The stabilized, extrapolated trapezoidal finite element method for the Navier-Stokes Equations

Alexandr Labovschii[5,6] William J. Layton[1,6]
Carolina C. Manica[2,6] Monika Neda[3,6] Leo G. Rebholz[4,6]
Department of Mathematics
University of Pittsburgh PA 15260

July 8, 2007

Abstract

We consider a fully discrete stabilized finite element method for the Navier-Stokes equations which is unconditionally stable and has second order temporal accuracy of $O(k^2+hk+$ spatial error$)$. The method involves a simple artificial viscosity stabilization of the linear system for the approximation of the new time level connected to anti-diffusion of its effects at the old time level. The method requires only the solution of one linear system (arising from an Oseen problem) per time step. The cell Reynolds number of this discrete linear Oseen problem is $O(1)$ and is thus amenable to standard iterative methods and preconditioners.

Keywords: Navier-Stokes equations, finite element discretization, artificial viscosity stabilization
AMS Subject Classification: 65M12, 65M60, 76D05, 76F65

1 Introduction

The accurate and reliable solution of fluid flow problems is important for many applications. In these one core problem is the Navier-Stokes equations, given by: find $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d\,(d = 2, 3),\ p : \Omega \times (0,T] \to \mathbb{R}$ satisfying

$$\begin{aligned}
&\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f},\ \text{for } \mathbf{x} \in \Omega,\ 0 < t \le T\\
&\nabla \cdot \mathbf{u} = 0,\ \mathbf{x} \in \Omega,\ \text{for } 0 \le t \le T,\\
&\mathbf{u} = 0,\ \text{on } \partial\Omega,\ \text{for } 0 < t \le T,\\
&\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}),\ \text{for } \mathbf{x} \in \Omega,
\end{aligned} \tag{1.1}$$

[1]wjl@pitt.edu, http://www.pitt.edu/~wjl
[2]cac15@pitt.edu, http://www.pitt.edu/~cac15
[3]mon5@pitt.edu, http://www.pitt.edu/~mon5
[4]ler6@math.pitt.edu, http://www.math.pitt.edu/~ler6
[5]ayl2@pitt.edu, http://www.pitt.edu/~ayl2
[6]Partially supported by NSF Grant DMS 0508260

with the usual normalization condition that $\int_\Omega p(\mathbf{x},t)\ d\mathbf{x} = 0$ for $0 < t \le T$ when (1.1) is discretized by accepted, accurate and stable methods, such as the finite element method in space and Crank-Nicolson in time, the approximation can still fail for many reasons. One common mode of failure is non-convergence of the iterative nonlinear and linear solvers used to compute the velocity and pressure at the new time levels. We consider herein a simple, second order accurate, and unconditionally stable method which addresses these failure modes. The method requires the solution of one *linear* system per time step.

This linear system is a discretized Oseen problem plus an $O(h)$ artificial viscosity operator - so the standard iterative solvers and well-tested preconditioners can be used successfully (the preconditioners are described, e.g., in chapter 8 of [ESW05]). Suppressing the spatial discretization, the method can be written as (with time step $k = \Delta t$ and tuning parameter $\alpha = O(1)$)

$$\nabla \cdot \mathbf{u}_{n+1} = 0 \text{ and}$$

$$\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{k} + \mathbf{U}_{n+1/2} \cdot \nabla(\frac{\mathbf{u}_{n+1} + \mathbf{u}_n}{2}) - \nu\Delta(\frac{\mathbf{u}_{n+1} + \mathbf{u}_n}{2}) - \alpha h \Delta \mathbf{u}_{n+1} + \nabla(\frac{p_{n+1} + p_n}{2}) = \mathbf{f}(t_{n+1/2}) - \alpha h \Delta \mathbf{u}_n. \tag{1.2}$$

Here $\mathbf{U}_{n+1/2} := \frac{3}{2}\mathbf{u}_n - \frac{1}{2}\mathbf{u}_{n-1}$ is the linear extrapolation of the velocity to $t_{n+1/2}$ from previous time levels. Thus, (1.2) is an extension of Baker's [B76] extrapolated Crank-Nicolson method. Artificial viscosity stabilization is introduced into the linear system for $\mathbf{u}_{n+1}$ by adding $-\alpha h \Delta \mathbf{u}_{n+1}$ to the LHS and correcting for it by $-\alpha h \Delta \mathbf{u}_n$ (the previous time level) on the RHS. This is a known idea[1] in practical CFD, and likely has been used in practical computations with many different timestepping methods. To our knowledge however, it has only been proven unconditionally stable in combination with first order, backward Euler time discretizations, e.g. E and Liu [EL01], Anitescu, Layton and Pahlevani [ALP04], Pahlevani [P06] for related stabilizations and also He[He02] for a two-level method based on Baker's extrapolated Crank-Nicolson method.

The increase in accuracy from first order Backward Euler with stabilization to second order in (1.2) (extrapolated CN with stabilization) is important. There is also a quite simple proof that (1.2) is unconditionally stable. We give the stability proof in Proposition 3.1 and then explore the effect the stabilization (and correction) in (1.2) have on the rates of convergence for various flow quantities.

No discretization is perfect. However, simple and stable ones leading to easily solvable linear systems can be very useful. We therefore conclude with numerical tests which verify accuracy and decrease in complexity in the linear equation solver.

Defining the method precisely requires a small amount of notation. The spatial part of (1.1) is naturally formulated in

$$\mathbf{X} := H_0^1(\Omega)^d, \quad Q := L_0^2(\Omega).$$

The finite element approximation begins by selecting conforming finite element spaces $\mathbf{X}^h \subset \mathbf{X}, Q^h \subset Q$ satisfying the usual discrete inf-sup condition (defined in Section 2). Denote the

[1] The author WL first saw it used as a numerical regularization in 1980 and it seems to have been known well before that. It is related to the simple Kelvin-Voight model of viscoelasticity, Oskolkov [O80], Kalantarev and Titi [KT07].

usual L^2 norm and inner product by $\|\cdot\|$ and $(\cdot,\cdot)$, and the space of discretely divergence free functions $\mathbf{V}^h$ by:

$$\mathbf{V}^h := \{\mathbf{v}^h \in \mathbf{X}^h : (q^h, \nabla \cdot \mathbf{v}^h) = 0, \ \forall \, q^h \in Q^h\}.$$

Define the explicitly skew-symmetrized trilinear form

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}), \tag{1.3}$$

and the extrapolation to $t_{n+\frac{1}{2}} := \frac{t_n+t_{n+1}}{2}$ by

$$E[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h] := \frac{3}{2}\mathbf{u}_n^h - \frac{1}{2}\mathbf{u}_{n-1}^h, \tag{1.4}$$

where $\mathbf{u}_j^h(x)$ is a *known* approximation to $\mathbf{u}(x,t_j)$.

The method studied is a 2-step method, so the initial condition and first step must be specified, but are not essential. We choose the Stokes Projection, defined in Section 2.

Algorithm 1.1 (Stabilized, extrapolated trapezoid rule)**.** *Let* $\mathbf{u}_0^h$ *be the Stokes Projection of* $\mathbf{u}_0(x)$ *into* $\mathbf{V}^h$*. At the first time level* $(\mathbf{u}_1^h, p_1^h) \in (\mathbf{X}^h, Q^h)$ *are sought, satisfying*

$$\begin{aligned}
(\frac{\mathbf{u}_1^h-\mathbf{u}_0^h}{k},\mathbf{v}^h) &+ \nu(\nabla(\frac{\mathbf{u}_1^h+\mathbf{u}_0^h}{2}),\nabla\mathbf{v}^h) + \alpha h(\nabla\mathbf{u}_1^h,\nabla\mathbf{v}^h)\\
&+b^*(\mathbf{u}_0^h,\frac{\mathbf{u}_1^h+\mathbf{u}_0^h}{2},\mathbf{v}^h) \ - (\frac{1}{2}(p_1^h+p_0^h),\nabla\cdot\mathbf{v}^h)\\
&= (\mathbf{f}(t_{\frac{1}{2}}),\mathbf{v}^h) + \alpha h(\nabla\mathbf{u}_0^h,\nabla\mathbf{v}^h), \qquad \forall\ \mathbf{v}^h \in \mathbf{X}^h,\\
&(\nabla\cdot\mathbf{u}_1^h,q^h) = 0, \qquad \forall\ q^h \in Q^h.
\end{aligned} \tag{1.5}$$

Given a time step $k>0$ *and an* $O(1)$ *constant* α*, the method computes* $\mathbf{u}_2^h,\mathbf{u}_3^h,\cdots,p_2^h,p_3^h,\cdots$ *where* $t_j = jk$ *and* $\mathbf{u}_j^h(x) \cong \mathbf{u}(x,t_j), p_j^h(x) \cong p(x,t_j)$*. For* $n \geq 1$*, given* $(\mathbf{u}_n^h,p_n^h) \in (\mathbf{X}^h,Q^h)$ *find* $(\mathbf{u}_{n+1}^h,p_{n+1}^h) \in (\mathbf{X}^h,Q^h)$ *satisfying*

$$\begin{aligned}
(\frac{\mathbf{u}_{n+1}^h-\mathbf{u}_n^h}{k},\mathbf{v}^h) &+ \nu(\nabla(\frac{\mathbf{u}_{n+1}^h+\mathbf{u}_n^h}{2}),\nabla\mathbf{v}^h) + \alpha h(\nabla\mathbf{u}_{n+1}^h,\nabla\mathbf{v}^h)\\
&+b^*(E[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\frac{\mathbf{u}_{n+1}^h+\mathbf{u}_n^h}{2},\mathbf{v}^h) \ - (\frac{1}{2}(p_{n+1}^h+p_n^h),\nabla\cdot\mathbf{v}^h)\\
&= (\mathbf{f}(t_{n+\frac{1}{2}}),\mathbf{v}^h) + \alpha h(\nabla\mathbf{u}_n^h,\nabla\mathbf{v}^h), \qquad \forall\ \mathbf{v}^h \in \mathbf{X}^h,\\
&(\nabla\cdot\mathbf{u}_{n+1}^h,q^h) = 0, \qquad \forall\ q^h \in Q^h.
\end{aligned} \tag{1.6}$$

We will refer to Algorithm 1.1 as CNLEStab (Crank-Nicolson with Linear Extrapolation Stabilized). If $\alpha = 0$, i.e. if no stabilization is used, Algorithm 1.1 reduces to one studied by G. Baker in 1976 [B76] and others, that we will refer to as CNLE.

We shall show that Algorithm 1.1 (CNLEStab) is unconditionally stable and second order accurate, $O(k^2 + hk + \text{spatial error})$. The extra stabilization terms added are $O(hk)$ because

$$\alpha h(\nabla(\mathbf{u}_{n+1}^h-\mathbf{u}_n^h),\nabla\mathbf{v}^h) = \alpha hk(\nabla(\frac{\mathbf{u}_{n+1}^h-\mathbf{u}_n^h}{k}),\nabla\mathbf{v}^h) \simeq hk(-\Delta\mathbf{u}_t) = O(hk).$$

As stated above, each time step of the method requires the solution of only one linear Oseen problem at cell Reynolds number $O(1)$.

