ANALYSIS OF A SECOND ORDER, UNCONDITIONALLY STABLE, PARTITIONED METHOD FOR THE EVOLUTIONARY STOKES-DARCY PROBLEM

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Abstract. In this work we study a new stabilization for abstract evolution equations applied to the numerical solution of the coupled, fully evolutionary Stokes-Darcy equations that model the interaction between groundwater and surface water flows. The method consists of uncoupling the fluid flow from the porous media subdomains by the Crank-Nicolson Leap-Frog (CNLF) method, studied by Kubacki in [13], with added stabilization terms that eliminate the CFL time step restriction of CNLF. We prove that the CNLF-stab method is unconditionally stable and second order convergent. We verify stability numerically. Numerical tests for convergence confirm second order convergence for the Stokes velocity and Darcy pressure variables as predicted.

Key words. Stokes, Darcy, surface-water, groundwater, partitioned, second order, stable

1. Introduction

One of the difficulties in solving the Stokes-Darcy problem arises from the coupling of two different physical processes in two adjacent domains. Using partitioned methods to uncouple the Stokes-Darcy equations resolves this issue and allows one to leverage existing solvers already optimized to solve the physical processes in each subdomain. Mu and Zhu first studied two, first order accurate partitioned methods for the evolutionary Stokes-Darcy equations in [16]. Layton, Trenchea, and Tran analyzed other first order partitioned methods in [14]. In [13], it was shown that the classical Crank-Nicolson Leap-Frog (CNLF) method results in a second order partitioned method for the Stokes-Darcy system. However, the conditional stability of CNLF makes the method impractical when faced with certain small problem parameters. In addition, even when the CFL type time step condition for stability holds, regular CNLF becomes unstable due to the unstable computational mode of Leap-Frog in some cases (see [13]).

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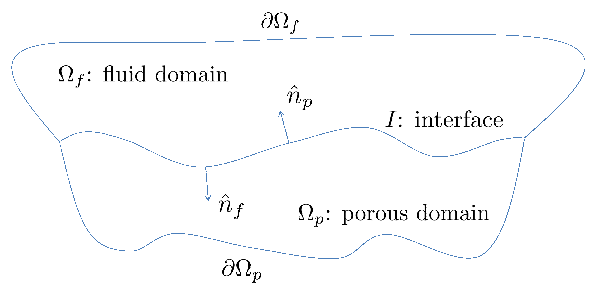


FIGURE 1. Fluid and porous media domains

By adding stabilization terms to both the Stokes as well as the groundwater flow equation, the proposed numerical scheme, denoted CNLF-stab and introduced in Section 3, equations (20)-(22), is unconditionally stable and second order convergent. More specifically, we give a proof that the stabilization eliminates the CFL type time step restriction without affecting the second order convergence of the Stokes velocity and Darcy pressure variables. In further contrast to classical CNLF, numerical tests in Section 5 demonstrate that CNLF-stab controls the unstable mode due to Leap-Frog in the cases in which regular CNLF stability fails. Analytic understanding of this attribute is an open question.

The system of equations modeling the interaction between surface water and groundwater flows follows. Let Ω_f, Ω_p denote two regular domains, the fluid and porous media regions respectively, and assume they lie across an interface, I, from each other (Figure 1). Assume that an incompressible fluid flows both ways across the interface into the two domains. We assume time-dependent Stokes flow in Ω_f and the time-dependent groundwater flow equation along with Darcy's law in Ω_p. The fluid velocity field $u = u(x,t)$ and pressure $p = p(x,t)$, defined on Ω_f, and porous media hydraulic head $\phi = \phi(x,t)$, defined on Ω_p, satisfy

$$\begin{aligned} u_t - \nu \Delta u + \nabla p = f_f(x,t), \nabla \cdot u = 0,\ \text{in } \Omega_f, \\ S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) = f_p(x,t),\ \text{in } \Omega_p, \\ u(x,0) = u_0,\ \text{in } \Omega_f,\ \phi(x,0) = \phi_0,\ \text{in } \Omega_p, \\ u(x,t) = 0,\ \text{in } \partial\Omega_f \backslash I, \phi(x,t) = 0,\ \text{in } \partial\Omega_p \backslash I, \\ + \text{ coupling conditions across } I, \end{aligned} \tag{1}$$

where we normalized the pressure p, as well as the body forces in the fluid region f_f by the fluid density. Denoted by f_p are the sinks and sources in the porous media region, $\nu > 0$ is the kinematic viscosity of the fluid, and $\mathcal{K}$ is the hydraulic conductivity tensor, assumed to be symmetric, positive definite with spectrum$(K) \in [k_{min}, k_{max}]$. We assume Dirichlet boundary conditions on the exterior boundaries of the two domains (not including the interface I). We discuss the coupling conditions in more detail in Section 2.

In the aforementioned equations, the parameter S_0 represents the specific storage, meaning the volume of water that a portion of a fully saturated porous medium releases from storage per unit volume and per unit drop in hydraulic head, see Freeze and Cherry [7], and Hantush [9]. Table 1 gives values of S_0 for different materials from Domenico and Mifflin [6] and Johnson [11]. In (2) we have the time step condition for stability in regular CNLF derived in [13], where g represents the gravitational acceleration constant and h the mesh size in the finite element discretization. The condition involves S_0, making the time step condition computationally restrictive. In the case of a confined aquifer, for instance, with $S_0 = \mathcal{O}(10^{-6})$, if we take $h = \mathcal{O}(0.1)$ then, because $g = O(10^1)$ the time step can at most be of order 10^{-5}. A small time step is prohibitive since studying flow in large aquifers with low conductivity necessitates accurate calculations over long time periods.

$$\Delta t \le C \max\{\min\{h^2, gS_0\}, \min\{h, gS_0 h\}\} \tag{2}$$

Table 1. Specific Storage (S_0) values for different materials

Material	**Specific Storage** S_0 (m^{-1})
Plastic clay	$2.0 \times 10^{-2} - 2.6 \times 10^{-3}$
Stiff clay	$2.6 \times 10^{-3} - 1.3 \times 10^{-3}$
Medium hard clay	$1.3 \times 10^{-3} - 9.2 \times 10^{-4}$
Loose sand	$1.0 \times 10^{-3} - 4.9 \times 10^{-4}$
Dense sand	$2.0 \times 10^{-4} - 1.3 \times 10^{-4}$
Dense sandy gravel	$1.0 \times 10^{-4} - 4.9 \times 10^{-5}$
Rock, fissured jointed	$6.9 \times 10^{-5} - 3.3 \times 10^{-6}$
Rock, sound	less than 3.3×10^{-6}

Another important parameter in our analysis is the hydraulic conductivity tensor in the porous medium, $\mathcal{K}$. In exact arithmetic, stability of CNLF does not depend upon $\mathcal{K}$. However, in the presence of round-off error, CNLF becomes unstable for small values of the minimum eigenvalue of $\mathcal{K}$, k_{min}. If $\Delta t = \mathcal{O}(10^{-2})$ and $k_{min} = \mathcal{O}(10^{-1})$ or smaller, classical CNLF method becomes unstable due to the unstable computational mode (see Kubacki [13] Section IV.B Figure 3). Since the hydraulic conductivity is often orders of magnitude smaller than 10^{-1} (see Table 2 for values of k_{min} from Bear [1]), this can be a frequent problem of using CNLF.

Table 2. Hydraulic conductivity (k_{min}) values for different materials

Material	**Hydraulic conductivity** k_{min} (m/s)
Well sorted gravel	$10^{-1} - 10^{0}$
Highly fractured rocks	$10^{-3} - 10^{0}$
Well sorted sand or sand & gravel	$10^{-4} - 10^{-2}$
Oil reservoir rocks	$10^{-6} - 10^{-4}$
Very fine sand, silt, loess, loam	$10^{-8} - 10^{-5}$
Layered clay	$10^{-8} - 10^{-6}$
Fresh sandstone, limestone, dolomite, granite	$10^{-12} - 10^{-7}$
Fat/Unweathered clay	$10^{-12} - 10^{-9}$

In Section 2 we present necessary preliminaries along with the corresponding weak formulation of the Stokes-Darcy problem. In Section 3 we introduce the CNLF-stab method for the evolutionary Stokes-Darcy problem and present the proof for unconditional stability. We prove second order convergence of the method in Section 4. Section 5 demonstrates the method's unconditional stability and second order convergence through a series of numerical tests. Finally, we present conclusions in Section 6.

2. Preliminaries

Before discussing the CNLF-stab method, we present the equivalent variational formulation along with some inequalities relevant to our analysis. To complete the system of equations in (1), we must add coupling conditions to describe the flow along the interface, I. The coupling conditions consist of conservation of mass across the interface

$$u \cdot \hat{n}_f - \mathcal{K}\nabla\phi \cdot \hat{n}_p = 0, \text{ on } I, \tag{3}$$

and balance of normal forces across the interface

$$p - \nu\hat{n}_f \cdot \nabla u \cdot \hat{n}_f = g\phi, \text{ on } I, \tag{4}$$

where $\hat{n}_p = -\hat{n}_f$ are the outward pointing unit normal vectors on $\Omega_{f/p}$ (Figure 1). The last condition is a condition on the tangential velocity on I. Let $\hat{\tau}_i$, $i = 1, \dots, d-1$, denote an orthonormal basis of tangent vectors on I, $d = 2$ or 3. We assume the Beavers-Joseph-Saffman condition, see Joseph [12] and Saffman [18]:

$$-\nu\hat{\tau}_i \cdot \nabla u \cdot \hat{n}_f = \frac{\alpha}{\sqrt{\hat{\tau}_i \cdot \mathcal{K} \cdot \hat{\tau}_i}} u \cdot \hat{\tau}_i, \text{ for } i = 1, \dots, d-1, \text{ on } I, \tag{5}$$

which is a simplification of the original and more physically realistic Beavers-Joseph condition, see Beavers and Joseph [2]. The parameter α in (5) is an experimentally determined constant. For more information on this condition see e.g., Mikelič and Jäger [10], and Payne and Straughan [17].

The equivalent variational formulation of equations (1)-(5) follows, see e.g., Discacciati, Miglio and Quarteroni [5]. Let the L^2 norm on $\Omega_{f/p}$ be denoted by $\|\cdot\|_{f/p}$ and the L^2 norm on the interface, I, by $\|\cdot\|_I$; denote the corresponding inner products on $\Omega_{f/p}$ by $(\cdot,\cdot)_{f/p}$. Furthermore, denote the H^1 norm on $\Omega_{f/p}$ by $\|\cdot\|_{1,f/p}$.

