{\delta^2 |k|^4}{(1 + \delta^2 |k|^2)^2} \right) \tilde{u}(k) = \sum_{k} \frac{|k|^2}{1 + \delta^2 |k|^2} \tilde{u}(k) \leq \frac{\| f \|^2}{\nu^2} \]
\[ \Rightarrow \| \nabla \tilde{u}_2 \|^2 = \sum_{k} \frac{|k|^2}{(1 + \delta^2 |k|^2)^2} \tilde{u}(k) \leq \frac{1}{1 + \delta^2} \sum_{k} \frac{|k|^2}{1 + \delta^2 |k|^2} \tilde{u}(k) \leq \frac{1}{1 + \delta^2} \frac{\| f \|^2}{\nu^2} \]  \hspace{1cm} (2.4)

Furthermore
\[ \| \tilde{\phi} \|_{1/2} + \delta^2 \| \tilde{\phi} \|_{3/2} = \sum_{k} \left( \frac{|k|}{(1 + \delta^2 |k|^2)^2} + \frac{\delta^2 |k|^3}{(1 + \delta^2 |k|^2)^2} \right) \tilde{\phi}(k) \]
\[ = \sum_{k} \frac{|k|}{1 + \delta^2 |k|^2} \tilde{\phi}(k) \]  \hspace{1cm} (2.5)
From (2.3)-(2.5)
\[
\sum_{k} \frac{|k|^2}{1 + \delta^2|k|^2} |\hat{\phi}(k)|^2 \leq \frac{1}{1 + \delta^2} \frac{c^2\|f\|_2^2}{\nu^4} \sum_{k} \frac{|k|^2}{1 + \delta^2|k|^2} |\hat{\phi}(k)|^2
\]
\[
= \sum_{k} \left[ \frac{|k|^2}{1 + \delta^2|k|^2} - \frac{1}{1 + \delta^2} \frac{c^2\|f\|_2^2}{\nu^4} \right] \frac{|k|^2}{1 + \delta^2|k|^2} |\hat{\phi}(k)|^2 \leq 0
\]

For $|k|$ large enough, the bracket term is positive. Then, the NS-$\alpha$ regularization will have a finite number of determining modes.

To estimate the number, let $k = |k| > 0$. Then, the number of determining modes is the greatest integer in the positive roots of
\[
\frac{k^2}{1 + \delta^2k^2} - \frac{1}{1 + \delta^2} \frac{c^2\|f\|_2^2}{\nu^4} \frac{k}{1 + \delta^2k^2} = 0
\]
\[
\Rightarrow k = \frac{1}{1 + \delta^2} \frac{c^2\|f\|_2^2}{\nu^4}
\]

**Remark 2.1.** $k_\alpha = \frac{1}{1 + \delta^2} \frac{c^2\|f\|_2^2}{\nu^4} \leq \frac{c^2\|f\|_2^2}{\nu^4} = k_{\text{NSE}}$.

When $\delta \to 0$, then $k_\alpha \to k_{\text{NSE}}$. Furthermore $k_\alpha \simeq (1 - \delta^2) \frac{c^2\|f\|_2^2}{\nu^4}$ if $\delta$ small

### 3. Estimate of determining modes for NS-$\omega$ model

Below is the same theorem for NS-$\omega$ models.

**Theorem 3.1.** Let $k_\omega(\delta)$ be the greatest integer in the positive root of
\[
k(1 + \delta^2k^2)^2 = \frac{c^2\|f\|_2^2}{\nu^4}
\]

Let $X_\omega$ denote the finite dimensional space
\[
X_\omega := \text{span} \{ e^{ikx} : |k| \leq k_\omega(\delta) \} \cap V
\]

If $u_1$ and $u_2$ are 2 solutions of the NS-$\omega$ regularization with $P_{X_\omega}(u_1 - u_2) = 0$ then $u_1 \equiv u_2$.