Remark 1.1. *At the first time level, a nonlinear treatment of the trilinear term can be used instead of extrapolation: find* $(\mathbf{u}_1^h, p_1^h) \in (\mathbf{X}^h, Q^h)$*, satisfying*

$$\begin{aligned}(\frac{\mathbf{u}_1^h - \mathbf{u}_0^h}{k}, \mathbf{v}^h) + \nu(\nabla(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}), \nabla \mathbf{v}^h) + \alpha h(\nabla \mathbf{u}_1^h, \nabla \mathbf{v}^h) \\ + b^*(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h) \ - (\frac{1}{2}(p_1^h + p_0^h), \nabla \cdot \mathbf{v}^h) \\ = (\mathbf{f}(t_{\frac{1}{2}}), \mathbf{v}^h) + \alpha h(\nabla \mathbf{u}_0^h, \nabla \mathbf{v}^h), \qquad \forall\ \mathbf{v}^h \in \mathbf{X}^h, \\ (\nabla \cdot \mathbf{u}_1^h, q^h) = 0, \qquad \forall\ q^h \in Q^h.\end{aligned} \tag{1.7}$$

We shall show that this modification affects neither the stability of the method nor the convergence rate of the velocity error approximation, but increases the convergence rate of pressure approximation.

The stabilization in the method alters the numerical method's kinetic energy rather than in its energy dissipation. Proposition 5.1 and Section 5 show that

$$\text{Kinetic Energy in CNLEStab} = \frac{1}{2L^3}[||\mathbf{u}_n^h||^2 + \alpha k h ||\nabla \mathbf{u}_n^h||^2],$$

$$\text{Energy Dissipation in CNLEStab} = \frac{\nu}{L^3}||\nabla \mathbf{u}_n^h||^2.$$

We shall show in Sections 5 and 6 that this has several interesting consequences.

Section 2 collects some mathematical preliminaries for the analysis that follows. Sections 3 and 4 present a convergence analysis of the method (1.2). The modification of the method's kinetic energy influences the norm in which convergence is proven. A basic convergence analysis is fundamental to a numerical method's usefulness but there are many important questions it does not answer. We try to address some of these in Section 5 and onward. In Section 5 we consider physical fidelity of a simulation produced by the method (1.2). One aspect of physical fidelity is conservation of important integral invariants of the Euler equations ($\nu = 0$) and near conservation when ν is small. The conservation of the method's kinetic energy when $\nu = 0$ is clear from the stability proof in Section 3. The second important integral invariant of the Euler equations in 3d is helicity, [MT92],[DG01],[CCE03] and in 2d, enstrophy. Approximate conservation of these is explored in Section 5. Section 6 gives some insight into the predictions of (1.1) of flow statistics in turbulent flows. In Section 7 we present the results of the computational tests. These confirm the rates of convergence, predicted in Section 3.

2 Mathematical Preliminaries

Recall that (1.1) is naturally formulated in

$$\mathbf{X} := H_0^1(\Omega)^d, \quad Q := L_0^2(\Omega).$$

The dual space of $\mathbf{X}$ is denoted by $\mathbf{X}^*$ (and its norm, by $||\cdot||_{-1}$), and $\mathbf{V} = \{\mathbf{v} \in \mathbf{X} : (q, \nabla \cdot \mathbf{v}) = 0,\ \forall\ q \in Q\}$ is the set of weakly divergence free functions in $\mathbf{X}$. Norms in the Sobolev spaces $H^k(\Omega)^d$ (or $W_2^k(\Omega)^d$) are denoted by $\|\cdot\|_k$, and seminorms by $|\cdot|_k$.

Later analysis will require upper bounds on the nonlinear term, given in the following lemma.

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^3$ *or* $\mathbb{R}^2$. *For all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(\Omega)\|\nabla \mathbf{u}\|\|\nabla \mathbf{v}\|\|\nabla \mathbf{w}\|,$$

and

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(\Omega)\sqrt{\|\mathbf{u}\|\|\nabla \mathbf{u}\|}\|\nabla \mathbf{v}\|\|\nabla \mathbf{w}\|.$$

If, in addition, $\mathbf{v}, \nabla \mathbf{v} \in L^\infty(\Omega)$,

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(\Omega)(\|\mathbf{v}\|_{L^\infty(\Omega)} + \|\nabla \mathbf{v}\|_{L^\infty(\Omega)})\|\mathbf{u}\|\|\nabla \mathbf{w}\|$$

and

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(\|\mathbf{u}\|\|\nabla \mathbf{v}\|_{L^\infty(\Omega)} + \|\nabla \mathbf{u}\|\|\mathbf{v}\|_{L^\infty(\Omega)})\|\mathbf{w}\|.$$

Proof. See Girault and Raviart [GR86] for a proof of the first inequality. The second inequality follows from Hölder's inequality, the Sobolev embedding theorem and an interpolation inequality, e.g., [LT98]. The third bound follows from the definition of the skew-symmetric form and Hölder's inequality

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \frac{1}{2}\|\nabla \mathbf{v}\|_{L^\infty(\Omega)}\|\mathbf{u}\|\|\mathbf{w}\| + \frac{1}{2}\|\mathbf{v}\|_{L^\infty(\Omega)}\|\mathbf{u}\|\|\nabla \mathbf{w}\|,$$

and Poincare's inequality, since $\mathbf{w} \in \mathbf{X}$. The proof of the last inequality can be found, e.g., in [LT98]. □

Throughout the paper, we shall assume that the velocity-pressure finite element spaces $\mathbf{X}^h \subset \mathbf{X}$ and $Q^h \subset Q$ are conforming, have approximation properties typical of finite element spaces commonly in use, and satisfy the discrete inf-sup, or LBB^h, condition

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in \mathbf{X}^h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{\|\nabla \mathbf{v}^h\|\|q^h\|} \geq \beta^h > 0, \tag{2.1}$$

where β^h is bounded away from zero uniformly in h. Examples of such spaces can be found in [GR79], [GR86], [G89]. In addition, we assume that an inverse inequality holds, i.e. there exists a constant C independent of h and k, such that

$$\|\nabla \mathbf{v}\| \leq Ch^{-1}\|\mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{X}^h. \tag{2.2}$$

We assume that $(\mathbf{X}^h, Q^h)$ satisfy the following approximation properties typical of piecewise polynomials of degree $(m, m-1)$, [BS94]:

$$\inf_{\mathbf{v} \in \mathbf{X}^h} \|\mathbf{u} - \mathbf{v}\| \leq Ch^{m+1}|\mathbf{u}|_{m+1}, \quad \mathbf{u} \in H^{m+1}(\Omega), \tag{2.3}$$

$$\inf_{\mathbf{v} \in \mathbf{X}^h} \|\nabla(\mathbf{u} - \mathbf{v})\| \leq Ch^{m}|\mathbf{u}|_{m+1}, \quad \mathbf{u} \in H^{m+1}(\Omega), \tag{2.4}$$

$$\inf_{q \in Q^h} \|p - q\| \leq Ch^{m}|p|_{m}, \quad p \in H^{m}(\Omega). \tag{2.5}$$

We will also use the following inequality, which holds under (2.1) and for all $\mathbf{u} \in \mathbf{V}$:

$$\inf_{\mathbf{v} \in \mathbf{V}^h} \|\nabla(\mathbf{u} - \mathbf{v})\| \leq C(\Omega) \inf_{\mathbf{v} \in \mathbf{X}^h} \|\nabla(\mathbf{u} - \mathbf{v})\|. \tag{2.6}$$

The proof of (2.6) can be found, e.g., in [GR79] (p.60, inequality (1.2)).

Throughout the paper we use the following Stokes Projection.

Definition 2.1 (Stokes Projection)**.** *The Stokes projection operator* P_S*:* $(\mathbf{X}, Q) \to (\mathbf{X}^h, Q^h)$*,* $P_S(\mathbf{u}, p) = (\tilde{\mathbf{u}}, \tilde{p})$*, satisfies*

$$\begin{aligned} \nu(\nabla(\mathbf{u}-\tilde{\mathbf{u}}), \nabla \mathbf{v}^h) - (p - \tilde{p}, \nabla \cdot \mathbf{v}^h) &= 0, \\ (\nabla \cdot (\mathbf{u}-\tilde{\mathbf{u}}), q^h) &= 0, \end{aligned} \tag{2.7}$$

for any $\mathbf{v}^h \in \mathbf{X}^h$*,* $q^h \in Q^h$*.*

In $(\mathbf{V}^h, Q^h)$ this formulation reads: given $(\mathbf{u}, p) \in (\mathbf{X}, Q)$, find $\tilde{\mathbf{u}} \in \mathbf{V}^h$ satisfying

$$\nu(\nabla(\mathbf{u}-\tilde{\mathbf{u}}), \nabla \mathbf{v}^h) - (p - q^h, \nabla \cdot \mathbf{v}^h) = 0, \tag{2.8}$$

for any $\mathbf{v}^h \in \mathbf{V}^h, q^h \in Q^h$. Under the discrete inf-sup condition (2.1), the Stokes projection is well defined.

Proposition 2.1 (Stability of the Stokes Projection)**.** *Let* $\mathbf{u}$*,* $\tilde{\mathbf{u}}$ *satisfy (2.8). The following bound holds*

$$\nu\|\nabla \tilde{\mathbf{u}}\|^2 \le 2[\nu\|\nabla \mathbf{u}\|^2 + d\nu^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2], \tag{2.9}$$

where d *is the dimension,* $d = 2, 3$*.*

Proof. Take $\mathbf{v}^h = \tilde{\mathbf{u}} \in \mathbf{V}^h$ in (2.8). This gives

$$\nu\|\nabla \tilde{\mathbf{u}}\|^2 = \nu(\nabla \mathbf{u}, \nabla \tilde{\mathbf{u}}) - (p - q^h, \nabla \cdot \tilde{\mathbf{u}}). \tag{2.10}$$

Using the Cauchy-Schwarz and Young inequalities, we obtain

$$\begin{aligned} \nu\|\nabla \tilde{\mathbf{u}}\|^2 \quad \le \quad & \nu\|\nabla \mathbf{u}\|^2 + \frac{\nu}{4}\|\nabla \tilde{\mathbf{u}}\|^2 \\ & + d\nu^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2 + \frac{\nu}{4d}\|\nabla \cdot \tilde{\mathbf{u}}\|^2. \end{aligned} \tag{2.11}$$

Next, use the obvious inequality $\|\nabla \cdot \tilde{\mathbf{u}}\|^2 \le d\|\nabla \tilde{\mathbf{u}}\|^2$. Combining the like terms in (2.11) concludes the proof. □

In the error analysis we shall use the error estimate of the Stokes Projection (2.8).