Define the spaces

$$X_f := \left\{ v \in (H^1(\Omega_f))^d : v = 0 \text{ on } \partial\Omega_f \backslash I \right\},$$
$$X_p := \left\{ \psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p \backslash I \right\},$$
$$Q := L^2(\Omega_f),$$
$$V_f := \{ v \in X_f : (\nabla \cdot v, q)_f = 0 \text{ for all } q \in Q \}.$$

The norms on the dual spaces X_f^* and X_p^* are given by

$$\|f\|_{-1,f/p} = \underset{0 \neq v \in X_{f/p}}{sup} \frac{(f,v)_{f/p}}{\|\nabla v\|_{f/p}}.$$

In the analysis to follow we use the following inequalities. The first is the Poincaré-Friedrichs inequality. The second is a trace inequality, see, for example, Brenner and Scott [3], chapter 1.6, p. 36-38. The first and second inequalities hold for any function w that belongs to either X_f or X_p and the third inequality holds for any function $u \in X_f$.

$$\|w\|_{f/p} \leq C_{P_{f/p}} \|\nabla w\|_{f/p}, \text{ for some constants } C_{P_{f/p}} > 0, \tag{6}$$

$$\|w\|_{L^2(\partial\Omega_{f/p})} \leq C_{\Omega_{f/p}} \|w\|_{f/p}^{\frac{1}{2}} \|\nabla w\|_{f/p}^{\frac{1}{2}}, \text{ for some constants } C_{\Omega_{f/p}} > 0, \tag{7}$$

$$\|\nabla \cdot u\|_f \leq \sqrt{d} \|\nabla u\|_f, \text{ where } d = 2, \text{ or } 3. \tag{8}$$

Define the bilinear forms

$$a_f(u,v) = (\nu \nabla u, \nabla v)_f + \sum_{i=1}^{d-1} \int_I \frac{\alpha}{\sqrt{\hat{\tau}_i \cdot \mathcal{K} \cdot \hat{\tau}_i}} (u \cdot \hat{\tau}_i)(v \cdot \hat{\tau}_i) \, ds,$$
$$a_p(\phi, \psi) = (\mathcal{K} \nabla \phi, \nabla \psi)_p,$$
$$c_I(u, \phi) = g \int_I \phi u \cdot \hat{n}_f \, ds.$$

The interface coupling term, $c_I(\cdot,\cdot)$, plays a key role in our analysis. The following inequalities hold for our bilinear forms.

Lemma 1. *The bilinear forms $a_f(\cdot,\cdot), a_p(\cdot,\cdot)$ and $c_I(u,\phi)$ satisfy*

$$a_f(u,v) \leq \max\left\{\nu+1, (1/2)C(\Omega_f)\alpha k_{min}^{-1/2}\right\}\|u\|_{1,f}\|v\|_{1,f}, \tag{9}$$

$$a_f(u,u) \geq \nu\|\nabla u\|_f^2 + \alpha k_{max}^{-1/2}\sum_{i=1}^{d-1}\int_I (u\cdot\hat{\tau}_i)^2 \, d\sigma =: \nu\|\nabla u\|_f^2 + \alpha k_{max}^{-1/2}\|u\cdot\hat{\tau}\|_I^2, \tag{10}$$

$$a_p(\phi,\psi) \leq k_{max}\|\nabla\phi\|_p\|\nabla\psi\|_p, \tag{11}$$

$$a_p(\phi,\phi) \geq k_{min}\|\nabla\phi\|_p^2, \tag{12}$$

$$|c_I(u,\phi)| \leq (1/2)gC(\Omega_f,\Omega_p)\|u\|_{1,f}\|\phi\|_{1,p} \tag{13}$$

for all $u, v \in X_f$ and all $\phi,\psi \in X_p$.

Proof. The proofs are straightforward. For the first four inequalities, see for example Kubacki [13] Section II Lemma 2.3. For the coupling inequality, see for example Moraiti [15] Section 2 Lemma 2.2. □

An additional inequality on the interface term is given below and holds under conditions on the domains Ω_f, Ω_p. The constant $C_{f,p}$ depends on $\Omega_{f/p}$ and in the special case of a flat interface I, with Ω_f and Ω_p being arbitrary domains, $C_{f,p}$ equals 1, see Moraiti in [15] Section 3 Lemmas 3.1 and 3.2.

$$|c_I(u,\phi)| \leq gC_{f,p}\|u\|_{DIV,f}\|\phi\|_{1,p}. \tag{14}$$

The variational formulation of the Stokes-Darcy problem (1), (3)-(5) reads: given $u(x,0)=u_0(x)$, $\phi(x,0)=\phi_0(x)$, find $u:[0,\infty)\to V_f$, $\phi:[0,\infty)\to X_p$ satisfying

$$(u_t,v)_f + a_f(u,v) + c_I(v,\phi) = (f_f,v)_f, \tag{15}$$

$$gS_0(\phi_t,\psi)_p + ga_p(\phi,\psi) - c_I(u,\psi) = g(f_p,\psi)_p, \tag{16}$$

for all $v\in V_f$, and $\psi \in X_p$. The existence and uniqueness of a solution (u,ϕ) to the problem (15)-(16) follows by the Hille - Yosida theorem, see Brézis [4].

We discretize in space using the Finite Element Method (FEM). Select a quasiuniform triangular mesh, $\mathcal{T}_h$ for the combined subdomains, $\Omega_f\cup\Omega_p$, where h denotes the maximum triangle diameter. Next, choose FEM spaces based on a conforming FEM triangulation:

$$\text{Fluid velocity: } X_f^h \subset X_f,$$

$$\text{Darcy Pressure: } X_p^h \subset X_p,$$

$$\text{Stokes Pressure: } Q_f^h \subset Q_f.$$

Additionally, we must select X_f^h and Q_f^h so that they satisfy the discrete inf-sup condition (LBB^h) (see, for example [8]) for stability of the discrete pressure. Notice that $V_f^h := \{v_h \in X_f^h : (q_h, \nabla \cdot v_h)_f = 0 \, \forall q_h \in Q_f^h\}$ is not necessarily a subset of V_f. Hence, we must include the incompressibility condition (18) in the semidiscretized formulation. Given $u_h(x,0) = u_0(x), \phi_h(x,0) = \phi_0(x)$, find $(u_h, p_h, \phi_h) : [0,\infty) \to (X_f^h, Q_f^h, X_p^h)$ such that

$$(u_{h,t}, v_h)_f + a_f(u_h, v_h) - (p_h, \nabla \cdot v_h)_f + c_I(v_h, \phi_h) = (f_f, v_h)_f, \tag{17}$$

$$(q_h, \nabla \cdot u_h)_f = 0, \tag{18}$$

$$gS_0(\phi_{h,t}, \psi_h)_p + ga_p(\phi_h, \psi_h) - c_I(u_h, \psi_h) = g(f_p, \psi_h i)_p, \tag{19}$$

for all $(v_h, q_h, \psi_h) \in (X_f^h, Q_f^h, X_p^h)$.

3. CNLF-stab method and Unconditional Stability

The CNLF-stab method for the numerical solution of the evolutionary Stokes-Darcy problem given in (1), (3)-(5) is introduced next.

Algorithm 2. *(CNLF-stab Method) Let $t^n := n\Delta t$ and $v^n := v(x, t^n)$ for any function $v(x,t)$. CNLF with added stabilization for the evolutionary Stokes-Darcy equations is as follows.*

Given $\left(u_h^k, p_h^k, \phi_h^k\right), \left(u_h^{k-1}, p_h^{k-1}, \phi_h^{k-1}\right) \in \left(X_f^h, Q_f^h, X_p^h\right)$, *find* $\left(u_h^{k+1}, p_h^{k+1}, \phi_h^{k+1}\right) \in \left(X_f^h, Q_f^h, X_p^h\right)$ *satisfying for all* $(v_h, q_h, \psi_h) \in \left(X_f^h, Q_f^h, X_p^h\right)$:

$$\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + \left(\nabla\cdot\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\right), \nabla\cdot v_h\right)_f + a_f\left(\frac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) - \left(\frac{p_h^{k+1}+p_h^{k-1}}{2}, \nabla\cdot v_h\right)_f + c_I(v_h, \phi_h^k) = (f_f^k, v_h)_f \tag{20}$$

$$\left(q_h, \nabla\cdot\left(\frac{u_h^{k+1}+u_h^{k-1}}{2}\right)\right)_f = 0, \tag{21}$$

$$gS_0\left(\frac{\phi_h^{n+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\frac{\phi_h^{k+1}-\phi_h^{k-1}}{2}, \psi_h\right) - c_I(u_h^k, \psi_h) + \Delta t g^2 C_{f,p}^2\left(\nabla(\phi_h^{k+1}-\phi_h^{k-1}), \nabla\psi_h\right)_p + \Delta t g^2 C_{f,p}^2\left(\phi_h^{k+1}-\phi_h^{k-1}, \psi_h\right)_p = g(f_p^k, \psi_h)_p \tag{22}$$

where $C_{f,p}$ *is the constant from inequality (14).*

CNLF-stab is a 3 level method. The zeroth terms, (u_h^0, p_h^0, ϕ_h^0), come from the initial conditions of the problem. We must obtain the first terms, (u_h^1, p_h^1, ϕ_h^1), by a one step method which uncouples the system, for example Backward Euler Leap-Frog (BELF). The errors in this first step will affect the overall convergence rate of the method. Notice that the added stability terms,

$$\left(\nabla\cdot\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\right), \nabla\cdot v_h\right)_f, \text{ in (20) and}$$

$$\Delta t g^2 C_{f,p}^2\left(\nabla(\phi_h^{k+1}-\phi_h^{k-1}), \nabla\psi_h\right)_p, \Delta t g^2 C_{f,p}^2\left(\phi_h^{k+1}-\phi_h^{k-1}, \psi_h\right)_p, \text{in (22)},$$

are $\mathcal{O}(\Delta t^2)$. Similar to CNLF, the CNLF-stab method uncouples the Stokes-Darcy equations into two subdomain problems by treating the coupling term explicitly with Leap-Frog. By adding the above stabilization terms to CNLF, we eliminate the need for a CFL type time step restriction for stability. The proof for unconditional stability of the CNLF-stab method (20)-(22) follows.

Theorem 3. *(Unconditional Stability of CNLF-stab) CNLF-stab is unconditionally stable: for any $N > 0$, there holds*

$$\begin{aligned}
&\frac{1}{2}\left(\|u_h^N\|^2_{DIV,f} + \|u_h^{N-1}\|^2_{DIV,f}\right) + gS_0\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right) \\
&+ \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\nu\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2 + k_{min}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2\right\} \\
&\le \|u_h^1\|^2_{DIV,f} + \|u_h^0\|^2_{DIV,f} + gS_0\left(\|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2\right) \qquad (23)\\
&+ 2\Delta t^2 g^2 C_{f,p}^2\left(\|\phi_h^1\|^2_{1,p} + \|\phi_h^0\|^2_{1,p}\right) + 2\Delta t\left\{c_I(\phi_h^0, u_h^1) - c_I(\phi_h^1, u_h^0)\right\} \\
&+ \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\frac{g}{k_{min}}\|f_p^k\|^2_{-1,p} + \frac{1}{\nu}\|f_f^k\|^2_{-1,f}\right\}.
\end{aligned}$$

Proof. In (20), (22) set $v_h = u_h^{k+1} + u_h^{k-1}$, $\psi_h = \phi_h^{k+1} + \phi_h^{k-1}$. Then the pressure term in (20) cancels by (21). Adding the two equations together and multiplying through by $2\Delta t$ yields

$$\begin{aligned}
&\|u_h^{k+1}\|^2_{DIV,f} - \|u_h^{k-1}\|^2_{DIV,f} + gS_0\left(\|\phi_h^{k+1}\|_p^2 - \|\phi_h^{k-1}\|_p^2\right) \\
&+ 2\Delta t^2 g^2 C_{f,p}^2\left(\|\phi_h^{k+1}\|^2_{1,p} - \|\nabla\phi_h^{k-1}\|^2_{1,p}\right) \\
&+ \Delta t\left\{a_p\left(\phi_h^{k+1}+\phi_h^{k-1}, \phi_h^{k+1}+\phi_h^{k-1}\right) + a_f\left(u_h^{k+1}+u_h^{k-1}, u_h^{k+1}+u_h^{k-1}\right)\right\} \\
&+ 2\Delta t\left(c_I(u_h^{k+1}+u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1}+\phi_h^{k-1})\right) \\
&\quad = 2\Delta t\left\{g\left(f_p^k, \phi_h^{k+1}+\phi_h^{k-1}\right)_p + \left(f_f^k, u_h^{k+1}+u_h^{k-1}\right)_f\right\}.
\end{aligned}$$