**Proof.** Let $u_1$ and $u_2$ be 2 solutions to the equilibrium problem and $\phi = u_1 - u_2$
\[
\begin{cases}
-\nu \Delta u_1 - u_1 \times (\nabla \times \bar{u}_1) + \nabla p_1 = f \\
-\nu \Delta u_2 - u_2 \times (\nabla \times \bar{u}_2) + \nabla p_2 = f
\end{cases}
\]

It turns out
\[
-\nu \Delta (u_1 - u_2) - (\phi \times (\nabla \times \bar{u}_1) + u_2 \times (\nabla \times \bar{\phi})) + \nabla (p_1 - p_2) = 0
\]
\[
\Rightarrow -\nu (\Delta \phi, \phi) = (\phi \times (\nabla \times \bar{u}_1), \phi) + (u_2 \times (\nabla \times \bar{\phi}), \phi) - (\nabla (p_1 - p_2), \phi)
\]
\[
\Rightarrow \nu \|\nabla \phi\|^2 = (u_2 \times (\nabla \times \bar{\phi}), \phi)
\]

(3.1)

Again applying identity (1.5) we have
\[
(u_2 \times (\nabla \times \bar{\phi}), \phi)
\]
\[
= - (\bar{\phi} \times (\nabla \times u_2), \phi) - (u_2 \cdot \nabla \bar{\phi}, \phi) - (\bar{\phi} \cdot \nabla u_2, \phi) + (\nabla (u_2 \cdot \bar{\phi}), \phi)
\]
\[
\leq c\|\bar{\phi}\|_{1/2} \|\nabla u_2\| \|\nabla \phi\|
\]

(3.2)

From (3.1) and (3.2), we have $\|\nabla \phi\|^2 \leq \frac{c^2}{\nu^2} \|\bar{\phi}\|_{1/2}^2 \|\nabla u_2\|^2$
But $- \nu \Delta u_2 - u_2 \times (\nabla \times \hat{u}_2) + \nabla p_2 = f$. Hence
\[
\nu \| \nabla u_2 \|^2 = (f, u_2) \leq \| f \| \| \nabla u_2 \| \text{ or } \| \nabla u_2 \|^2 \leq \frac{\| f \|^2}{\nu^2}
\]
And we get $\| \nabla \phi \|^2 \leq \frac{c^2\| f \|^2}{\nu^4} \| \bar{\phi} \|^2 \| \phi \|^2$

Using Fourier transform
\[
\sum_k |k|^2 |\hat{\phi}(k)|^2 \leq \frac{c^2\| f \|^2}{\nu^4} \sum_k \frac{|k|}{(1 + \delta^2|k|^2)^2} |\hat{\phi}(k)|^2
\]
\[
= \sum_k \left[ |k|^2 - \frac{c^2\| f \|^2}{\nu^4} \frac{|k|}{(1 + \delta^2|k|^2)^2} \right] |\hat{\phi}(k)|^2 \leq 0
\]

For $|k|$ large enough, the bracket term is positive. Then, the NS-$\omega$ regularization will have a finite number of determining modes.

To estimate the number, let $k = |k| > 0$. Then, the number of determining modes is the greatest integer in the positive roots of
\[
k^2 - \frac{c^2\| f \|^2}{\nu^4} \frac{k}{(1 + \delta^2k^2)^2} = 0
\]
\[
\Rightarrow k(1 + \delta^2k^2)^2 = \frac{c^2\| f \|^2}{\nu^4}
\]

**Remark 3.1.** $k_\omega \leq \frac{c^2\| f \|^2}{\nu^4} = k_{NSE}$.

When $\delta \to 0$, then $k_\omega \to k_{NSE}$. Furthermore $k_\omega \simeq (1 - 2\delta^2)\frac{c^2\| f \|^2}{\nu^4}$ if $\delta$ small

**Remark 3.2.** By computation in this report, $k_\omega \leq k_\alpha$. We could expect that NS-$\omega$ regularization need fewer modes than NS-$\alpha$.

**Acknowledgements**

I would like to thank Professor William Layton for several helpful discussions during this work.

**References**