Proposition 2.2 (Error estimate for the Stokes Projection)**.** *Suppose the discrete inf-sup condition (2.1) holds. Then the error in the Stokes Projection satisfies*

$$\nu\|\nabla(\mathbf{u}-\tilde{\mathbf{u}})\|^2 \le C[\nu \inf_{\mathbf{v}^h \in \mathbf{X}^h} \|\nabla(\mathbf{u}-\mathbf{v}^h)\|^2 + \nu^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2], \tag{2.12}$$

where C *is a constant independent of* h *and* ν*.*

Proof. Decompose the projection error $\mathbf{e} = \mathbf{u} - \tilde{\mathbf{u}}$ into $\mathbf{e} = \mathbf{u} - I(\mathbf{u}) - (\tilde{\mathbf{u}} - I(\mathbf{u})) = \boldsymbol{\eta} - \boldsymbol{\phi}$, where $\boldsymbol{\eta} = \mathbf{u} - I(\mathbf{u})$, $\boldsymbol{\phi} = \tilde{\mathbf{u}} - I(\mathbf{u})$, and $I(\mathbf{u})$ approximates $\mathbf{u}$ in $\mathbf{V}^h$. Take $\mathbf{v}^h = \boldsymbol{\phi} \in \mathbf{V}^h$ in (2.8). This gives

$$\nu\|\nabla \boldsymbol{\phi}\|^2 = \nu(\nabla \boldsymbol{\eta}, \nabla \boldsymbol{\phi}) - (p - q^h, \nabla \cdot \boldsymbol{\phi}). \tag{2.13}$$

The Cauchy-Schwarz and Young inequalities lead to

$$\nu\|\nabla \boldsymbol{\phi}\|^2 \le 2\nu\|\nabla \boldsymbol{\eta}\|^2 + C\nu^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2. \tag{2.14}$$

Since $I(\mathbf{u})$ is an approximation of $\mathbf{u}$ in $\mathbf{V}^h$, we can take infimum over $\mathbf{V}^h$. The proof is concluded by applying (2.6) and the triangle inequality. □

Remark 2.1. *Using the Aubin-Nitsche lift, one can obtain (see, e.g., [BDK82])*

$$\|\mathbf{u} - \tilde{\mathbf{u}}\| \le Ch \left(\inf_{\mathbf{v}^h \in \mathbf{X}^h} \|\nabla(\mathbf{u} - \mathbf{v}^h)\| + \inf_{q^h \in Q^h} \|p - q^h\| \right), \tag{2.15}$$

where $C = C(\nu, \Omega)$.

The following variation on the discrete Gronwall Lemma is given in [HR90] as a remark to Lemma 5.1. In this estimate, the first sum on the right hand side is only up to the next-to-last time step, which allows for an estimate with no smallness condition on k.

Lemma 2.2 (Discrete Gronwall)**.** *Let* k, B, a_n, b_n, c_n, d_n *for integers* $n \ge 0$ *be nonnegative numbers such that for* $N \ge 1$*, if*

$$a_N + k \sum_{n=0}^{N} b_n \le k \sum_{n=0}^{N-1} d_n a_n + k \sum_{n=0}^{N} c_n + B,$$

then for all $k > 0$*,*

$$a_N + k \sum_{n=0}^{N} b_n \le \exp(k \sum_{n=0}^{N-1} d_n) \left(k \sum_{n=0}^{N} c_n + B \right).$$

The following results are readily obtained by Taylor series expansion.

Lemma 2.3. *Let* $k = t_{n+1} - t_n$ *for all* i *and denote* $t_{n+1/2} = \frac{t_{n+1}+t_n}{2}$*. Let* $\psi(\cdot, t)$ *be a function such that* $\psi_t \in \mathcal{C}^0(0, T; L^2(\Omega))$*. Then there exists* $\theta \in (0, 1)$ *such that*

$$\|\frac{\psi(\cdot, t_{n+1}) - \psi(\cdot, t_n)}{k}\| \le C\|\psi_t(\cdot, t_{n+\theta})\|.$$

If $\psi_{tt} \in \mathcal{C}^0(0, T; L^2(\Omega))$*, then there exist* $\theta_1, \theta_2 \in (0, 1)$ *such that*

$$\|\frac{\psi(\cdot, t_{n+1}) + \psi(\cdot, t_n)}{2} - \psi(\cdot, t_{n+1/2})\| \le Ck^2\|\psi_{tt}(\cdot, t_{n+\theta_1})\|$$

and

$$\|\frac{3}{2}\psi(\cdot, t_n) - \frac{1}{2}\psi(\cdot, t_{n-1}) - \psi(\cdot, t_{n+1/2})\| \le Ck^2\|\psi_{tt}(\cdot, t_{n+\theta_2})\|.$$

If $\psi_{ttt} \in \mathcal{C}^0(0, T; L^2(\Omega))$*, then there exists* $\theta_3 \in (0, 1)$ *such that*

$$\|\frac{\psi(\cdot, t_{n+1}) - \psi(\cdot, t_n)}{k} - \psi_t(\cdot, t_{n+1/2})\| \le Ck^2\|\psi_{ttt}(\cdot, t_{n+\theta_3})\|.$$

3 Stability and Convergence of the Stabilized Method

We start with the proof of unconditional stability, which is the mathematical key to the good properties of the method, and motivates the more technical error analysis that follows.

The unconditional stability of Algorithm 1.1 is proven in the following proposition.

Proposition 3.1. *[Stability of extrapolated trapezoidal method] Let* $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega))$*. The stabilized, extrapolated trapezoid scheme (1.5)-(1.6) (and the scheme (1.6)-(1.7)) is unconditionally stable. For any* $h,k>0$ *and* $\alpha \geq 0, n \geq 0$

$$||\mathbf{u}^h_{n+1}||^2 + \alpha kh||\nabla \mathbf{u}^h_{n+1}||^2 + \nu k \sum_{i=0}^{n} ||\nabla(\frac{\mathbf{u}^h_{i+1}+\mathbf{u}^h_i}{2})||^2$$
$$\leq ||\mathbf{u}^h_0||^2 + \alpha kh||\nabla \mathbf{u}^h_0||^2 + \nu^{-1} k \sum_{i=0}^{n} ||\mathbf{f}(t_{i+\frac{1}{2}})||^2_{-1}.$$

Proof. Taking $\mathbf{v}^h = \frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2} \in \mathbf{V}^h$ in (1.5) (and in (1.7)) gives

$$(\frac{\mathbf{u}^h_1-\mathbf{u}^h_0}{k}, \frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2}) + \nu\|\nabla(\frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2})\|^2 + \alpha hk(\nabla\frac{\mathbf{u}^h_1-\mathbf{u}^h_0}{k}, \nabla\frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2})$$
$$= (\mathbf{f}(t_{\frac{1}{2}}), \frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2}). \tag{3.1}$$

Apply the Cauchy-Schwarz and Young inequalities. This gives

$$\frac{\|\mathbf{u}^h_1\|^2-\|\mathbf{u}^h_0\|^2}{2k} + \nu\|\nabla(\frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2})\|^2 + \alpha hk\frac{\|\nabla\mathbf{u}^h_1\|^2-\|\nabla\mathbf{u}^h_0\|^2}{2k}$$
$$\leq \frac{1}{2}\nu^{-1}\|\mathbf{f}(t_{\frac{1}{2}})\|^2_{-1} + \frac{1}{2}\nu\|\nabla(\frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2})\|^2. \tag{3.2}$$

Thus, on the first time level we obtain the stability bound

$$\|\mathbf{u}^h_1\|^2 + \nu k\|\nabla(\frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2})\|^2 + \alpha hk\|\nabla\mathbf{u}^h_1\|^2 \leq \|\mathbf{u}^h_0\|^2 + \alpha hk\|\nabla\mathbf{u}^h_0\|^2 + \nu^{-1}k\|\mathbf{f}(t_{\frac{1}{2}})\|^2_{-1}. \tag{3.3}$$

Now consider (1.6) for $n \geq 1$; let $\mathbf{v}^h = \frac{\mathbf{u}^h_{n+1}+\mathbf{u}^h_n}{2} \in \mathbf{V}^h$. This gives

$$(\frac{\mathbf{u}^h_{n+1}-\mathbf{u}^h_n}{k}, \frac{\mathbf{u}^h_{n+1}+\mathbf{u}^h_n}{2}) + \nu\|\nabla(\frac{\mathbf{u}^h_{n+1}+\mathbf{u}^h_n}{2})\|^2 + \alpha hk(\nabla(\frac{\mathbf{u}^h_{n+1}-\mathbf{u}^h_n}{k}), \nabla(\frac{\mathbf{u}^h_{n+1}+\mathbf{u}^h_n}{2}))$$
$$= (\mathbf{f}(t_{n+\frac{1}{2}}), \frac{\mathbf{u}^h_{n+1}+\mathbf{u}^h_n}{2}). \tag{3.4}$$

Applying Cauchy-Schwarz and Young inequalities leads to

$$\frac{\|\mathbf{u}^h_{n+1}\|^2-\|\mathbf{u}^h_n\|^2}{2k} + \nu\|\nabla(\frac{\mathbf{u}^h_{n+1}+\mathbf{u}^h_n}{2})\|^2 + \alpha hk\frac{\|\nabla\mathbf{u}^h_{n+1}\|^2-\|\nabla\mathbf{u}^h_n\|^2}{2k}$$
$$\leq \frac{1}{2}\nu^{-1}\|\mathbf{f}(t_{n+\frac{1}{2}})\|^2_{-1} + \frac{1}{2}\nu\|\nabla(\frac{\mathbf{u}^h_{n+1}+\mathbf{u}^h_n}{2})\|^2. \tag{3.5}$$

Simplifying (3.5) gives

$$(\|\mathbf{u}^h_{n+1}\|^2-\|\mathbf{u}^h_n\|^2) + \nu k\|\nabla(\frac{\mathbf{u}^h_{n+1}+\mathbf{u}^h_n}{2})\|^2 + \alpha hk(\|\nabla\mathbf{u}^h_{n+1}\|^2-\|\nabla\mathbf{u}^h_n\|^2)$$
$$\leq \nu^{-1}k\|\mathbf{f}(t_{n+\frac{1}{2}})\|^2_{-1}. \tag{3.6}$$

Summing (3.6) over the time levels gives

$$\|\mathbf{u}_{n+1}^h\|^2 + k\sum_{i=1}^{n} \nu\|\nabla(\frac{\mathbf{u}_{i+1}^h+\mathbf{u}_i^h}{2})\|^2 + \alpha hk\|\nabla\mathbf{u}_{n+1}^h\|^2$$
$$\leq \|\mathbf{u}_1^h\|^2 + \alpha hk\|\nabla\mathbf{u}_1^h\|^2 + k\sum_{i=1}^{n}\nu^{-1}\|\mathbf{f}(t_{i+\frac{1}{2}})\|_{-1}^2. \tag{3.7}$$

Finally, using the bound on $(\|\mathbf{u}_1^h\|^2 + \alpha hk\|\nabla\mathbf{u}_1^h\|^2)$ from (3.3), we obtain that for all $n \geq 1$

$$\|\mathbf{u}_{n+1}^h\|^2 + k\sum_{i=0}^{n} \nu\|\nabla(\frac{\mathbf{u}_{i+1}^h+\mathbf{u}_i^h}{2})\|^2 + \alpha hk\|\nabla\mathbf{u}_{n+1}^h\|^2$$
$$\leq \|\mathbf{u}_0^h\|^2 + \alpha hk\|\nabla\mathbf{u}_0^h\|^2 + k\sum_{i=0}^{n}\nu^{-1}\|\mathbf{f}(t_{i+\frac{1}{2}})\|_{-1}^2. \tag{3.8}$$

This result, combined with Proposition 2.1, proves the Proposition. □

Hence the method is unconditionally stable. The question remains: how fast does $\mathbf{u}^h$ converge to $\mathbf{u}$? To evaluate the rates of convergence as $h \to 0$, we must make a specific choice of $\mathbf{X}^h, Q^h$.