If we let

$$C^{k+1/2} = c_I(\phi_h^k, u_h^{k+1}) - c_I(\phi_h^{k+1}, u_h^k),$$

then the interface terms in the equation above become

$$c_I(u_h^{k+1}+u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1}+\phi_h^{k-1}) = C^{k+\frac{1}{2}} - C^{k-\frac{1}{2}}.$$

Using coercivity of the bilinear forms $a_{f/p}(\cdot,\cdot)$, the dual norms on X_p, X_f and Young's inequality we obtain, after rearranging,

$$\begin{aligned} &\|u_h^{k+1}\|^2_{DIV,f} - \|u_h^{k-1}\|^2_{DIV,f} + gS_0\left(\|\phi_h^{k+1}\|_p^2 - \|\phi_h^{k-1}\|_p^2\right) \\ &+ 2\Delta t^2 g^2 C_{f,p}^2\left(\|\phi_h^{k+1}\|^2_{1,p} - \|\phi_h^{k-1}\|^2_{1,p}\right) + 2\Delta t\left\{C^{k+\frac{1}{2}} - C^{k-\frac{1}{2}}\right\} \\ &+ \Delta t\left\{\tfrac{k_{min}}{2}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2 + \tfrac{\nu}{2}\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2\right\} \\ &\quad \le \Delta t\frac{1}{2\nu}\|f_f^k\|^2_{-1,f} + \Delta t\frac{g^2}{2k_{min}}\|f_p^k\|^2_{-1,p}. \end{aligned} \tag{24}$$

Denote the energy terms by

$$\begin{aligned} E^{k+1/2} = \|u_h^{k+1}\|^2_{DIV,f} + \|u_h^k\|^2_{DIV,f} + gS_0\left(\|\phi_h^{k+1}\|_p^2 + \|\phi_h^k\|_p^2\right) \\ +2\Delta t^2 g^2 C_{f,p}^2\left(\|\phi_h^{k+1}\|^2_{1,p} + \|\phi_h^k\|^2_{1,p}\right). \end{aligned}$$

Then (24) becomes

$$\begin{aligned} E^{k+1/2} - E^{k-1/2} + \Delta t\left\{\tfrac{k_{min}}{2}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2 + \tfrac{\nu}{2}\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2\right\} \\ +2\Delta t\left\{C^{k+1/2} - C^{k-1/2}\right\} \le \Delta t\frac{g^2}{2k_{min}}\|f_p^k\|^2_{-1,p} + \Delta t\frac{1}{2\nu}\|f_f^k\|^2_{-1,f}. \end{aligned}$$

Sum up the inequality from $k = 1$ to $N - 1$ to find

$$\begin{aligned} E^{N-1/2} + \Delta t\sum_{k=1}^{N-1}\left\{\tfrac{k_{min}}{2}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2 + \tfrac{\nu}{2}\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2\right\} \\ +2\Delta t C^{N-1/2} \le E^{1/2} + C^{1/2} + \Delta t\frac{g^2}{2k_{min}}\|f_p^k\|^2_{-1,p} + \Delta t\frac{1}{2\nu}\|f_f^k\|^2_{-1,f}. \end{aligned} \tag{25}$$

Applying inequality (14) to the interface terms involved in the term $C^{N-1/2}$ gives

$$\begin{aligned} &|c_I(u_h^N, \phi_h^{N-1})| \le gC_{f,p}\|u_h^N\|_{DIV,f}\|\phi_h^{N-1}\|_{1,p} \text{ and} \\ &|c_I(u_h^{N-1}, \phi_h^N)| \le gC_{f,p}\|u_h^{N-1}\|_{DIV,f}\|\phi_h^N\|_{1,p}. \end{aligned}$$

Therefore, we may bound the term $C^{N-1/2}$ by the Cauchy-Schwarz and Young inequalities as follows.

$$|2\Delta t C^{N-1/2}| \le \frac{1}{2}\left(\|u_h^N\|^2_{DIV,f} + \|u_h^{N-1}\|^2_{DIV,f}\right) + 2\Delta t^2 g^2 C_{f,p}^2\left(\|\phi_h^{N-1}\|^2_{1,p} + \|\phi_h^N\|^2_{1,p}\right).$$

Thus,

$$E^{N-1/2} + 2\Delta t C^{N-1/2} \geq \frac{1}{2}\left(\|u_h^N\|_{DIV,f}^2 + \|u_h^{N-1}\|_{DIV,f}^2\right) + gS_0\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right). \tag{26}$$

After combining inequalities (25) and (26) we achieve the desired unconditional stability bound (23). □

Corollary 4. *If $f_f = f_p \equiv 0$, then the averages, $(u_h^{n+1} + u_h^{n-1})/2$ and $(\phi_h^{n+1} + \phi_h^{n-1})/2$, converge to zero as $n \to \infty$.*

Proof. The bound (23) implies that the series $\sum_{n=1}^{\infty}\{\|\nabla(u_h^{n+1}+u_h^{n-1})\|\}$ converges. Thus, $\|\nabla(u_h^{n+1}+u_h^{n-1})\| \to 0$, as $n \to \infty$, and by the Poincaré-Friedrichs inequality, $\|u_h^{n+1} + u_h^{n-1}\| \to 0$ as well. (Similarly for $(\phi_h^{n+1} + \phi_h^{n-1})/2$.) □

This shows that the CNLF-stab method controls the stable mode of Leap-Frog. However, Theorem 3 does not imply control over the unstable mode. We check the behavior of the unstable mode in the numerical experiments in Section 5.

4. Error Analysis of CNLF-stab

In this section, in Theorem 6, we establish the method's second order convergence over long time intervals. An essential feature of the error analysis is that no form of Gronwall's inequality is available as a tool.

We assume that the FEM spaces, X_f^h, X_p^h and Q_f^h, satisfy approximation properties of piecewise polynomials of degree $r-1$, r, and $r+1$:

$$\begin{aligned}
\inf_{u_h \in X_f^h} \|u - u_h\|_f &\leq Ch^{r+1}\|u\|_{H^{r+1}(\Omega_f)} \\
\inf_{u_h \in X_f^h} \|u - u_h\|_{H^1(\Omega_f)} &\leq Ch^{r}\|u\|_{H^{r+1}(\Omega_f)} \\
\inf_{\phi_h \in X_p^h} \|\phi - \phi_h\|_p &\leq Ch^{r+1}\|\phi\|_{H^{r+1}(\Omega_p)} \\
\inf_{\phi_h \in X_p^h} \|\phi - \phi_h\|_{H^1(\Omega_p)} &\leq Ch^{r}\|\phi\|_{H^{r+1}(\Omega_p)} \\
\inf_{p_h \in Q_f^h} \|p - p_h\|_f &\leq Ch^{r+1}\|p\|_{H^{r+1}(\Omega_f)}.
\end{aligned} \tag{27}$$

Moreover, we assume that the spaces X_f^h and Q_f^h satisfy the (LBB^h) condition. As a consequence, there exists some constant C such that if $u \in V_f$, then

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega_f)} \le C \inf_{x_h \in X_f^h} \|u - x_h\|_{H^1(\Omega_f)}, \tag{28}$$

(see, for example, Girault and Raviart [8], Chapter II, Proof of Theorem 1.1, Equation (1.12)). Let $N \in \mathbf{N}$ be given and denote $t^n = n\Delta t$ and $T = N\Delta t$. If $T = \infty$ then $N = \infty$. We introduce the following discrete norms.

$$|||v|||^2_{L^2(0,T;H^s(\Omega_{f,p}))} := \sum_{k=1}^{N} \|v^k\|^2_{H^s(\Omega_{f,p})}\Delta t,$$

$$|||v|||_{L^\infty(0,T;H^s(\Omega_{f,p}))} := \max_{0\le k\le N} \|v^k\|_{H^s(\Omega_{f,p})}.$$

In the proof of convergence to follow, we will need the bounds given in the next lemma.

Lemma 5. *(Consistency Errors) The following inequalities hold:*

$$\Delta t \sum_{k=1}^{N-1} \left\| u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 \le \frac{(\Delta t)^4}{20} \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{29}$$

$$\Delta t \sum_{k=1}^{N-1} \|\phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t}\|_p^2 \le \frac{(\Delta t)^4}{20} \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{30}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(u^k - \frac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 \le \frac{7(\Delta t)^4}{6} \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))}, \tag{31}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 \le \frac{7(\Delta t)^4}{6} \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}. \tag{32}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right) \right\|_f^2 \le \frac{(\Delta t)^4}{20} \|\nabla u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{33}$$

$$\Delta t \sum_{k=1}^{N-1} \|\phi^{k+1} - \phi^{k-1}\|^2_{1,p} \le 4\Delta t^2 \|\phi_t\|^2_{L^2(0,T,H^1(\Omega_p))} \tag{34}$$

Proof. For inequalities (29)-(33) see Kubacki in [13] Section 3 Lemma 3.2. For the proof of (34), we have

$$\begin{aligned}
\Delta t \sum_{k=1}^{N-1} \|\phi^{k+1} - \phi^{k-1}\|_p^2 &= \Delta t \sum_{k=1}^{N-1} \int_{\Omega_f} \left(\int_{t_{k-1}}^{t^{k+1}} \phi_t \; dt \right) \; dx \\
&\leq \Delta t \int_{\Omega_f} \sum_{k=1}^{N-1} \int_{t^{k-1}}^{t^{k+1}} \; dt \int_{t^{k-1}}^{t^{k+1}} \phi_t^2 \; dt \; dx \\
&= \Delta t \int_{\Omega_f} \sum_{k=1}^{N-1} 2\Delta t \int_{t^{k-1}}^{t^{k+1}} \phi_t^2 \; dt \; dx \\
&\leq 2\Delta t^2 \int_{\Omega_f} 2 \sum_{k=1}^{N} \int_{t^{k-1}}^{t^k} \phi_t^2 \; dt \; dx \\
&= 4\Delta t^2 \|\phi_t\|^2_{L^2(0,T,L^2(\Omega_p))}, \qquad (35)
\end{aligned}$$

Similarly,

$$\Delta t \sum_{k=1}^{N-1} \|\nabla \left(\phi^{k+1} - \phi^{k-1}\right)\|_p^2 = 4\Delta t^2 \|\nabla \phi_t\|^2_{L^2(0,T,L^2(\Omega_p))}. \qquad (36)$$

Inequalities (35) and (36) combined give (34). □

Denote the errors by $e_f^n = u^n - u_h^n$ and $e_p^n = \phi^n - \phi_h^n$.

Theorem 6. *(Second Order Convergence of CNLF-stab) Consider the CNLF-stab method* (20)-(22)*. For any* $0 < t_N = T \leq \infty$*, if* u*,* p*,* ϕ *satisfy the regularity conditions*

$$u \in L^2(0,T;H^{r+2}(\Omega_f)) \cap L^\infty(0,T;H^{r+1}(\Omega_f)) \cap H^3(0,T;H^1(\Omega_f)),$$

$$p \in L^2(0,T;H^{r+1}(\Omega_f)),$$

$$\phi \in L^2(0,T;H^{r+2}(\Omega_p)) \cap L^\infty(0,T;H^{r+1}(\Omega_p)) \cap H^3(0,T;H^1(\Omega_p)),$$

then there exists a constant $\widehat{C} > 0$, independent of the mesh width h, time step Δt and final time $t_N = T$, such that

$$\begin{aligned}
&\frac{1}{2}(\|e_f^N\|^2_{DIV,f} + \|e_f^{N-1}\|^2_{DIV,f}) + gS_0(\|e_p^N\|^2_p + \|e_p^{N-1}\|^2_p) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2} \|\nabla(e_f^{k+1} + e_f^{k-1})\|^2_f + \frac{gk_{min}}{2} \|\nabla(e_p^{k+1} + e_p^{k-1})\|^2_p \right) \\
&\quad \le \widehat{C} \Big\{ h^{2r} \{ \|u_t\|^2_{L^2(0,T;H^{r+1}(\Omega_f))} + \||u|\|^2_{L^2(0,T;H^{r+1}(\Omega_f))} + \||u|\|^2_{L^\infty(0,T;H^{r+1}(\Omega_f))} \\
&\qquad + \Delta t^4 \|\phi_t\|^2_{L^2(0,T;H^{r+1}(\Omega_p))} + \||\phi|\|^2_{L^\infty(0,T;H^{r+1}(\Omega_p))} \} \\
&\qquad + h^{2r+2} \{ \||p|\|^2_{L^2(0,T;H^{r+1}(\Omega_p))} + \|\phi_t\|^2_{L^2(0,T;H^{r+1}(\Omega_p))} + \||\phi|\|^2_{L^\infty(0,T;H^{r+1}(\Omega_p))} \} \\
&\qquad + \Delta t^4 \{ \|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad + \|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \} + \|e_f^1\|^2_{DIV,f} + \|e_p^1\|^2_{1,p} \Big\}.
\end{aligned} \tag{37}$$

Proof. Consider CNLF-stab (20)-(22) over the discretely divergence free space $V^h := \{v_h \in X_f^h : (q_h, \nabla \cdot v_h)_f = 0 \, \forall q_h \in Q_f^h\}$, instead of X_f^h, so that the pressure term $\left(\frac{p_h^{k+1}+p_h^{k-1}}{2}, \nabla \cdot v_h\right)$ cancels out. Subtract (20) and (22) from the variational form (15) and (16) evaluated at time t^k to get:

$$\begin{aligned}
&\left(u_t^k - \tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f - \left(\nabla \cdot \left(\tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\right), \nabla \cdot v_h\right)_f + a_f\left(u^k - \tfrac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) \\
&\qquad - \left(p^k, \nabla \cdot v_h\right)_f + c_I\left(v_h, \phi^k - \phi_h^k\right) = 0, \\
&gS_0\left(\phi_t^k - \tfrac{\phi_h^{k+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\phi^k - \tfrac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) \\
&-\Delta t g^2 C_{f,p}^2 (\nabla(\phi_h^{k+1} - \phi_h^{k-1}), \nabla\psi_h)_p - \Delta t g^2 C_{f,p}^2 (\phi_h^{k+1} - \phi_h^{k-1}, \psi_h)_p - c_I\left(u^k - u_h^k, \psi_h\right) = 0.
\end{aligned}$$

Since v_h is discretely divergence free, we have that $\left(p^k, \nabla \cdot v_h\right)_f = \left(p^k - \lambda_h^k, \nabla \cdot v_h\right)_f$, for any $\lambda_h \in Q_f^h$. Further, $\left(\nabla \cdot u_t^k, v_h\right) = 0$. Thus, after rearranging we get:

$$\begin{aligned}
&\left(\tfrac{e_f^{k+1}-e_f^{k-1}}{2\Delta t}, v_h\right)_f + \left(\nabla \cdot \left(\tfrac{e_f^{k+1}-e_f^{k-1}}{2\Delta t}\right), \nabla \cdot v_h\right)_f + a_f\left(\tfrac{e_f^{k+1}+e_f^{k+1}}{2}, v_h\right) + c_I\left(v_h, e_p^k\right) \\
&\quad = -\left(u_t^k - \tfrac{u^{k+1}-u^{k-1}}{2\Delta t}, v_h\right)_f - \left(\nabla \cdot \left(u_t^k - \tfrac{u^{k+1}-u^{k-1}}{2\Delta t}\right), \nabla \cdot v_h\right)_f \\
&\qquad - a_f\left(u^k - \tfrac{u^{k+1}+u^{k-1}}{2}, v_h\right) + \left(p^k - \lambda_h^k, \nabla \cdot v_h\right)_f,
\end{aligned}$$

$$gS_0\left(\frac{e_p^{k+1}-e_p^{k-1}}{2\Delta t},\psi_h\right)_p + a_p\left(\frac{e_p^{k+1}+e_p^{k-1}}{2},\psi_h\right) + \Delta t g^2 C_{f,p}^2(\nabla(e_p^{k+1}-e_p^{k-1}),\nabla\psi_h)_p$$
$$+\Delta t g^2 C_{f,p}^2(e_p^{k+1}-e_p^{k-1},\psi_h)_p - c_I\left(e_f^k,\psi_h\right)$$
$$= -gS_0\left(\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t},\psi_h\right)_p - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2},\psi_h\right)$$
$$+\Delta t g^2 C_{f,p}^2(\nabla(\phi^{k+1}-\phi^{k-1}),\nabla\psi_h)_p + \Delta t g^2 C_{f,p}^2(\phi^{k+1}-\phi^{k-1},\psi_h)_p.$$

Denote the consistency errors by:

$$\varepsilon_f^k(v_h) = -\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t},v_h\right)_f - \left(\nabla\cdot\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right),\nabla\cdot v_h\right)_f$$
$$- a_f\left(u^k - \frac{u^{k+1}+u^{k-1}}{2},v_h\right),$$
$$\varepsilon_p^k(\psi_h) = -gS_0\left(\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t},\psi_h\right)_p + \Delta t g^2 C_{f,p}^2(\nabla(\phi^{k+1}-\phi^{k-1}),\nabla\psi_h)_p$$
$$+\Delta t g^2 C_{f,p}^2(\phi^{k+1}-\phi^{k-1},\psi_h)_p - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2},\psi_h\right).$$

Decompose the error terms into

$$e_f^{k+1} = u^{k+1}-u_h^{k+1} = (u^{k+1}-\tilde{u}^{k+1}) + (\tilde{u}^{k+1}-u_h^{k+1}) = \eta_f^{k+1}+\xi_f^{k+1},$$
$$e_p^{k+1} = \phi^{k+1}-\phi_h^{k+1} = (\phi^{k+1}-\tilde{\phi}^{k+1}) + (\tilde{\phi}^{k+1}-\phi_h^{k+1}) = \eta_p^{k+1}+\xi_p^{k+1},$$

and take $\tilde{u}^{k+1}\in V^h$ and $\tilde{\phi}^{k+1}\in X_p^h$, so that $\xi_f^{k+1}\in V^h$. Then the error equations become:

$$\left(\frac{\xi_f^{k+1}-\xi_f^{k-1}}{2\Delta t},v_h\right)_f + \left(\nabla\cdot\left(\frac{\xi_f^{k+1}-\xi_f^{k-1}}{2\Delta t}\right),v_h\right)_f + a_f\left(\frac{\xi_f^{k+1}+\xi_f^{k-1}}{2},v_h\right) + c_I(v_h,\xi_p^k)$$
$$= -\left(\frac{\eta_f^{k+1}-\eta_f^{k-1}}{2\Delta t},v_h\right)_f - \left(\nabla\cdot\left(\frac{\eta_f^{k+1}-\eta_f^{k-1}}{2\Delta t}\right),\nabla\cdot v_h\right)_f - a_f\left(\frac{\eta_f^{k+1}+\eta_f^{k-1}}{2},v_h\right)$$
$$- c_I(v_h,\eta_p^k) + \varepsilon_f^k(v_h) + \left(p^k-\lambda_h^k,\nabla\cdot v_h\right)_f,$$

$$gS_0\left(\frac{\xi_p^{k+1}-\xi_p^{k-1}}{2\Delta t},\psi_h\right)_p+a_p\left(\frac{\xi_p^{k+1}+\xi_p^{k-1}}{2},\psi_h\right)+\Delta t g^2C_{f,p}^2(\nabla(\xi_p^{k+1}-\xi_p^{k-1}),\nabla\psi_h)_p$$
$$+\Delta t g^2C_{f,p}^2(\xi_p^{k+1}-\xi_p^{k-1},\psi_h)_p-c_I(\xi_f^k,\psi_h)$$
$$=-gS_0\left(\frac{\eta_p^{k+1}-\eta_p^{k-1}}{2\Delta t},\psi_h\right)_p-a_p\left(\frac{\eta_p^{k+1}+\eta_p^{k-1}}{2},\psi_h\right)+c_I(\eta_f^k,\psi_h)$$
$$-\Delta t g^2C_{f,p}^2(\nabla(\eta_p^{k+1}-\eta_p^{k-1}),\nabla\psi_h)_p-\Delta t g^2C_{f,p}^2(\eta_p^{k+1}-\eta_p^{k-1},\psi_h)_p+\varepsilon_p^k(\psi_h).$$

Pick $v_h=\xi_f^{k+1}+\xi_f^{k-1}\in V^h$ and $\psi_h=\xi_p^{k+1}+\xi_p^{k-1}\in X_p^h$ in the equations above and add to obtain:

$$\frac{1}{2\Delta t}\left(\|\xi_f^{k+1}\|_{DIV,f}^2+gS_0\|\xi_p^{k+1}\|_p^2+\Delta t^2g^2C_{f,p}^2\|\xi_p^{k+1}\|_{H^1(\Omega_p)}^2\right)$$
$$-\frac{1}{2\Delta t}\left(\|\xi_f^{k-1}\|_{DIV,f}^2+gS_0\|\xi_p^{k-1}\|_p^2+\Delta t^2g^2C_{f,p}^2\|\xi_p^{k-1}\|_{H^1(\Omega_p)}^2\right)$$
$$+\left[c_I(\xi_f^{k+1}+\xi_f^{k-1},\xi_p^k)-c_I(\xi_f^k,\xi_p^{k+1}+\xi_p^{k-1})\right]$$
$$+\frac{1}{2}\left[a_f(\xi_f^{k+1}+\xi_f^{k-1},\xi_f^{k+1}+\xi_f^{k-1})+a_p(\xi_p^{k+1}+\xi_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1})\right]$$
$$=-\frac{1}{2\Delta t}\left[\left(\eta_f^{k+1}-\eta_f^{k-1},\xi_f^{k+1}+\xi_f^{k-1}\right)_f+\left(\nabla\cdot(\eta_f^{k+1}-\eta_f^{k-1}),\nabla\cdot(\xi_f^{k+1}-\xi_f^{k+1})\right)_f\right]$$
$$-\frac{1}{2\Delta t}[gS_0\left(\eta_p^{k+1}-\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1}\right)_p$$
$$+\Delta t g^2C_{f,p}^2(\nabla(\eta_p^{k+1}-\eta_p^{k-1}),\nabla(\xi_p^{k+1}+\xi_p^{k-1}))_p+\Delta t g^2C_{f,p}^2(\eta_p^{k+1}-\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1})_p]$$
$$-\frac{1}{2}\left[a_f\left(\eta_f^{k+1}+\eta_f^{k-1},\xi_f^{k+1}+\xi_f^{k-1}\right)+a_p\left(\eta_p^{k+1}+\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1}\right)\right]$$
$$-\left[c_I(\xi_f^{k+1}+\xi_f^{k-1},\eta_p^k)-c_I(\eta_f^k,\xi_p^{k+1}+\xi_p^{k-1})\right]$$
$$+\varepsilon_f^k(\xi_f^{k+1}+\xi_f^{k-1})+\left(p^k-\lambda_h^k,\nabla\cdot(\xi_f^{k+1}+\xi_f^{k-1})\right)_f+\varepsilon_p^k(\xi_p^{k+1}+\xi_p^{k-1}).$$