Theorem 3.1 (Velocity Convergence Rates)**.** *Let the finite-element spaces* $(\mathbf{X}^h, Q^h)$ *include continuous piecewise polynomials of degree* m *and* $m-1$ *respectively* ($m \geq 2$)*, and satisfy the discrete inf-sup condition (2.1) and approximation properties (2.3)-(2.5). Let* $C\|\mathbf{u}\|_{L^\infty(0,T;H^{m+1}(\Omega))}kh^{m-\frac{3}{2}} \leq 1/2$*, and*

$$\mathbf{u} \in L^\infty(0,T;H^{m+1}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)) \cap \mathcal{C}^0(0,T;H^1(\Omega)),$$
$$\nabla\mathbf{u} \in L^\infty(0,T;L^\infty(\Omega)),$$
$$\mathbf{u}_t \in L^2(0,T;H^{m+1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \nabla\mathbf{u}_{tt} \in L^2(0,T;H^1(\Omega)),$$
$$p_{tt} \in L^2(0,T;L^2(\Omega)).$$

Then there is a $C = C(\nu,\mathbf{u},p,T) < \infty$ *such that* $\forall n \in \{0,1,...,N-1\}$ *the error in Algorithm 1.1 satisfies*

$$\|\mathbf{u}(t_{n+1}) - \mathbf{u}_{n+1}^h\| + \left(k\sum_{i=0}^{n}\nu\|\nabla(\frac{(\mathbf{u}(t_{i+1})-\mathbf{u}_{i+1}^h)+(\mathbf{u}(t_i)-\mathbf{u}_i^h)}{2})\|^2\right)^{\frac{1}{2}}$$
$$+\alpha^{\frac{1}{2}}h^{\frac{1}{2}}k^{\frac{1}{2}}\|\nabla(\mathbf{u}(t_{n+1})-\mathbf{u}_{n+1}^h)\| \leq C(\nu,\mathbf{u},p)\left(h^m + \alpha hk + k^2\right).$$

The rest of this section will be devoted to proving this theorem.

Proof. Consider the variational formulation corresponding to the Navier-Stokes equations (1.1), for any time t, in $\mathbf{X}^h$,

$$(\mathbf{u}_t, \mathbf{v}^h) + b^*(\mathbf{u},\mathbf{u},\mathbf{v}^h) + \nu(\nabla\mathbf{u},\nabla\mathbf{v}^h) - (p, \nabla\cdot\mathbf{v}^h) = (\mathbf{f},\mathbf{v}^h), \quad \forall\mathbf{v}^h \in \mathbf{X}^h. \tag{3.9}$$

Then subtract (1.6) from (3.9), taken at $t = t_{n+\frac{1}{2}}$, to get

$$
\begin{aligned}
&(\mathbf{u}_t(t_{n+\frac{1}{2}}) - \frac{\mathbf{u}^h_{n+1} - \mathbf{u}^h_n}{k}, \mathbf{v}^h) + \nu(\nabla \mathbf{u}(t_{n+\frac{1}{2}}) - \nabla(\frac{\mathbf{u}^h_{n+1} + \mathbf{u}^h_n}{2}), \nabla \mathbf{v}^h) \\
&- \quad \alpha h(\nabla(\mathbf{u}^h_{n+1} - \mathbf{u}^h_n), \nabla \mathbf{v}^h) + b^*(\mathbf{u}(t_{n+\frac{1}{2}}), \mathbf{u}(t_{n+\frac{1}{2}}), \mathbf{v}^h) \\
&- \quad b^*(E[\mathbf{u}^h_n, \mathbf{u}^h_{n-1}], \frac{\mathbf{u}^h_{n+1} + \mathbf{u}^h_n}{2}, \mathbf{v}^h) - (p(t_{n+\frac{1}{2}}) - \frac{p(t_{n+1}) + p(t_n)}{2}, \nabla \cdot \mathbf{v}^h) = 0. \quad (3.10)
\end{aligned}
$$

Let the velocity error be decomposed as

$$
\mathbf{e}_n := \mathbf{u}(t_n) - \mathbf{u}^h_n = (\mathbf{u}(t_n) - \mathbf{U}_n) - (\mathbf{u}^h_n - \mathbf{U}_n) =: \boldsymbol{\eta}_n - \boldsymbol{\phi}^h_n, \quad (3.11)
$$

where $\mathbf{U}_n$ is the Stokes Projection of $\mathbf{u}_n$ into $\mathbf{V}^h$ (therefore $\boldsymbol{\phi}^h_n \in \mathbf{V}^h$, but $\boldsymbol{\eta}_n \notin \mathbf{V}^h$). For $\boldsymbol{\xi} = \mathbf{e}, \boldsymbol{\phi}^h$ or $\boldsymbol{\eta}$, define $\boldsymbol{\xi}_{n+\frac{1}{2}} := \frac{\boldsymbol{\xi}_{n+1} + \boldsymbol{\xi}_n}{2}$.

Add and subtract

$$
\begin{aligned}
&(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{k}, \mathbf{v}^h) + \nu(\nabla(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{2}), \nabla \mathbf{v}^h) \\
&\quad + \alpha h(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)), \nabla \mathbf{v}^h) - (\frac{p(t_{n+1}) + p(t_n)}{2}, \nabla \cdot \mathbf{v}^h) \\
&\quad + b^*(\mathbf{u}(t_{n+\frac{1}{2}}) + E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})] + E[\mathbf{u}^h_n, \mathbf{u}^h_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h)
\end{aligned}
$$

to (3.10) to obtain the error equation (recall also that $(q^h, \nabla \cdot \mathbf{v}^h) = 0, \forall q^h \in Q^h$)

$$
\begin{aligned}
(\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k}, \mathbf{v}^h) \quad + \quad & \nu(\nabla \mathbf{e}_{n+1/2}, \nabla \mathbf{v}^h) + \alpha h(\nabla(\mathbf{e}_{n+1} - \mathbf{e}_n), \nabla \mathbf{v}^h) \\
&= (\frac{p(t_{n+1}) + p(t_n)}{2} - q^h, \nabla \cdot \mathbf{v}^h) - b^*(E[\mathbf{u}^h_n, \mathbf{u}^h_{n-1}], \mathbf{e}_{n+1/2}, \mathbf{v}^h) \\
&\quad + b^*(E[\mathbf{e}_n, \mathbf{e}_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h) + T(\mathbf{u}, p; \mathbf{v}^h), \quad (3.12)
\end{aligned}
$$

where

$$
\begin{aligned}
T(\mathbf{u}, p; \mathbf{v}^h) \quad = \quad & (\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{k} - \mathbf{u}_t(t_{n+\frac{1}{2}}), \mathbf{v}^h) + \nu(\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}) - \nabla \mathbf{u}(t_{n+\frac{1}{2}}), \nabla \mathbf{v}^h) \\
& - \alpha h(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)), \nabla \mathbf{v}^h) - (\frac{p(t_{n+1}) + p(t_n)}{2} - p(t_{n+\frac{1}{2}}), \nabla \cdot \mathbf{v}^h) \\
& + b^*(\mathbf{u}(t_{n+\frac{1}{2}}), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2} - \mathbf{u}(t_{n+\frac{1}{2}}), \mathbf{v}^h) \\
& - b^*(E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})] - \mathbf{u}(t_{n+\frac{1}{2}}), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h). \quad (3.13)
\end{aligned}
$$

Using the error decomposition (3.11) and setting $\mathbf{v}^h = \boldsymbol{\phi}^h_{n+1/2}$ in (3.12) gives

$$
\begin{aligned}
&\frac{1}{2k}(\|\boldsymbol{\phi}^h_{n+1}\|^2 - \|\boldsymbol{\phi}^h_n\|^2) + \nu\|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + \frac{\alpha h}{2}(\|\nabla \boldsymbol{\phi}^h_{n+1}\|^2 - \|\nabla \boldsymbol{\phi}^h_n\|^2) \\
&\quad = (\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}, \boldsymbol{\phi}^h_{n+1/2}) + \nu(\nabla \boldsymbol{\eta}_{n+1/2}, \nabla \boldsymbol{\phi}^h_{n+1/2}) \\
&\quad + \alpha hk(\nabla(\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}), \nabla \boldsymbol{\phi}^h_{n+1/2}) - (\frac{p(t_{n+1}) + p(t_n)}{2} - q^h, \nabla \cdot \boldsymbol{\phi}^h_{n+1/2}) \\
&\quad + b^*(E[\mathbf{u}^h_n, \mathbf{u}^h_{n-1}], \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) + b^*(E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}^h_{n+1/2}) \\
&\quad + b^*(E[\boldsymbol{\phi}^h_n, \boldsymbol{\phi}^h_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}^h_{n+1/2}) + T(\mathbf{u}, p; \boldsymbol{\phi}^h_{n+1/2}), \quad (3.14)
\end{aligned}
$$

since $b^*(E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \boldsymbol{\phi}_{n+1/2}^h, \boldsymbol{\phi}_{n+1/2}^h) = 0$.

Also it follows from the choice of the projection $\mathbf{U}_n$ that

$$\nu(\nabla \boldsymbol{\eta}_{n+1/2}, \nabla \boldsymbol{\phi}_{n+1/2}^h) - (\frac{p(t_{n+1}) + p(t_n)}{2} - q^h, \nabla \cdot \boldsymbol{\phi}_{n+1/2}^h) = 0.$$

Applying the Cauchy-Schwarz and Young's inequalities to the linear terms on the right hand side of (3.14) gives

$$\begin{aligned}
&\frac{1}{2k}(\|\boldsymbol{\phi}_{n+1}^h\|^2 - \|\boldsymbol{\phi}_n^h\|^2) + \frac{3\nu}{4}\|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + \frac{\alpha h}{2}(\|\nabla \boldsymbol{\phi}_{n+1}^h\|^2 - \|\nabla \boldsymbol{\phi}_n^h\|^2) \\
&\quad \leq C\nu^{-1}\|\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}\|^2 + C\nu^{-1}\alpha h k\|\nabla(\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k})\|^2 \\
&\quad + |\, b^*(E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h)\,| \\
&\quad + |\, b^*(E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}_{n+1/2}^h)\,| \\
&\quad + |\, b^*(E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}_{n+1/2}^h)\,| + |\, T(\mathbf{u}, p; \boldsymbol{\phi}_{n+1/2}^h)\,|, \qquad (3.15)
\end{aligned}$$

For clarity, we analyze each of the remaining nonlinear terms on the RHS of (3.15) individually. Here we use frequently Lemma 2.1 and the inverse estimate (2.2), together with Young's inequality.