Rewrite the coupling terms on the left hand side equivalently as follows:

$$c_I(\xi_f^{k+1}+\xi_f^{k-1},\xi_p^k)-c_I(\xi_f^k,\xi_p^{k+1}+\xi_p^{k-1})$$
$$=\left(c_I(\xi_f^{k+1},\xi_p^k)-c_I(\xi_f^k,\xi_p^{k+1})\right)-\left(c_I(\xi_f^k,\xi_p^{k-1})-c_I(\xi_f^{k-1},\xi_p^k)\right)$$
$$=C_\xi^{k+\frac{1}{2}}-C_\xi^{k-\frac{1}{2}}.$$

If we denote the ξ energy terms by

$$E_\xi^{k+1/2} := \|\xi_f^{k+1}\|^2_{DIV,f} + gS_0\|\xi_p^{k+1}\|_p^2 + \Delta t^2 g^2 C_{f,p}^2 \|\xi_p^{k+1}\|^2_{H^1(\Omega_p)} + \|\xi_f^{k}\|^2_{DIV,f} + gS_0\|\xi_p^{k}\|_p^2 + \Delta t^2 g^2 C_{f,p}^2 \|\xi_p^{k}\|^2_{H^1(\Omega_p)}$$

and also apply the coercivity of the forms $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$, the inequality becomes

$$\begin{aligned}
&E_\xi^{k+1/2} + 2\Delta t C_\xi^{k+\frac{1}{2}} - E_\xi^{k-1/2} - 2\Delta t C_\xi^{k-\frac{1}{2}} \\
&\quad + \Delta t \left(\nu\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2 + gk_{min}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1}\|_p^2\right) \\
&\leq -\left[\left(\eta_f^{k+1}-\eta_f^{k-1}, \xi_f^{k+1}+\xi_f^{k-1}\right)_f + \left(\nabla\cdot\left(\eta_f^{k+1}-\eta_f^{k-1}\right), \nabla\cdot\left(\xi_f^{k+1}+\xi_f^{k-1}\right)\right)_f\right] \\
&- [gS_0\left(\eta_p^{k+1}-\eta_p^{k-1}, \xi_p^{k+1}+\xi_p^{k-1}\right)_p + 2\Delta t^2 g^2 C_{f,p}^2 (\nabla(\eta_p^{k+1}-\eta_p^{k-1}), \nabla(\xi_p^{k+1}+\xi_p^{k-1}))_p \\
&\quad + 2\Delta t^2 g^2 C_{f,p}^2 (\eta_p^{k+1}-\eta_p^{k-1}, \xi_p^{k+1}+\xi_p^{k-1})_p] \\
&\quad - \Delta t\left[a_f\left(\eta_f^{k+1}+\eta_f^{k-1}, \xi_f^{k+1}+\xi_f^{k-1}\right) + a_p\left(\eta_p^{k+1}+\eta_p^{k-1}, \xi_p^{k+1}+\xi_p^{k-1}\right)\right] \\
&\quad - 2\Delta t\left[c_I(\xi_f^{k+1}+\xi_f^{k-1}, \eta_p^k) - c_I(\eta_f^k, \xi_p^{k+1}+\xi_p^{k-1})\right] \\
&\quad + 2\Delta t\left[\varepsilon_f^k(\xi_f^{k+1}+\xi_f^{k-1}) + (p^k-\lambda_h^k, \nabla\cdot(\xi_f^{k+1}+\xi_f^{k-1}))_f + \varepsilon_p^k(\xi_p^{k+1}+\xi_p^{k-1})\right],
\end{aligned} \tag{38}$$

where we multiplied through by $2\Delta t$. Next, we bound each term on the right hand side of the above inequality. We bound the first two terms by the standard Cauchy-Schwarz and Young inequalities along with the Poincaré inequality (6) and inequality (8).

$$\begin{aligned}
&\left(\eta_f^{k+1}-\eta_f^{k-1}, \xi_f^{k+1}+\xi_f^{k-1}\right)_f + \left(\nabla\cdot\left(\eta_f^{k+1}-\eta_f^{k-1}\right), \nabla\cdot\left(\xi_f^{k+1}+\xi_f^{k-1}\right)\right)_f \\
&\leq \frac{6C_{P,f}^2}{\nu\Delta t}\|\eta_f^{k+1}-\eta_f^{k-1}\|_f^2 + \frac{6d^2}{\nu\Delta t}\|\nabla\left(\eta_f^{k+1}-\eta_f^{k-1}\right)\|_f^2 + \Delta t\frac{\nu}{12}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2 \ ,
\end{aligned}$$

$$
\begin{aligned}
gS_0(\eta_p^{k+1}-\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1})_p &+ 2\Delta t^2 g^2 C_{f,p}^2 \left(\nabla\left(\eta_p^{k+1}-\eta_p^{k-1}\right),\nabla\left(\xi_p^{k+1}+\xi_p^{k-1}\right)\right)_p \\
&+ 2\Delta t^2 g^2 C_{f,p}^2 \left(\eta_p^{k+1}-\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1}\right)_p \\
&\le \frac{15 g C_{P,p}^2}{2k_{min}\Delta t}\left(S_0^2+4\Delta t^4 g^2 C_{f,p}^4\right)\|\eta_p^{k+1}-\eta_p^{k-1}\|_p^2 \\
&\quad + \frac{30\Delta t^3 g^3 C_{f,p}^4}{k_{min}}\|\nabla\left(\eta_p^{k+1}-\eta_p^{k-1}\right)\|_p^2 + \Delta t\frac{gk_{min}}{10}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2.
\end{aligned}
$$

To bound the second term, we apply the continuity of the bilinear forms $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$. Letting $M=\nu+\alpha C k_{min}^{-1/2}$ gives:

$$
\begin{aligned}
&a_f(\eta_f^{k+1}+\eta_f^{k-1},\xi_f^{k+1}+\xi_f^{k-1}) + a_p(\eta_p^{k+1}+\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1}) \\
&\le M\|\nabla(\eta_f^{k+1}+\eta_f^{k-1})\|_f\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f + gk_{max}\|\nabla(\eta_p^{k+1}+\eta_p^{k-1})\|_p\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p \\
&\le \frac{3M^2}{\nu}\|\nabla(\eta_f^{k+1}+\eta_f^{k-1})\|_f^2 + \frac{5gk_{max}^2}{2k_{min}}\|\nabla(\eta_p^{k+1}+\eta_p^{k-1})\|_p^2 \\
&\quad + \frac{\nu}{12}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2 + \frac{gk_{min}}{10}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2.
\end{aligned}
$$

We bound the coupling terms on the right hand side using the trace (7), Poincaré (6) and Young inequalities. Letting $C=C_{\Omega_f}^2 C_{\Omega_p}^2 C_{P,f}C_{P,p}g^2$, this yields

$$
\begin{aligned}
&c_I(\xi_f^{k+1}+\xi_f^{k-1},\eta_p^k) - c_I(\eta_f^k,\xi_p^{k+1}+\xi_p^{k-1}) \\
&\le g\left(\|(\xi_f^{k+1}+\xi_f^{k-1})\cdot\hat{n}_f\|_I\|\eta_p^k\|_I + \|\eta_f^k\cdot\hat{n}_f\|_I\|\xi_p^{k+1}+\xi_p^{k-1}\|_I\right) \\
&\le C_{\Omega_f}C_{\Omega_p}g\left(\|\xi_f^{k+1}+\xi_f^{k-1}\|_f^{1/2}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^{1/2}\|\eta_p^k\|_p^{1/2}\|\nabla\eta_p^k\|_p^{1/2}\right) \\
&\quad + \left(\|\xi_p^{k+1}+\xi_p^{k-1}\|_p^{1/2}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^{1/2}\|\eta_f^k\|_f^{1/2}\|\nabla\eta_f^k\|_f^{1/2}\right) \\
&\le \sqrt{C}\left(\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f\|\nabla\eta_p^k\|_p + \|\nabla\eta_f^k\|_f\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p\right) \\
&\le \frac{6C}{\nu}\|\nabla\eta_f^k\|_f^2 + \frac{5C}{gk_{min}}\|\nabla\eta_p^k\|_p^2 + \frac{\nu}{24}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2 + \frac{gk_{min}}{20}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2.
\end{aligned}
$$

Finally, we bound the consistency errors, ε_f^k and ε_p^k, and the pressure term as follows. We use the Cauchy-Schwarz, Young and Poincaré (6) inequalities as well as inequality (8).

$$\begin{aligned}
\varepsilon_f^k(\xi_f^{k+1}+\xi_f^{k-1}) &= -\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}, \xi_f^{k+1}+\xi_f^{k-1}\right) \\
&\quad - a_f\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}, \xi_f^{k+1}+\xi_f^{k-1}\right) \\
&\quad - \left(\nabla\cdot\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right), \nabla\cdot\left(\xi_f^{k+1}+\xi_f^{k-1}\right)\right) \\
&\le C_{P,f}\left(\left\|u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f + d\left\|\nabla\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right)\right\|_f\right. \\
&\quad \left. + M\left\|\nabla\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f\right)\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f \\
&\le \frac{9C_{P,f}^2}{\nu}\left\|u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f^2 + \frac{9M^2}{\nu}\left\|\nabla\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f^2 \\
&\quad + \frac{9d^2}{\nu}\left\|\nabla\left(u_t^k - \frac{u^{k+1}+u^{k-1}}{2\Delta t}\right)\right\|_f^2 + \frac{\nu}{12}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2,
\end{aligned}$$

$$\begin{aligned}
\varepsilon_p^k(\xi_p^{k+1}+\xi_p^{k-1}) &= -gS_0\left(\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}, \xi_p^{k+1}+\xi_p^{k-1}\right)_p \\
&\quad + \Delta t g^2 C_{f,p}^2(\nabla(\phi^{k+1}-\phi^{k-1}), \nabla(\xi_p^{k+1}+\xi_p^{k-1}))_p \\
&+ \Delta t g^2 C_{f,p}^2(\phi^{k+1}-\phi^{k-1}, \xi_p^{k+1}+\xi_p^{k-1})_p - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}, \xi_p^{k+1}+\xi_p^{k-1}\right) \\
&\le \left(C_{P,p}\left\|\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p + gk_{max}\left\|\nabla\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p\right. \\
&\quad \left. + \Delta t g^2 C_{f,p}^2\left(\|\nabla(\phi^{k+1}-\phi^{k-1})\|_p + C_{P,p}\|\phi^{k+1}-\phi^{k-1}\|_p\right)\right)\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p \\
&\le \frac{10gS_0^2C_{P,p}^2}{k_{min}}\left\|\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p^2 + \frac{10\Delta t^2 g^3 C_{f,p}^4}{k_{min}}\|\nabla(\phi^{k+1}-\phi^{k-1})\|_p^2 \\
&\quad + \frac{10\Delta t^2 g^3 C_{f,p}^4 C_{P,p}^2}{k_{min}}\|\phi^{k+1}-\phi^{k-1}\|_p^2 + \frac{10gk_{max}}{k_{min}}\left\|\nabla\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p^2 \\
&\quad + \frac{gk_{min}}{10}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2,
\end{aligned}$$

$$\begin{aligned}
\left(p^k - \lambda_h^k, \nabla\cdot(\xi_f^{k+1}+\xi_f^{k-1})\right)_f &\le \|p^k-\lambda_h^k\|_f\|\nabla\cdot(\xi_f^{k+1}+\xi_f^{k-1})\|_f \\
&\le \frac{6d}{\nu}\|p^k-\lambda_h^k\|_f^2 + \frac{\nu}{24}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2.
\end{aligned}$$

After absorbing all the resulting ξ terms into the left hand side of inequality (38) and grouping together the remaining terms, the inequality becomes

$$
\begin{aligned}
&E_\xi^{k+\frac{1}{2}} + 2\Delta t C_\xi^{k+\frac{1}{2}} - E_\xi^{k-\frac{1}{2}} - 2\Delta t C_\xi^{k-\frac{1}{2}} \\
&\quad + \Delta t \left(\frac{\nu}{2} \|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2 + \frac{g k_{min}}{2} \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2 \right) \\
&\leq (\Delta t)^{-1} \Bigg\{ \frac{6C_{P,f}^2}{\nu} \|\eta_f^{k+1} - \eta_f^{k-1}\|_f^2 + \frac{15 g C_{P,p}^2}{2k_{min}} \left(S_0^2 + 4\Delta t^4 g^2 C_{f,p}^4 \right) \|\eta_p^{k+1} - \eta_p^{k-1}\|_p^2 \\
&\qquad\qquad + \frac{6d^2}{\nu} \|\nabla \left(\eta_f^{k+1} - \eta_f^{k-1} \right) \|_f^2 \Bigg\} \\
&+ \Delta t \Bigg\{ \frac{30\Delta t^2 g^3 C_{f,p}^4}{k_{min}} \|\nabla \left(\eta_p^{k+1} - \eta_p^{k-1} \right) \|_p^2 + \frac{3M^2}{\nu} \|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f^2 \\
&\qquad + \frac{5 g k_{max}^2}{2k_{min}} \|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_p^2 + \frac{12C}{\nu} \|\nabla \eta_f^k\|_f^2 + \frac{10C}{g k_{min}} \|\nabla \eta_p^k\|_p^2 \\
&\quad + \frac{18C_{P,f}^2}{\nu} \left\| u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 + \frac{18M^2}{\nu} \left\| \nabla \left(u^k - \frac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 \\
&\quad + \frac{18d^2}{\nu} \left\| \nabla \left(u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right) \right\|_f^2 + \frac{12d}{\nu} \|p^k - \lambda_h^k\|_f^2 \\
&\quad + \frac{20 g S_0^2 C_{P,p}^2}{k_{min}} \left\| \phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t} \right\|_p^2 + \frac{20\Delta t^2 g^3 C_{f,p}^4}{k_{min}} \|\nabla(\phi^{k+1} - \phi^{k-1})\|_p^2 \\
&\quad + \frac{20\Delta t^2 g^3 C_{f,p}^4 C_{P,p}^2}{k_{min}} \|\phi^{k+1} - \phi^{k-1}\|_p^2 + \frac{20 g k_{max}^2}{k_{min}} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 \Bigg\}.
\end{aligned}
$$

Now, we sum this inequality from $k = 1, ..., N-1$. This yields

$$
\begin{aligned}
& E_\xi^{N-\frac{1}{2}} + 2\Delta t C_\xi^{N-\frac{1}{2}} - E_\xi^{\frac{1}{2}} - 2\Delta t C_\xi^{\frac{1}{2}} \\
& \quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2} \|\nabla \left(\xi_f^{k+1} + \xi_f^{k-1} \right) \|_f^2 + \frac{g k_{min}}{2} \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2 \right) \\
& \le (\Delta t)^{-1} \sum_{k=1}^{N-1} \Bigg\{ \frac{6C_{P,f}^2}{\nu} \|\eta_f^{k+1} - \eta_f^{k-1}\|_f^2 + \frac{15gC_{P,p}^2}{2k_{min}} \left(S_0^2 + 4\Delta t^4 g^2 C_{f,p}^4 \right) \|\eta_p^{k+1} - \eta_p^{k-1}\|_p^2 \\
& \qquad\qquad + \frac{6d^2}{\nu} \|\nabla \left(\eta_f^{k+1} - \eta_f^{k-1} \right) \|_f^2 \Bigg\} \\
& \quad + \Delta t \sum_{k=1}^{N-1} \Bigg\{ \frac{30\Delta t^2 g^3 C_{f,p}^4}{k_{min}} \|\nabla \left(\eta_p^{k+1} - \eta_p^{k-1} \right) \|_p^2 + \frac{3M^2}{\nu} \|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f^2 \\
& \qquad + \frac{5gk_{max}^2}{2k_{min}} \|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_p^2 + \frac{12C}{\nu} \|\nabla \eta_f^k\|_f^2 + \frac{10C}{gk_{min}} \|\nabla \eta_p^k\|_p^2 \\
& \quad + \frac{18C_{P,f}^2}{\nu} \left\| u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 + \frac{18M^2}{\nu} \left\| \nabla \left(u^k - \frac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 \\
& \quad + \frac{18d^2}{\nu} \left\| \nabla \left(u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right) \right\|_f^2 + \frac{12d}{\nu} \|p^k - \lambda_h^k\|_f^2 \\
& \quad + \frac{20gS_0^2 C_{P,p}^2}{k_{min}} \left\| \phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t} \right\|_p^2 + \frac{20\Delta t^2 g^3 C_{f,p}^4}{k_{min}} \|\nabla(\phi^{k+1} - \phi^{k-1})\|_p^2 \\
& \quad + \frac{20\Delta t^2 g^3 C_{f,p}^4 C_{P,p}^2}{k_{min}} \|\phi^{k+1} - \phi^{k-1}\|_p^2 + \frac{20gk_{max}^2}{k_{min}} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 \Bigg\}.
\end{aligned}
$$

To obtain a bound involving norms instead of summations, we use the Cauchy-Schwarz and other basic inequalities to bound each term on the right hand side as follows. For the first term, we have:

$$
\begin{aligned}
\sum_{k=1}^{N-1} \|\eta_f^{k+1} - \eta_f^{k-1}\|_f^2 &= \sum_{k=1}^{N-1} \left\| \int_{t^{k-1}}^{t^{k+1}} \eta_{f,t} dt \right\|_f^2 \\
&\le \sum_{k=1}^{N-1} \int_{\Omega_f} (2\Delta t) \int_{t^{k-1}}^{t^{k+1}} |\eta_{f,t}|^2 dt \quad dx \\
&\le 4\Delta t \|\eta_{f,t}\|_{L^2(0,T;L^2(\Omega_f))}^2. \qquad (39)
\end{aligned}
$$

Likewise, we treat the second term,

$$
\sum_{k=1}^{N-1} \|\eta_p^{k+1} - \eta_p^{k-1}\|_f^2 \le 4\Delta t \|\eta_{p,t}\|_{L^2(0,T;L^2(\Omega_p))}^2. \qquad (40)
$$

In a similar manner we bound the third and fourth terms.

$$\sum_{k=1}^{N-1} \|\nabla \left(\eta_f^{k+1} - \eta_f^{k-1}\right)\|_f^2 \leq 4\Delta t \|\nabla \eta_{f,t}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{41}$$

$$\sum_{k=1}^{N-1} \|\nabla \left(\eta_p^{k+1} - \eta_p^{k-1}\right)\|_p^2 \leq 4\Delta t \|\nabla \eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} \ . \tag{42}$$

Inequalities (39) and (41) imply the following.

$$\sum_{k=1}^{N-1} \left\{\|\eta_f^{k+1} - \eta_f^{k-1}\|_f^2 + \|\nabla \left(\eta_f^{k+1} - \eta_f^{k-1}\right)\|_f^2\right\} \leq 4\Delta t \|\eta_{f,t}\|^2_{L^2(0,T;H^1(\Omega_f))} \tag{43}$$

The rest of the η terms are bounded using Cauchy-Schwarz and the discrete norms.

$$\begin{aligned}\sum_{k=1}^{N-1} \|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f^2 &\leq 2\sum_{k=1}^{N-1}\left(\|\nabla \eta_f^{k+1}\|_f^2 + \|\nabla \eta_f^{k-1}\|_f^2\right) \\ &\leq 4\sum_{k=0}^{N} \|\nabla \eta_f^k\|_f^2 \leq 4(\Delta t)^{-1} \||\nabla \eta_f|\|^2_{L^2(0,T;L^2(\Omega_f))},\end{aligned} \tag{44}$$

$$\sum_{k=1}^{N-1} \|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_f^2 \leq 4(\Delta t)^{-1} \||\nabla \eta_p|\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{45}$$

$$\sum_{k=1}^{N-1} \|\nabla \eta_f^k\|_f^2 \leq (\Delta t)^{-1} \||\nabla \eta_f|\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{46}$$

$$\sum_{k=1}^{N-1} \|\nabla \eta_p^k\|_p^2 \leq (\Delta t)^{-1} \||\nabla \eta_p|\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{47}$$

$$\sum_{k=1}^{N-1} \|p^k - \lambda_h^k\|_f^2 \leq (\Delta t)^{-1} \||p - \lambda_h|\|^2_{L^2(0,T;L^2(\Omega_f))}. \tag{48}$$

After applying bounds (39)-(48), along with (29)-(34), and the bound (26) from the stability proof, and after absorbing all the constants into one constant, $\widehat{C}_1$, the inequality becomes

$$
\begin{aligned}
&\frac{1}{2}(\|\xi_f^N\|^2_{DIV,f} + \|\xi_f^{N-1}\|^2_{DIV,f}) + gS_0(\|\xi_p^N\|^2_p + \|\xi_p^{N-1}\|^2_p) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2} \|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|^2_f + \frac{gk_{min}}{2} \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|^2_p \right) \\
&\quad \leq \widehat{C}_1 \Big\{ \|\eta_{f,t}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + \Delta t^4 \|\nabla \eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\quad\quad + |||\nabla \eta_f|||^2_{L^2(0,T;L^2(\Omega_f))} + |||\nabla \eta_p|||^2_{L^2(0,T;L^2(\Omega_p))} + \Delta t^4 \Big(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} \\
&\quad\quad + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} + \|\phi_t\|^2_{L^2(0,T,H^1(\Omega_p))} \\
&\quad\quad + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big) + |||p - \lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} \Big\} + E_\xi^{1/2} + 2\Delta t C_\xi^{1/2}.
\end{aligned}
\tag{49}
$$

Recall that the error terms equal $e_f^N = u^N - u_h^N = \eta_f^N + \xi_f^N$ and $e_p^N = \phi^N - \phi_h^N = \eta_p^N + \xi_p^N$. Applying the triangle inequality we have

$$
\begin{aligned}
&\frac{1}{4}(\|e^N{}_f\|^2_{DIV,f} + \|e_f^{N-1}\|^2_{DIV,f}) + \frac{gS_0}{2}(\|e_p^N\|^2_p + \|e_p^{N-1}\|^2_p) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{4} \|\nabla(e_f^{k+1} + e_f^{k-1})\|^2_f + \frac{gk_{min}}{4} \|\nabla(e_p^{k+1} + e_p^{k-1})\|^2_p \right) \\
&\quad \leq \frac{1}{2}(\|\xi_f^N\|^2_{DIV,f} + \|\xi_f^{N-1}\|^2_{DIV,f}) + gS_0(\|\xi_p^N\|^2_p + \|\xi_p^{N-1}\|^2_p) \\
&\quad\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2} \|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|^2_f + \frac{gk_{min}}{2} \|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|^2_p \right) \\
&\quad\quad + \frac{1}{2}(\|\eta_f^N\|^2_{DIV,f} + \|\eta_f^{N-1}\|^2_{DIV,f}) + gS_0(\|\eta_p^N\|^2_p + \|\eta_p^{N-1}\|^2_p) \\
&\quad\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2} \|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|^2_f + \frac{gk_{min}}{2} \|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|^2_p \right).
\end{aligned}
$$