We start with the first nonlinear term in (3.15). Adding and subtracting the quantity $b^*(E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})], \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h)$, and using Lemma 2.1, followed by Young's inequality, we get

$$\begin{aligned}
&\left| b^*(E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \right| \\
&\quad \leq \frac{\nu}{16}\|\boldsymbol{\phi}_{n+1/2}^h\|^2 + C\nu^{-1}\|\nabla E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})]\|^2\, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
&\quad + C\nu^{-1}\|\nabla E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}]\|^2\, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
&\quad + C\|E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\|^{1/2}\|\nabla E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\|^{1/2}\|\nabla \boldsymbol{\eta}_{n+1/2}\|\, \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|. \qquad (3.16)
\end{aligned}$$

The first two terms involving the operator $E[\cdot, \cdot]$ can be bounded by using its definition (1.4) and regularity assumptions on $\mathbf{u}$,

$$\|\nabla E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})]\| \leq C \qquad \text{and} \qquad \|\nabla E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}]\| \leq \frac{3}{2}\|\nabla \boldsymbol{\eta}_n\| + \frac{1}{2}\|\nabla \boldsymbol{\eta}_{n-1}\|. \quad (3.17)$$

For the third and fourth terms, we also need the inverse estimate (2.2), resulting in

$$\begin{aligned}
\|E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\|\, \|\nabla E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\| &\leq C(\|\boldsymbol{\phi}_n^h\| + \|\boldsymbol{\phi}_{n-1}^h\|)\,(\|\nabla \boldsymbol{\phi}_n^h\| + \|\nabla \boldsymbol{\phi}_{n-1}^h\|), \\
&\leq Ch^{-1}(\|\boldsymbol{\phi}_n^h\| + \|\boldsymbol{\phi}_{n-1}^h\|)^2,
\end{aligned}$$

so that

$$\begin{aligned}
&\|E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\|^{1/2}\, \|\nabla E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\|^{1/2}\|\nabla \boldsymbol{\eta}_{n+1/2}\|\, \|\nabla \boldsymbol{\phi}_{n+1/2}\| \\
&\qquad \leq Ch^{-3/2}\|\nabla \boldsymbol{\eta}_{n+1/2}\|\,(\|\boldsymbol{\phi}_n^h\| + \|\boldsymbol{\phi}_{n-1}^h\|)\,(\|\boldsymbol{\phi}_n^h\| + \|\boldsymbol{\phi}_{n+1}^h\|), \qquad (3.18)
\end{aligned}$$

Putting (3.17) and (3.18) back into (3.16), we have

$$\begin{aligned}
&\left|b^*(E[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\boldsymbol{\eta}_{n+1/2},\boldsymbol{\phi}_{n+1/2}^h)\right| \\
&\qquad \le \frac{\nu}{16}\|\boldsymbol{\phi}_{n+1/2}^h\|^2 + C\nu^{-1}\,\|\nabla\boldsymbol{\eta}_{n+1/2}\|^2 \\
&\qquad\quad + C\nu^{-1}(\|\nabla\boldsymbol{\eta}_n\|^2+\|\nabla\boldsymbol{\eta}_{n-1}\|^2)\,\|\nabla\boldsymbol{\eta}_{n+1/2}\|^2 \\
&\qquad\quad + Ch^{-3/2}\|\nabla\boldsymbol{\eta}_{n+1/2}\|\,(\|\boldsymbol{\phi}_{n-1}^h\|^2+\|\boldsymbol{\phi}_n^h\|^2+\|\boldsymbol{\phi}_{n+1}^h\|^2). \qquad (3.19)
\end{aligned}$$

For the second trilinear term, use Lemma 2.1 and the assumption that $\|\nabla\mathbf{u}(t)\|$ is bounded for any $t\in[0,T]$. Then we apply Young's inequality and (3.17), resulting in

$$\begin{aligned}
\left|b^*(E[\boldsymbol{\eta}_n,\boldsymbol{\eta}_{n-1}],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\boldsymbol{\phi}_{n+1/2}^h)\right| &\le C\|\nabla E[\boldsymbol{\eta}_n,\boldsymbol{\eta}_{n-1}]\|\|\nabla\boldsymbol{\phi}_{n+1/2}^h\| \\
&\le C\nu^{-1}(\|\nabla\boldsymbol{\eta}_n\|^2+\|\nabla\boldsymbol{\eta}_{n-1}\|^2) \\
&\quad + \frac{\nu}{16}\|\nabla\boldsymbol{\phi}_{n+1/2}^h\|^2. \qquad (3.20)
\end{aligned}$$

The third trilinear term is bounded with the help of the third inequality in Lemma 2.1 and the regularity assumptions on $\mathbf{u}$. As a result,

$$\begin{aligned}
\left|b^*(E[\boldsymbol{\phi}_n^h,\boldsymbol{\phi}_{n-1}^h],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\boldsymbol{\phi}_{n+1/2}^h)\right| &\le C\|E[\boldsymbol{\phi}_n^h,\boldsymbol{\phi}_{n-1}^h]\|\|\nabla\boldsymbol{\phi}_{n+1/2}^h\| \\
&\le C\nu^{-1}(\|\boldsymbol{\phi}_n^h\|^2+\|\boldsymbol{\phi}_{n-1}^h\|^2) \\
&\quad + \frac{\nu}{16}\|\nabla\boldsymbol{\phi}_{n+1/2}^h\|^2, \qquad (3.21)
\end{aligned}$$

where the last step follows from Young's inequality.

Now, with (3.19), (3.20) and (3.21), the error equation (3.15) can be rewritten as

$$\begin{aligned}
&\frac{1}{2k}(\|\boldsymbol{\phi}_{n+1}^h\|^2-\|\boldsymbol{\phi}_n^h\|^2)+\frac{9\nu}{16}\|\nabla\boldsymbol{\phi}_{n+1/2}^h\|^2+\frac{\alpha h}{2}(\|\nabla\boldsymbol{\phi}_{n+1}^h\|^2-\|\nabla\boldsymbol{\phi}_n^h\|^2) \\
&\quad \le C\nu^{-1}\|\frac{\boldsymbol{\eta}_{n+1}-\boldsymbol{\eta}_n}{k}\|^2 + C\nu^{-1}\alpha hk\|\nabla(\frac{\boldsymbol{\eta}_{n+1}-\boldsymbol{\eta}_n}{k})\|^2 + C\nu^{-1}\|\nabla\boldsymbol{\eta}_{n+1/2}\|^2 \\
&\quad + C\nu^{-1}(\|\nabla\boldsymbol{\eta}_n\|^2+\|\nabla\boldsymbol{\eta}_{n-1}\|^2)\|\nabla\boldsymbol{\eta}_{n+1/2}\|^2 \\
&\quad + C\nu^{-1}(\|\nabla\boldsymbol{\eta}_n\|^2+\|\nabla\boldsymbol{\eta}_{n-1}\|^2) + C\nu^{-1}(\|\boldsymbol{\phi}_n^h\|^2+\|\boldsymbol{\phi}_{n-1}^h\|^2) \\
&\quad + Ch^{-3/2}\|\nabla\boldsymbol{\eta}_{n+1/2}\|(\|\boldsymbol{\phi}_{n-1}^h\|^2+\|\boldsymbol{\phi}_n^h\|^2+\|\boldsymbol{\phi}_{n+1}^h\|^2) + |\,T(\mathbf{u},p;\boldsymbol{\phi}_{n+1/2}^h)\,|, \qquad (3.22)
\end{aligned}$$

and what is left is to bound $|\,T(\mathbf{u},p;\boldsymbol{\phi}_{n+1/2}^h)\,|$. Each of its four linear terms can be bounded by the Cauchy-Schwarz and Young's inequalities, together with the estimates in Lemma 2.3. We take care of one at a time below.

$$|(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{k}-\mathbf{u}_t(t_{n+1/2}),\boldsymbol{\phi}_{n+1/2}^h)| \le \frac{\nu}{80}\|\nabla\boldsymbol{\phi}_{n+1/2}^h\|^2 + C\nu^{-1}k^4\|\mathbf{u}_{ttt}(t_{n+\theta_1})\|^2, \qquad (3.23)$$

$$\begin{aligned}
\nu|(\nabla(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbf{u}(t_{n+\frac12})),\nabla\boldsymbol{\phi}_{n+1/2}^h)| &\le \frac{\nu}{80}\|\nabla\boldsymbol{\phi}_{n+1/2}^h\|^2 \\
&\quad + C\nu k^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta_2})\|^2, \qquad (3.24)
\end{aligned}$$

$$\alpha hk|(\nabla(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{k}),\nabla\boldsymbol{\phi}^h_{n+1/2})| \le \frac{\nu}{80}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\nu^{-1}\alpha^2h^2k^2\|\nabla\mathbf{u}_t(t_{n+\theta_3})\|^2, \tag{3.25}$$

$$|(\frac{p(t_{n+1})+p(t_n)}{2} - p(t_{n+1/2}),\nabla\cdot\boldsymbol{\phi}^h_{n+1/2})| \le \frac{\nu}{80}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\nu^{-1}k^4\|p_{tt}(t_{n+\theta_4})\|^2, \tag{3.26}$$

for some $\theta_1,\theta_2,\theta_3,\theta_4 \in (0,1)$.

For the two nonlinear terms in $|\,T(\mathbf{u},p;\boldsymbol{\phi}^h_{n+1/2})\,|$, use Lemma 2.1, Lemma 2.3 and Young's inequality, together with $\|\nabla\mathbf{u}(t)\| \le C$, for any $t\in[0,T]$. This gives

$$\begin{aligned}
&\left|b^*(E[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})]-\mathbf{u}(t_{n+1/2}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\boldsymbol{\phi}^h_{n+1/2})\right| \\
&\quad+\left|b^*(\mathbf{u}(t_{n+1/2}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbf{u}(t_{n+1/2}),\boldsymbol{\phi}^h_{n+1/2})\right| \\
&\quad\le C(\Omega)\|\nabla(\frac{3}{2}\mathbf{u}(t_n)-\frac{1}{2}\mathbf{u}(t_{n-1})-\mathbf{u}(t_{n+1/2}))\|\|\nabla(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2})\|\|\nabla\boldsymbol{\phi}^h_{n+1/2}\| \\
&\quad+C(\Omega)\|\nabla(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbf{u}(t_{n+1/2}))\|\|\nabla\mathbf{u}(t_{n+1/2})\|\|\nabla\boldsymbol{\phi}^h_{n+1/2}\| \\
&\quad\le C\nu^{-1}k^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta_5})\|^2+\frac{\nu}{80}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2,
\end{aligned} \tag{3.27}$$

for some $\theta_5\in(0,1)$.