Notice that $\|\eta_{f,p}^N\|^2_{f,p}, \|\eta_{f,p}^{N-1}\|^2_{f,p} \leq |||\eta_{f,p}|||^2_{L^\infty(0,T;L^2(\Omega_{f,p}))}$ and therefore $\|\eta_f^N\|^2_{DIV,f} \leq d|||\eta_f|||^2_{L^\infty(0,T;H^1(\Omega_f))}$. This fact, together with the previous bounds for η terms and inequality (49) result in

$$
\begin{aligned}
&\frac{1}{4}(\|e^N{}_f\|^2_{DIV,f} + \|e_f^{N-1}\|^2_{DIV,f}) + \frac{gS_0}{2}(\|e_p^N\|_p^2 + \|e_p^{N-1}\|_p^2) \\
&\qquad + \Delta t \sum_{k=1}^{N-1}\left(\frac{\nu}{4}\|\nabla(e_f^{k+1}+e_f^{k-1})\|_f^2 + \frac{gk_{min}}{4}\|\nabla(e_p^{k+1}+e_p^{k-1})\|_p^2\right) \\
&\qquad \leq \widehat{C}_2\Big\{\|\eta_{f,t}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad + \Delta t^4\|\nabla\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + |||\nabla\eta_f|||^2_{L^2(0,T;L^2(\Omega_f))} + |||\nabla\eta_p|||^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad + \Delta t^4\Big(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad + \|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}\Big) + |||p-\lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&\qquad + |||\eta_f|||^2_{L^\infty(0,T;H^1(\Omega_f))} + |||\eta_p|||^2_{L^\infty(0,T;L^2(\Omega_p))}\Big\} + \|\xi_f^1\|^2_{DIV,f} + \|\xi_f^0\|^2_{DIV,f} \\
&\qquad + gS_0(\|\xi_p^1\|_p^2 + \|\xi_p^0\|_p^2) + \Delta t^2 g^2 C_{f,p}^2(\|\xi_p^1\|^2_{1,p} + \|\xi_p^0\|^2_{1,p}) + 2\Delta t C_\xi^{1/2},
\end{aligned}
\tag{50}
$$

where we absorbed all constants into a new constant, $\widehat{C}_2 > 0$. Now, we bound the coupling terms on the right hand side as follows:

$$
C_\xi^{1/2} \leq \frac{C}{2}\left(\|\xi_p^0\|^2_{1,p} + \|\xi_p^1\|^2_{1,p} + \|\xi_f^0\|^2_{DIV,f} + \|\xi_f^1\|^2_{DIV,f}\right). \tag{51}
$$

Inequality (50) holds for any $\tilde{u} \in V^h$, $\lambda_h \in Q_f^h$, and $\tilde{\phi} \in X_p^h$. Taking the infimum over the spaces V^h, Q_f^h, and X_p^h, using (28) to bound the infimum over V^h by the infimum over X_f^h and using bound (51), we have the following for some positive constant $\widehat{C}_3$:

$$
\begin{aligned}
\frac{1}{2}(\|e^N{}_f\|^2_{DIV,f} + \|e_f^{N-1}\|^2_{DIV,f}) + gS_0(\|e_p^N\|_p^2 + \|e_p^{N-1}\|_p^2) \\
+ \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2} \|\nabla(e_f^{k+1} + e_f^{k-1})\|_f^2 + \frac{gk_{min}}{2} \|\nabla(e_p^{k+1} + e_p^{k-1})\|_p^2 \right) \\
\leq \widehat{C}_3 \Bigg\{ \inf_{\tilde{u} \in X_f^h} \{ \|\eta_{f,t}\|^2_{L^2(0,T;H^1(\Omega_f))} + |||\nabla \eta_f|||^2_{L^2(0,T;L^2(\Omega_f))} \\
+ |||\eta_f|||^2_{L^\infty(0,T;H^1(\Omega_f))} + \|\xi_f^1\|^2_{DIV,f} + \|\xi_f^0\|^2_{DIV,f} \} \\
+ \inf_{\lambda_h \in Q_f^h} |||p - \lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} + \inf_{\tilde{\phi} \in X_p^h} \{ \|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
+ \Delta t^4 \|\nabla \eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + |||\nabla \eta_p|||^2_{L^2(0,T;L^2(\Omega_p))} + |||\eta_p|||^2_{L^\infty(0,T;L^2(\Omega_p))} \\
+ \|\xi_p^1\|^2_{1,p} + \|\xi_p^0\|^2_{1,p} \} + \Delta t^4 \Big(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} \\
+ \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} + \|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big) \Bigg\}.
\end{aligned}
$$

The result of the theorem now immediately follows by applying the approximation assumptions given in (27). □

Corollary 7. *Under the same regularity conditions as in Theorem 6, the temporal growth of the error is at most*

$$\|e_f^N\|_{DIV,f}, \|e_p^N\|_p = \mathcal{O}(\sqrt{t_N}).$$

Proof. For any function $v : [0, \infty) \to X$ and any spatial norm $\|\cdot\|_X$ we have

$$\int_0^{t_N} \|v\|_X^2 \, dt \leq t_N \|v\|^2_{L^\infty(0,\infty;X)},$$

for any $0 < T \leq \infty$. Similarly for the discrete norms we have

$$\sum_{k=1}^N \|v^k\|_X^2 \Delta t \leq \|v\|^2_{L^\infty(0,\infty;X)} \sum_{k=1}^N \Delta t = t_N \|v\|^2_{L^\infty(0,\infty;X)}.$$

Applying the above to the terms on the RHS of (37) gives the claim of the Corollary.

□

5. Numerical tests

We verify the method's unconditional stability and rate of convergence in a series of numerical tests. For these experiments we use the exact solutions introduced by Mu and Zhu in [16], recalled next. All experiments were conducted using FreeFEM++ [19].

$$\Omega_f = (0,1)\times(1,2), \qquad \Omega_p = (0,1)\times(0,1), \qquad I = \{(x,1) : x\in(0,1)\}$$

$$u(x,y,t) = \Big((x^2(y-1)^2+y)\cos(t), (\frac{2}{3}x(1-y)^3+2-\pi\sin(\pi x))\cos(t)\Big)$$

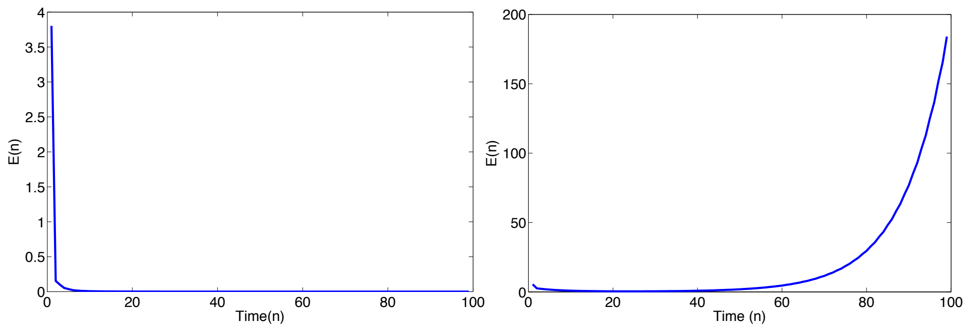
$$p(x,y,t) = (2-\pi\sin(\pi x))\sin(\frac{\pi}{2}y)\cos(t)$$

$$\phi(x,y,t) = (2-\pi\sin(\pi x))(1-y-\cos(\pi y))\cos(t)$$

To confirm unconditional stability of the CNLF-stab method we set the body force and source functions, f_f and f_p equal to zero. We also enforce homogeneous Dirichlet boundary conditions, except along the interface I.

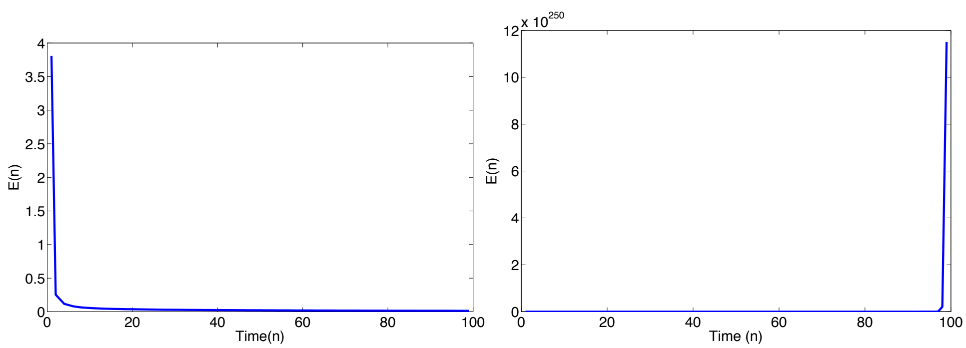
5.1. Test 1 - Unconditional Stability (CNLF's CFL (2) violated). We set $h=\Delta t=0.1$. We calculate the energy of the system over the time interval $[0,10]$. In Figure 2 we take $S_0=0.1$ and $k_{min}=1.0$, while in Figure 3 we consider the case of a confined aquifer and set $S_0=10^{-6}$ and $k_{min}=10^{-4}$. The values for S_0, Δt and h in both cases violate the stability condition (2) for original CNLF. The energy of CNLF-stab decays to zero over time, as expected, while CNLF blows up in both cases. In particular, CNLF experiences a drastic increase in system energy in the second case (Figure 3b).

5.2. Test 2 - Control of Unstable Mode (CFL (2) holds). We test the effect of CNLF-stab on the unstable mode of Leap-Frog, given by $\|w_h^{n+1}-w_h^{n-1}\|_{f,p}^2$ for $w=u,\phi$, in Figures 4 and 5. For these tests we set $h=\Delta t=0.05$ and further $S_0=1.0$ and $k_{min}=0.1$. In [13], it was shown that decreasing the value of k_{min} from 1.0 to 0.1 led to instability even though condition (2) holds. Numerical tests showed the sudden rise in energy corresponded to spurious oscillations in the unstable modes. We calculate these unstable modes in both the Stokes velocity and Darcy pressure



(A) CNLF-stab (B) CNLF

FIGURE 2. Energy with $S_0 = 0.1, k_{min} = 0.1, h = \Delta t = 0.1, T = 10 \Longrightarrow$ (2) violated



(A) CNLF-stab (B) CNLF

FIGURE 3. Energy with $S_0 = 10^{-6}, k_{min} = 10^{-4}, h = \Delta t = 0.1, T = 10 \Longrightarrow$ (2) violated

variables and compare them to the total energy of the method at each time step. Notice in Figure 4 that the energy and unstable modes of CNLF-stab decay to zero, while we observe a rise in the energy for CNLF in Figure 5 corresponding to oscillations in the unstable modes. Therefore, numerical tests indicate that CNLF-stab damps the unstable mode of Leap-Frog. Theoretical verification that CNLF-stab does control the unstable mode is an open question.

5.3. Test 3 - Convergence Rate Verification. We next test the convergence rate of the CNLF-stab method. We set the parameters α, ν, S_0, $\mathcal{K}$, g, equal to 1 and apply inhomogeneous Dirichlet external boundary conditions: $u_h = u$ on $\Omega_f \setminus I$, $\phi_h = \phi$ on $\Omega_p \setminus I$. We chose the initial conditions, as well as the first terms in the method, to match the exact solutions. We set $h = \Delta t$ and calculate the

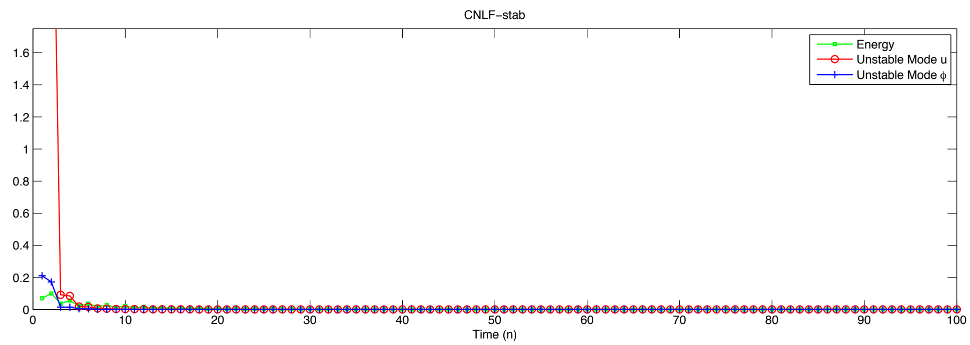


FIGURE 4. CNLF-stab Energy and Unstable Modes with $S_0 = 1.0, k_{min} = 0.1, h = \Delta t = 0.05, T = 5$.

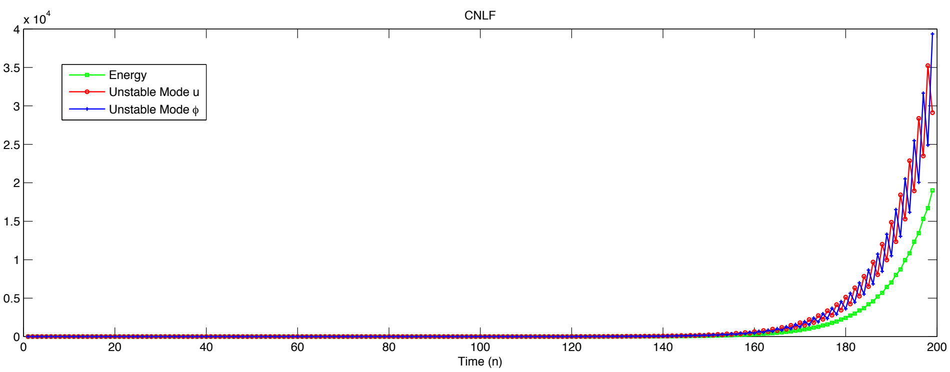


FIGURE 5. CNLF Energy and Unstable Modes with $S_0 = 1.0, k_{min} = 0.1, h = \Delta t = 0.05, T = 10$.

errors and convergence rates for the variables u, p, and ϕ in Table 3 over the time interval $[0, 10]$. Define the norms for the errors, $E(u)$, $E(p)$, and $E(\phi)$, as follows.

$$E(u) = |||u - u_h|||_{L^\infty(0,T;DIV(\Omega_f))},$$

$$E(p) = |||p - p_h|||_{L^\infty(0,T;L^2(\Omega_f))},$$

$$E(\phi) = |||\phi - \phi_h|||_{L^\infty(0,T;L^2(\Omega_p))}.$$

We let $r_{u,\phi}$ denote the calculated rate of convergence. As expected, we have second order convergence for the Stokes velocity, u, and Darcy pressure, ϕ. However, we do not have second order convergence for the Stokes pressure, p. A further numerical investigation of this effect on the convergence of the Stokes pressure follows.

$h=\Delta t$	$E(u)$	r_u	$E(\phi)$	r_ϕ	$E(p)$
1/4	6.98×10^{-2}		1.54×10^{-1}		2.65
1/8	1.58×10^{-2}	2.14	3.85×10^{-2}	2.00	2.56
1/16	4.15×10^{-3}	1.93	9.62×10^{-3}	2.00	2.35
1/32	1.12×10^{-3}	1.89	2.40×10^{-3}	2.00	1.34

TABLE 3. Errors for CNLF+Stab

5.4. Test 4 - An Anomaly in the Stokes Pressure. We examing the lack of second order convergence for the Stokes pressure p in the above table by conducting a series of tests to isolate the anomaly. We ran the same convergence tests to analyze the errors in regular CNLF in Table 4, CNLF with added Grad-Div stabilization terms in the Stokes equation (CNLF-GradDiv) in Table 5, and CNLF with the added $O(\Delta t^2)$ stabilization terms in the Darcy equation (CNLF-StabDarcy) in Table 6. The Stokes pressure is second order convergent in both CNLF and CNLF-StabDarcy. In tests for CNLF-GradDiv, we obtained the same results for the Stokes pressure error as we found for CNLF-stab. This suggests that the added Grad-Div stabilization term in the Stokes equation adversely affects the convergence rate of the Stokes pressure.

Surprisingly, when we calculated $|||\nabla\cdot u_h|||_{L^2(0,T;L^2(\Omega_f))}$, $|||\nabla\cdot u_{h,t}|||^2_{L^2(0,T;L^2(\Omega_f))} = \sum_{k=1}^{N-1}\|\nabla\cdot\frac{u_f^{n+1}-u_f^{n-1}}{2\Delta t}\|_f^2$, and $|||p_h|||_{L^2(0,T;L^2(\Omega_f))}$ for CNLF-stab in Table 7, we see that $|||\nabla\cdot u_h|||_{L^2(0,T;L^2(\Omega_f))}$ and $|||\nabla\cdot u_{h,t}|||_{L^2(0,T;L^2(\Omega_f))}$ converge to zero while the discrete norm of the pressure is still stabilizing. We obtained the same numerical results for CNLF-GradDiv. When we calculated those same norms for CNLF in Table 8 we see that $|||\nabla\cdot u_h|||_{L^2(0,T;L^2(\Omega_f))}$ and $|||\nabla\cdot u_{h,t}|||_{L^2(0,T;L^2(\Omega_f))}$ converge to zero and the discrete pressure norm converges to 1.56. We obtained similar results for CNLF-StabDarcy, given in Table 9. We have no theoretical explanation for this effect. It is another important open question.

$h=\Delta t$	$E(u)$	$E(\phi)$	$E(p)$
1/4	1.18×10^{-1}	5.27×10^{-2}	1.52
1/8	1.97×10^{-2}	1.12×10^{-2}	2.88×10^{-1}
1/16	4.84×10^{-3}	2.29×10^{-3}	5.96×10^{-2}
1/32	1.22×10^{-3}	5.72×10^{-4}	1.44×10^{-2}

TABLE 4. Errors for CNLF

$h = \Delta t$	$E(u)$	$E(\phi)$	$E(p)$
1/4	6.98×10^{-2}	5.27×10^{-2}	2.64
1/8	1.58×10^{-2}	1.12×10^{-2}	2.56
1/16	4.15×10^{-3}	2.29×10^{-3}	2.35
1/32	1.12×10^{-3}	5.72×10^{-4}	1.34

TABLE 5. Errors for CNLF-GradDiv

$h = \Delta t$	$E(u)$	$E(\phi)$	$E(p)$
1/4	8.41×10^{-2}	3.85×10^{-2}	1.02
1/8	1.91×10^{-2}	3.85×10^{-2}	2.56×10^{-1}
1/16	4.84×10^{-3}	9.62×10^{-3}	6.09×10^{-2}
1/32	1.22^{-3}	2.40×10^{-3}	1.45×10^{-2}

TABLE 6. Error for CNLF-StabDarcy

$h = \Delta t$	$\|\|\|\nabla \cdot u_h\|\|\|_f$	$\|\|\|\nabla \cdot u_{h,t}\|\|\|_f$	$\|\|\|p_h\|\|\|_f$
1/4	1.6×10^{-1}	2.2×10^{-1}	5.13
1/8	3.51×10^{-2}	5.51×10^{-3}	4.92
1/16	4.15×10^{-3}	2.35×10^{-3}	2.35
1/32	1.12×10^{-3}	9.55×10^{-4}	1.34

TABLE 7. Discrete Norms for CNLF-Stab

$h = \Delta t$	$\|\|\|\nabla \cdot u_h\|\|\|_f$	$\|\|\|\nabla \cdot u_{h,t}\|\|\|_f$	$\|\|\|p_h\|\|\|_f$
1/4	2.18×10^{-1}	5.78×10^{-1}	2.71
1/8	4.47×10^{-2}	2.14×10^{-1}	1.63
1/16	1.12×10^{-2}	1.02×10^{-1}	1.56
1/32	2.81×10^{-3}	4.24×10^{-2}	1.56

TABLE 8. Discrete norms for CNLF

$h = \Delta t$	$\|\|\|\nabla \cdot u\|\|\|_f$	$\|\|\|\nabla \cdot u_t\|\|\|_f$	$\|\|\|p\|\|\|$
1/4	1.98×10^{-1}	4.53×10^{-1}	2.40
1/8	4.41×10^{-2}	2.06×10^{-1}	1.62
1/16	1.11×10^{-2}	4.16×10^{-4}	1.56
1/32	1.22×10^{-3}	9.59×10^{-4}	1.56

TABLE 9. Discrete Norms for CNLF-StabDarcy

6. Conclusions

The added stabilization terms in the CNLF-stab method correct one of the two shortcomings of original CNLF, namely the conditional stability. Theoretical and numerical analysis of the CNLF-stab method showed that the stabilization maintains second order convergence in the Stokes velocity and Darcy pressure variables while eliminating the dependence on the specific storage parameter, S_0, for stability. Numerical tests suggest that the added stabilization terms dampen the effect of the unstable mode from Leap-Frog, in contrast to regular CNLF. Our tests reveal two important open theoretical questions: (1) Whether the added stabilization terms in CNLF-stab control the unstable mode of Leap-Frog, and (2) Why the fluid pressure p fails to be second order convergent with CNLF-stab.

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References

[1] J. Bear. *Dynamics of fluids in porous media*. Dover, 1988.

[2] G. Beavers and D. Joseph. Boundary conditions at a naturally impermeable wall. *Journal of Fluid Mechanics*, 30:197–207, 1967.

[3] S. Brenner and R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer, 3 edition, 2008.

[4] H. Brézis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext Series. Springer, 2010.

[5] M. Discacciati, E. Miglio, and A. Quarteroni. Mathematical and numerical models for coupling surface and groundwater flows. *Appl. Numer. Math.*, 43(1-2):57–74, 2002. 19th Dundee Biennial Conference on Numerical Analysis (2001).

[6] P. Domenico and M. Mifflin. Water from low-permeability sediments and land subsidence. *Water Resources Research*, 1(4):563–576, 1965.

[7] R. A. Freeze and J. A. Cherry. *Groundwater*. Prentice-Hall, 1979.

[8] V. Girault and P.A. Raviart. *Finite Element Approximation of the Navier-Stokes Equations*. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1979.

[9] M. S. Hantush. *Flow to an eccentric well in a leaky circular aquifer*. American Geophysical Union, 1960.

[10] W. Jager and A. Mikelic. On the boundary condition at the interface between a porous medium and a free fluid. *SIAM Journal of Applied Mathematics*, 60:1111–1127, 2000.

[11] A. I. Johnson. *Specific yield - compilation of specific yields for various materials*. Hydrologic properties of earth materials; Geological Survey water-supply paper. U.S. Gov. Print. Off., Washington D.C., 1967.

[12] I. Jones. Low Reynolds number flow past a porous spherical shell. *Mathematical Proceedings of the Cambridge Philosophical Society*, 73(01):231–238, 1973.

[13] M. Kubacki. Uncoupling groundwater-surface water flow using the Crank-Nicolson Leapfrog method. To appear in *Numerical Methods for Partial Differential Equations*, 2012.

[14] W. Layton, C. Trenchea, and H. Tran. Analysis of long time stability and errors of two partitioned methods for uncoupling evolutionary groundwater-surface water flows. *SINUM*, 2012.

[15] M. Moraiti. On the quasistatic approximation in the Stokes-Darcy model of groundwater-surface water flows. *JMAA*, 394:796–808, 2012.

[16] M. Mu and X. Zhu. Decoupled schemes for a non-stationary mixed Stokes-Darcy model. *Mathematics of Computation*, 79(270):707–731, 2010.

[17] L. Payne and B. Straughan. Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modeling questions. *Journal de Mathématiques Pures et Appliquées*, 77:317–354, 1998.

[18] P. Saffman. On the boundary condition at the interface of a porous medium. *Stud. Appl. Math.*, 1:93–101, 1971.

[19] FreeFEM++ software. www.freefem.org/ff++/index.htm.

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