Combining (3.23)-(3.27), we have

$$\begin{aligned}
|\,T(\mathbf{u},p;\boldsymbol{\phi}^h_{n+1/2})\,| \;&\le\; \frac{\nu}{16}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\nu^{-1}k^4(\|\mathbf{u}_{ttt}(t_{n+\theta_1})\|^2+\|p_{tt}(t_{n+\theta_4})\|^2) \\
&\quad+C\nu k^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta_5})\|^2+C\nu^{-1}\alpha^2h^2k^2\|\nabla\mathbf{u}_t(t_{n+\theta_3})\|^2,
\end{aligned} \tag{3.28}$$

so that error equation (3.22) gives

$$\begin{aligned}
&\frac{1}{2k}(\|\boldsymbol{\phi}^h_{n+1}\|^2-\|\boldsymbol{\phi}^h_n\|^2)+\frac{\nu}{2}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2+\frac{\alpha h}{2}(\|\nabla\boldsymbol{\phi}^h_{n+1}\|^2-\|\nabla\boldsymbol{\phi}^h_n\|^2) \\
&\quad\le C\nu^{-1}\|\frac{\boldsymbol{\eta}_{n+1}-\boldsymbol{\eta}_n}{k}\|^2+C\nu^{-1}\alpha hk\|\nabla(\frac{\boldsymbol{\eta}_{n+1}-\boldsymbol{\eta}_n}{k})\|^2+C\nu^{-1}\|\nabla\boldsymbol{\eta}_{n+1/2}\|^2 \\
&\quad+C\nu^{-1}(\|\nabla\boldsymbol{\eta}_n\|^2+\|\nabla\boldsymbol{\eta}_{n-1}\|^2)\|\nabla\boldsymbol{\eta}_{n+1/2}\|^2 \\
&\quad+C\nu^{-1}(\|\nabla\boldsymbol{\eta}_n\|^2+\|\nabla\boldsymbol{\eta}_{n-1}\|^2)+C\nu^{-1}(\|\boldsymbol{\phi}^h_n\|^2+\|\boldsymbol{\phi}^h_{n-1}\|^2) \\
&\quad+Ch^{-3/2}\|\nabla\boldsymbol{\eta}_{n+1/2}\|(\|\boldsymbol{\phi}^h_{n-1}\|^2+\|\boldsymbol{\phi}^h_n\|^2+\|\boldsymbol{\phi}^h_{n+1}\|^2) \\
&\quad+C\nu^{-1}k^4(\|\mathbf{u}_{ttt}(t_{n+\theta_1})\|^2+\|p_{tt}(t_{n+\theta_4})\|^2) \\
&\quad+C\nu k^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta_5})\|^2+C\nu^{-1}\alpha^2h^2k^2\|\nabla\mathbf{u}_t(t_{n+\theta_3})\|^2.
\end{aligned} \tag{3.29}$$

Multiply both sides of (3.29) by $2k$ and use (2.12),(2.15) together with the approximation properties (2.3)-(2.5) of the spaces $(\mathbf{X}^h,Q^h)$. Then sum over the time levels from 1 to n, choosing $\mathbf{U}_0 = \mathbf{u}_0^h$, which gives $\boldsymbol{\phi}_0^h = 0$, and

$$
\begin{aligned}
&\|\boldsymbol{\phi}_{n+1}^h\|^2 + k\sum_{i=1}^{n} \nu \|\nabla \boldsymbol{\phi}_{i+1/2}^h\|^2 + \alpha h k \|\nabla \boldsymbol{\phi}_{n+1}^h\|^2 \\
&\quad \leq \ \|\boldsymbol{\phi}_1^h\|^2 + \alpha h k\|\nabla \boldsymbol{\phi}_1^h\|^2 + C\nu^{-1} h^{2m+2}\|\mathbf{u}_t\|^2_{L^2(0,T;H^{m+1}(\Omega))} \\
&\qquad + C\nu^{-1}\alpha h^{2m+1} k \|\mathbf{u}_t\|^2_{L^2(0,T;H^{m+1}(\Omega))} + C\nu^{-1} h^{2m}\|\mathbf{u}\|^2_{L^2(0,T;H^{m+1}(\Omega))} \\
&\qquad + C\nu^{-1} h^{4m}\|\mathbf{u}\|^2_{L^2(0,T;H^{m+1}(\Omega))} \\
&\qquad + C\nu^{-1} h^{2m}\|\mathbf{u}\|^2_{L^2(0,T;H^{m+1}(\Omega))} + C\nu^{-1}k^4(\|\mathbf{u}_{ttt}\|^2_{L^2(0,T;L^2(\Omega))} + \|p_{tt}\|^2_{L^2(0,T;L^2(\Omega))}) \\
&\qquad + C\nu k^4 \|\nabla \mathbf{u}_{tt}\|^2_{L^2(0,T;L^2(\Omega))} + C\nu^{-1}\alpha^2 h^2 k^2 \|\nabla \mathbf{u}_t\|^2_{L^2(0,T;L^2(\Omega))} \\
&\qquad + C\nu^{-1} k \sum_{i=1}^{n} (\|\boldsymbol{\phi}_{i-1}^h\|^2 + \|\boldsymbol{\phi}_i^h\|^2) \\
&\qquad + C h^{m-3/2} k \sum_{i=1}^{n} |\, \mathbf{u}(t_{i+1/2})\,|_{m+1} (\|\boldsymbol{\phi}_{i-1}^h\|^2 + \|\boldsymbol{\phi}_i^h\|^2 + \|\boldsymbol{\phi}_{i+1}^h\|^2). \qquad (3.30)
\end{aligned}
$$

Since $\mathbf{u} \in L^\infty(0,T;H^{m+1}(\Omega))$, the last two sums in (3.30) can be combined as

$$
C\|\mathbf{u}\|_{L^\infty(0,T;H^{m+1}(\Omega))} h^{m-3/2} k \|\boldsymbol{\phi}_{n+1}^h\|^2 + C(h^{m-3/2} + \nu^{-1}) k \sum_{i=1}^{n} \|\boldsymbol{\phi}_i^h\|^2.
$$

Using the regularity of $\mathbf{u}$ and p, and the assumption that $C\|\mathbf{u}\|_{L^\infty(0,T;H^{m+1}(\Omega))} h^{m-3/2} k \leq 1/2$, the error equation finally takes the form

$$
\begin{aligned}
&\frac{1}{2}\|\boldsymbol{\phi}_{n+1}^h\|^2 + k\sum_{i=1}^{n} \nu\|\nabla \boldsymbol{\phi}_{i+1/2}^h\|^2 + \alpha h k\|\nabla \boldsymbol{\phi}_{n+1}^h\|^2 \\
&\quad \leq \|\boldsymbol{\phi}_1^h\|^2 + \alpha h k \|\nabla \boldsymbol{\phi}_1^h\|^2 + C\nu^{-1}(2 + h^2 + \alpha h k + h^{2m}) h^{2m} \\
&\qquad + C\nu^{-1}\alpha^2 h^2 k^2 + C(\nu^{-1} + \nu) k^4 \\
&\qquad + Ck\sum_{i=1}^{n} (\nu^{-1} + h^{m-3/2}) \|\boldsymbol{\phi}_i^h\|^2. \qquad (3.31)
\end{aligned}
$$

To complete the proof bounds are needed for $\boldsymbol{\phi}_1^h$ in the above estimates. These bounds depend upon the way the first time step is taken, and there are two possibilities (1.5) and (1.7); we shall analyze both. Both lead to an optimal velocity error estimate. The more expensive method (1.7) also leads to an optimal pressure error estimate (in Theorem 4.3 below). The error equation for $\boldsymbol{\phi}_1^h$ is the same as for $\boldsymbol{\phi}_n^h$ except for the nonlinear terms, and is treated in the same way, except for the nonlinear term. Therefore, we go directly to the treatment of the nonlinear term in both cases (1.5) and (1.7).

We start with formulation (1.5). Adding and subtracting $b^*(\mathbf{u}_0^h - \mathbf{u}(t_0), \frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2}, \mathbf{v}^h)$ to the nonlinear terms in (3.10), we have

$$
\begin{aligned}
b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \mathbf{v}^h) - b^*(\mathbf{u}_0^h, \frac{\mathbf{u}_0^h + \mathbf{u}_1^h}{2}, \mathbf{v}^h) \ &= \ b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \mathbf{v}^h) \\
&+ \ b^*(\mathbf{u}_0^h, \mathbf{e}_{1/2}, \mathbf{v}^h) - b^*(\mathbf{e}_0, \frac{\mathbf{u}(t_0) + \mathbf{u}(t_1)}{2}, \mathbf{v}^h) \\
&+ \ b^*(\mathbf{u}(t_0), \frac{\mathbf{u}(t_0) + \mathbf{u}(t_1)}{2}, \mathbf{v}^h). \qquad (3.32)
\end{aligned}
$$

Taking $\mathbf{v}^h = \boldsymbol{\phi}^h_{1/2}$, the second and third terms in (3.37) can be treated exactly as in (3.16), (3.20) and (3.21). The first and last are bounded as follows. Using Lemma 2.3 and the fact that there exists $t_\theta \in (0,k)$ such that $\mathbf{u}(t_{1/2}) - \mathbf{u}(t_0) = k\mathbf{u}_t(t_\theta)$, we obtain

$$\begin{aligned}
&|b^*(\mathbf{u}(t_0), \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}, \boldsymbol{\phi}^h_{1/2}) - b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}^h_{1/2})| \\
&= |b^*(\mathbf{u}(t_0), \mathbf{u}(t_{1/2}) + Ck^2\mathbf{u}_{tt}(t_\theta), \boldsymbol{\phi}^h_{1/2}) - b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}^h_{1/2})| \\
&\le |b^*(\mathbf{u}(t_0) - \mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}^h_{1/2})| + Ck^2|b^*(\mathbf{u}(t_0), \mathbf{u}_{tt}(t_\theta), \boldsymbol{\phi}^h_{1/2})| \\
&\le k|b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}^h_{1/2})| + Ck^2|b^*(\mathbf{u}(t_0), \mathbf{u}_{tt}(t_\theta), \boldsymbol{\phi}^h_{1/2})| \\
&\le k|b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}^h_{1/2})| + \epsilon\nu\|\nabla\boldsymbol{\phi}^h_{1/2}\|^2 + C\nu^{-1}k^4. \qquad (3.33)
\end{aligned}$$

In order to bound the first term in (3.33), we use integration by parts and Hölder's inequality to obtain

$$b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}^h_{1/2}) = (\mathbf{u}_t(t_\theta)\cdot\nabla\mathbf{u}(t_{1/2}), \boldsymbol{\phi}^h_{1/2}) + \frac{1}{2}(\nabla\cdot\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2})\cdot\boldsymbol{\phi}^h_{1/2}). \quad (3.34)$$

Thus,

$$\begin{aligned}
k|b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}^h_{1/2})| &\le Ck(\|\mathbf{u}_t(t_\theta)\|\|\nabla\mathbf{u}(t_{1/2})\|_{L^\infty(\Omega)} \\
&\quad + \|\nabla\mathbf{u}_t(t_\theta)\|\|\mathbf{u}(t_{1/2})\|_{L^\infty(\Omega)})\|\boldsymbol{\phi}^h_{1/2}\| \\
&\le Ck^3 + \frac{1}{4k}\|\boldsymbol{\phi}^h_{1/2}\|^2. \qquad (3.35)
\end{aligned}$$

Now use the bounds (3.33) and (3.35) in the error analysis at the first time level (note that $\boldsymbol{\phi}^h_{1/2} = \frac{1}{2}\boldsymbol{\phi}^h_1$, since $\boldsymbol{\phi}^h_0 = 0$) to get

$$\begin{aligned}
\|\boldsymbol{\phi}^h_1\|^2 + \nu k\|\nabla\boldsymbol{\phi}^h_1\|^2 + \alpha hk\|\nabla\boldsymbol{\phi}^h_1\|^2 &\le C[\nu^{-1}kh^{2m} + \nu^{-1}kh^{2m} + \nu^{-1}kh^{2m+2} \\
&\quad + \nu^{-1}\alpha^2h^2k^3 + \nu^{-1}\alpha h^{2m+1}k^2 + \nu^{-1}kh^{4m} \\
&\quad + \nu^{-1}k^5 + \nu k^5 + k^4]. \qquad (3.36)
\end{aligned}$$

If formulation (1.7) is used, then, instead of (3.37), we obtain, by adding and subtracting $b^*(\frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2} - \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2} + \mathbf{u}(t_{1/2}), \frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2}, \mathbf{v}^h)$ to the nonlinear terms in first time level analog of (3.10), the following

$$\begin{aligned}
&b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \mathbf{v}^h) - b^*(\frac{\mathbf{u}^h_0+\mathbf{u}^h_1}{2}, \frac{\mathbf{u}^h_0+\mathbf{u}^h_1}{2}, \mathbf{v}^h) \\
&\quad = b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}) - \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}, \mathbf{v}^h) \\
&\qquad + b^*(\frac{\mathbf{u}^h_0+\mathbf{u}^h_1}{2}, \mathbf{e}_{1/2}, \mathbf{v}^h) - b^*(\mathbf{e}_0, \frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2}, \mathbf{v}^h) \\
&\qquad + b^*(\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2} - \mathbf{u}(t_{1/2}), \frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2}, \mathbf{v}^h) \qquad (3.37)
\end{aligned}$$

Taking $\mathbf{v}^h = \boldsymbol{\phi}^h_{1/2}$, the second and third terms in (3.37) can be treated exactly as in (3.16), (3.20) and (3.21). The first and last are similar, since, after application of Lemma 2.1 and regularity assumptions on $\mathbf{u}$, both can be bounded as

$$C\|\nabla(\mathbf{u}(t_{1/2} - \frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2})\|\|\nabla\boldsymbol{\phi}^h_{1/2}\| \le \epsilon\nu\|\nabla\boldsymbol{\phi}^h_{1/2}\|^2 + C\nu^{-1}k^4,$$

with the help of Lemma 2.3 and Young's inequality. This leads to the upper bound

$$\|\phi_1^h\|^2 + \nu k\|\nabla\phi_1^h\|^2 + \alpha hk\|\nabla\phi_1^h\|^2 \le C[\nu^{-1}kh^{2m} + \nu^{-1}kh^{2m} + \nu^{-1}kh^{2m+2} + \nu^{-1}\alpha^2h^2k^3 + \nu^{-1}\alpha h^{2m+1}k^2 + \nu^{-1}kh^{4m} + \nu^{-1}k^5 + \nu k^5]. \quad (3.38)$$

This bound is sharper than (3.36), but it will not contribute to a higher order estimate. We thus insert the bound for $\|\phi_1^h\|^2 + \alpha hk\|\nabla\phi_1^h\|^2$, obtained in (3.36), into (3.31), which gives

$$\|\phi_{n+1}^h\|^2 + 2k\sum_{i=0}^{n}\nu\|\nabla(\frac{\phi_{i+1}^h + \phi_i^h}{2})\|^2 + 2\alpha hk\|\nabla\phi_{n+1}^h\|^2 \le C(\nu+\nu^{-1})\left(h^{2m} + \alpha^2h^2k^2 + k^4\right) + C\nu^{-1}(k\sum_{i=0}^{n}\|\phi_i^h\|^2). \quad (3.39)$$

Hence, it follows from the discrete Gronwall Lemma, that there exists $C = C(\nu, \Omega, T, \mathbf{u}, p)$ such that for any $n \ge 0$

$$\|\phi_{n+1}^h\|^2 + k\sum_{i=0}^{n}\nu\|\nabla(\frac{\phi_{i+1}^h + \phi_i^h}{2})\|^2 + \alpha hk\|\nabla\phi_{n+1}^h\|^2 \le C\left(h^{2m} + \alpha^2h^2k^2 + k^4\right). \quad (3.40)$$

Finally, the statement of the theorem follows from the triangle inequality. □

4 Error estimates for time derivatives and pressure

In order to prove pressure stability and convergence, we need to derive a bound on the time difference of the velocity error $\|\frac{\mathbf{e}_{n+1}-\mathbf{e}_n}{k}\|$.

Theorem 4.1. *Let the finite-element spaces $(\mathbf{X}^h, Q^h)$ include continuous piecewise polynomials of degree m and $m-1$ respectively ($m \ge 2$) and satisfy the discrete inf-sup condition. Let the assumptions of Theorem 3.1 be satisfied and*

$$\nabla\mathbf{u}_{tt} \in L^2(0,T;L^\infty(\Omega)), \Delta\mathbf{u}_{tt} \in L^2(0,T;L^2(\Omega)),$$

$$\mathbf{u}_{ttt} \in L^\infty(0,T;L^2(\Omega)),$$

$$\nabla p_{tt} \in L^2(0,T;L^2(\Omega)).$$

Then, if the finite element approximation $\mathbf{u}_n^h$ is defined by (1.5)-(1.6), there exists a constant $C = C(\nu, \mathbf{u}, p, T) < \infty$ such that

$$\nu k^2\|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\|^2 + \nu\|\nabla(\frac{\mathbf{e}_n + \mathbf{e}_{n-1}}{2})\|^2 + k\sum_{i=0}^{n-1}\|\frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{k}\|^2 + \alpha hk\cdot k\sum_{i=0}^{n-1}\|\nabla(\frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{k})\|^2 \le C(h^{2m} + \alpha^2h^2k^2 + h^{-3}k^8 + k^3). \quad (4.1)$$

If the finite element approximation $\mathbf{u}_n^h$ is defined via (1.7)-(1.6), then there exists a $C = C(\nu, \mathbf{u}, p, T) < \infty$ such that

$$\begin{aligned}
&\nu k^2 \|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\|^2 + \nu\|\nabla(\frac{\mathbf{e}_n + \mathbf{e}_{n-1}}{2})\|^2 \\
&+ \quad k\sum_{i=0}^{n-1} \|\frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{k}\|^2 + \alpha h k \cdot k \sum_{i=0}^{n-1} \|\nabla(\frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{k})\|^2 \\
&\leq C(h^{2m} + \alpha^2 h^2 k^2 + h^{-3}k^8 + k^4). \qquad (4.2)
\end{aligned}$$

Proof. Consider the error decomposition (3.11). Take $\mathbf{v}^h = \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k} \in \mathbf{V}^h$ in (3.12),(3.13) to obtain

$$\begin{aligned}
\|\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}\|^2 \quad &+ \quad \nu\frac{\|\nabla\boldsymbol{\phi}_{n+1}^h\|^2 - \|\nabla\boldsymbol{\phi}_n^h\|^2}{2k} + \alpha h k \|\nabla(\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k})\|^2 \\
&= \quad (\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) + \nu(\nabla(\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}), \nabla(\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k})) \\
&\quad -(\frac{p(t_{n+1}) + p(t_n)}{2} - q^h, \nabla\cdot\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +b^*(E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})], \mathbf{e}_{n+1/2}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad -b^*(E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}], \mathbf{e}_{n+1/2}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +b^*(E[\boldsymbol{\phi}_n, \boldsymbol{\phi}_{n-1}], \mathbf{e}_{n+1/2}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +b^*(E[\mathbf{e}_n, \mathbf{e}_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +T(\mathbf{u}, p; \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}), \qquad (4.3)
\end{aligned}$$

where, using Taylor expansion,

$$\begin{aligned}
T(\mathbf{u}, p; \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \quad &= \quad (\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{k} - \mathbf{u}_t(t_{n+1/2}), \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +\alpha h k(\nabla(\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}), \nabla(\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k})) \\
&\quad +Ck^2\nu(\nabla\mathbf{u}_{tt}(t_{n+\theta}), \nabla(\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k})) + Ck^2(\mathbf{u}_{ttt}(t_{n+\theta}), \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +Ck^2 b^*(\mathbf{u}_{tt}(t_{n+\theta}), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +Ck^2 b^*(\mathbf{u}(t_{n+1/2}), \mathbf{u}_{tt}(t_{n+\theta}), \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +\alpha h k(\nabla(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}), \nabla(\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k})) \\
&\quad +Ck^2(p_{tt}(t_{n+\theta}), \nabla\cdot(\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k})), \qquad (4.4)
\end{aligned}$$

for some $\theta \in (0, 1)$ and $\forall q^h \in Q^h$.

Also it follows from the definition of Stokes Projection that

$$\nu(\nabla(\frac{\boldsymbol{\eta}_{n+1}-\boldsymbol{\eta}_n}{k}),\nabla(\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}))-(\frac{p(t_{n+1})+p(t_n)}{2}-q^h,\nabla\cdot\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})=0. \quad (4.5)$$

We bound the four nonlinear terms on the right-hand side of (4.3), using Lemma 2.1 and Cauchy-Schwarz and Young's inequalities. For the first term integrating by parts and applying Hölder's inequality gives

$$\begin{aligned}
&|b^*(E[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})],\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\
&\quad= |(E[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})]\cdot\nabla\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})\\
&\qquad+\frac{1}{2}(\nabla\cdot E[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})],\mathbf{e}_{n+1/2}\cdot\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\
&\quad= |(E[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})]\cdot\nabla\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\
&\quad\le C\|\nabla\mathbf{e}_{n+1/2}\|\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|\\
&\quad\le \epsilon\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2+C\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2. \qquad (4.6)
\end{aligned}$$

Using the first bound from Lemma 2.1 and the inverse inequality (2.2), we obtain the bounds on the second and third nonlinear terms

$$\begin{aligned}
|b^*(E[\boldsymbol{\eta}_n,\boldsymbol{\eta}_{n-1}],\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})| &\le Ch^{-1}\|\nabla E[\boldsymbol{\eta}_n,\boldsymbol{\eta}_{n-1}]\|\|\nabla\mathbf{e}_{n+1/2}\|\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|\\
&\le \epsilon\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2\\
&+ Ch^{-2}\|\nabla(\frac{3}{2}\boldsymbol{\eta}_n-\frac{1}{2}\boldsymbol{\eta}_{n-1})\|^2\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2, \qquad (4.7)
\end{aligned}$$

and, using also the intermediate result (3.40) of Theorem 3.1,

$$\begin{aligned}
&|b^*(E[\boldsymbol{\phi}_n,\boldsymbol{\phi}_{n-1}],\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\
&\quad\le Ch^{-3/2}\|E[\boldsymbol{\phi}_n,\boldsymbol{\phi}_{n-1}]\|\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|\\
&\quad\le \epsilon\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2+Ch^{-3}(h^{2m}+\alpha^2h^2k^2+k^4)\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2. \qquad (4.8)
\end{aligned}$$

Finally, consider the fourth nonlinear term. Use the obvious identity $\frac{3}{2}\mathbf{e}_n-\frac{1}{2}\mathbf{e}_{n-1}=\frac{\mathbf{e}_n+\mathbf{e}_{n-1}}{2}+(\mathbf{e}_n-\mathbf{e}_{n-1})$ and the regularity of $\mathbf{u}$. It follows from the last inequality of Lemma

2.1 that

$$
\begin{aligned}
&|b^*(E[\mathbf{e}_n,\mathbf{e}_{n-1}],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})| \\
&\quad\le |b^*(\frac{\mathbf{e}_n+\mathbf{e}_{n-1}}{2},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})| \\
&\qquad+|b^*(\boldsymbol{\eta}_n-\boldsymbol{\eta}_{n-1},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})| \\
&\qquad+|b^*(k\frac{\boldsymbol{\phi}^h_n-\boldsymbol{\phi}^h_{n-1}}{k},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})| \\
&\quad\le \epsilon\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2+C\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2 \\
&\qquad+C(h^{2m}+\alpha^2h^2k^2+k^4)+Ck^2\|\nabla(\frac{\boldsymbol{\phi}^h_n-\boldsymbol{\phi}^h_{n-1}}{k})\|^2. \qquad (4.9)
\end{aligned}
$$

Insert these bounds in (4.3). The bound on $|T(\mathbf{u},p;\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|$ is obtained as in the proof of Theorem 3.1. Choosing $\epsilon=\frac{1}{24}$ gives

$$
\begin{aligned}
\frac12\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2 &+ \nu\frac{\|\nabla\boldsymbol{\phi}^h_{n+1}\|^2-\|\nabla\boldsymbol{\phi}^h_n\|^2}{2k}+\frac{\alpha hk}{2}\|\nabla(\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})\|^2 \\
&\le C\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2+Ck^2\|\nabla(\frac{\boldsymbol{\phi}^h_n-\boldsymbol{\phi}^h_{n-1}}{k})\|^2+C(h^{2m}+\alpha^2h^2k^2+k^4) \\
&\quad+Ch^{-3}(h^{2m}+\alpha^2h^2k^2+k^4)\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2. \qquad (4.10)
\end{aligned}
$$

At the first time level, take $\mathbf{v}^h=\frac{\boldsymbol{\phi}^h_1-\boldsymbol{\phi}^h_0}{k}$; taking $\mathbf{U}_0=\mathbf{u}^h_0$ in the initial error decomposition gives $\boldsymbol{\phi}^h_0=0$. For the constant extrapolation (1.5) we obtain

$$
\begin{aligned}
\frac12\|\frac{\boldsymbol{\phi}^h_1-\boldsymbol{\phi}^h_0}{k}\|^2 &+ \nu\frac{\|\nabla\boldsymbol{\phi}^h_1\|^2-\|\nabla\boldsymbol{\phi}^h_0\|^2}{2k}+\frac{\alpha hk}{2}\|\nabla(\frac{\boldsymbol{\phi}^h_1-\boldsymbol{\phi}^h_0}{k})\|^2 \\
&\le C(h^{2m}+\alpha^2h^2k^2+k^4)+kb^*(\mathbf{u}_t(t_\theta),\mathbf{u}_{1/2},\frac{\boldsymbol{\phi}^h_1-\boldsymbol{\phi}^h_0}{k}). \qquad (4.11)
\end{aligned}
$$

If we use (1.7) instead of (1.5) at the first time level, we have

$$
\begin{aligned}
\frac12\|\frac{\boldsymbol{\phi}^h_1-\boldsymbol{\phi}^h_0}{k}\|^2 &+ \nu\frac{\|\nabla\boldsymbol{\phi}^h_1\|^2-\|\nabla\boldsymbol{\phi}^h_0\|^2}{2k}+\frac{\alpha hk}{2}\|\nabla(\frac{\boldsymbol{\phi}^h_1-\boldsymbol{\phi}^h_0}{k})\|^2 \\
&\le C(h^{2m}+\alpha^2h^2k^2+k^4)+k^2b^*(\mathbf{u}_t(t_\theta),\mathbf{u}_{1/2},\frac{\boldsymbol{\phi}^h_1-\boldsymbol{\phi}^h_0}{k}). \qquad (4.12)
\end{aligned}
$$

Sum (4.10) over the time levels $n\ge1$ and add to (4.11) (or to (4.12) in the case of linear extrapolation). Multiply by $2k$ to obtain

$$
\begin{aligned}
k\sum_{i=0}^{n}\|\frac{\boldsymbol{\phi}^h_{i+1}-\boldsymbol{\phi}^h_i}{k}\|^2 &+ \nu\|\nabla\boldsymbol{\phi}^h_{n+1}\|^2+\alpha hk\cdot k\sum_{i=0}^{n}\|\nabla(\frac{\boldsymbol{\phi}^h_{i+1}-\boldsymbol{\phi}^h_i}{k})\|^2 \\
&\le Ck^2\cdot k\sum_{i=0}^{n-1}\|\nabla(\frac{\boldsymbol{\phi}^h_{i+1}-\boldsymbol{\phi}^h_i}{k})\|^2+C(h^{2m}+\alpha^2h^2k^2+k^4+h^{-3}k^8) \\
&\quad+k^{2+\sigma}b^*(\mathbf{u}_t(t_\theta),\mathbf{u}(t_{1/2}),\frac{\boldsymbol{\phi}^h_1-\boldsymbol{\phi}^h_0}{k}), \qquad (4.13)
\end{aligned}
$$

where $\sigma = 0$ for the constant extrapolation (1.5) and $\sigma = 1$ for the linear extrapolation (1.7).

For any $n \geq 1$, add the inequalities (4.13) at the time levels $n+1$ and n. Use the identity

$$\|\nabla \boldsymbol{\phi}^h_{n+1}\|^2 + \|\nabla \boldsymbol{\phi}^h_n\|^2 = \frac{1}{2}k^2\|\nabla(\frac{\boldsymbol{\phi}^h_{n+1} - \boldsymbol{\phi}^h_n}{k})\|^2 + 2\|\nabla(\frac{\boldsymbol{\phi}^h_{n+1} + \boldsymbol{\phi}^h_n}{2})\|^2. \tag{4.14}$$

At any time level $n \geq 1$ we obtain

$$\begin{aligned}
\frac{1}{2}\nu k^2\|\nabla(\frac{\boldsymbol{\phi}^h_{n+1} - \boldsymbol{\phi}^h_n}{k})\|^2 \quad &+ \quad 2\nu\|\nabla(\frac{\boldsymbol{\phi}^h_{n+1} + \boldsymbol{\phi}^h_n}{2})\|^2 \\
&+k\sum_{i=0}^{n}\|\frac{\boldsymbol{\phi}^h_{i+1} - \boldsymbol{\phi}^h_i}{k}\|^2 + \alpha hk \cdot k\sum_{i=0}^{n}\|\nabla(\frac{\boldsymbol{\phi}^h_{i+1} - \boldsymbol{\phi}^h_i}{k})\|^2 \\
&\leq \quad C\nu^{-1} \cdot k\sum_{i=0}^{n-1}\frac{1}{2}\nu k^2\|\nabla(\frac{\boldsymbol{\phi}^h_{i+1} - \boldsymbol{\phi}^h_i}{k})\|^2 \\
&\quad +C(h^{2m} + \alpha^2h^2k^2 + k^4 + h^{-3}k^8) \\
&\quad +k^{2+\sigma}b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \frac{\boldsymbol{\phi}^h_1 - \boldsymbol{\phi}^h_0}{k}).
\end{aligned} \tag{4.15}$$

Next, decompose the last term in the right-hand side of (4.15), using Lemma 2.1 and Young's inequality. This yields

$$k^{2+\sigma}|b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \frac{\boldsymbol{\phi}^h_1 - \boldsymbol{\phi}^h_0}{k})| \leq \frac{1}{2}k\|\frac{\boldsymbol{\phi}^h_1 - \boldsymbol{\phi}^h_0}{k}\|^2 + Ck^{3+2\sigma}. \tag{4.16}$$

Hence it follows from the discrete Gronwall Lemma that

$$\begin{aligned}
\nu k^2\|\nabla(\frac{\boldsymbol{\phi}^h_{n+1} - \boldsymbol{\phi}^h_n}{k})\|^2 + \nu\|\nabla(\frac{\boldsymbol{\phi}^h_{n+1} + \boldsymbol{\phi}^h_n}{2})\|^2 + k\sum_{i=0}^{n}\|\frac{\boldsymbol{\phi}^h_{i+1} - \boldsymbol{\phi}^h_i}{k}\|^2 \\
+\alpha hk \cdot k\sum_{i=0}^{n}\|\nabla(\frac{\boldsymbol{\phi}^h_{i+1} - \boldsymbol{\phi}^h_i}{k})\|^2 \leq C(h^{2m} + \alpha^2h^2k^2 + k^4 + k^{3+2\sigma} + h^{-3}k^8).
\end{aligned} \tag{4.17}$$

The proof of the theorem is now concluded by the triangle inequality. □

For the stability of pressure we will need the following *á priori* bounds

Lemma 4.1. *Let the assumptions of Theorem 4.1 hold. Then there exists a constant* $C = C(\nu, \mathbf{u}, p, T)$ *such that for any* n

$$k\sum_{i=0}^{n}\|\frac{\mathbf{u}^h_{i+1} - \mathbf{u}^h_i}{k}\| \leq k\sum_{i=0}^{n}\|\frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{k}\| + k\sum_{i=0}^{n}\|\frac{\mathbf{u}(t_{i+1}) - \mathbf{u}(t_i)}{k}\| \leq C,$$

$$k^2\|\nabla(\frac{\mathbf{u}^h_{n+1} - \mathbf{u}^h_n}{k})\|^2 \leq k^2\|\nabla(\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k})\|^2 + k^2\|\nabla(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{k})\|^2 \leq C.$$

Proof. Use the decomposition $\mathbf{u}^h_i = \mathbf{u}(t_i) - \mathbf{e}_i$. The triangle inequality completes the proof. □

Theorem 4.2 (Pressure Stability). *Let $(\mathbf{u}_n^h, p_n^h)$ satisfy (1.5)-(1.6) (or (1.7)-(1.6)). Let $f \in L^2(0,T;H^{-1}(\Omega))$ and let the assumptions of Theorem 4.1 be satisfied. Then,*

$$k\sum_{i=0}^{n-1}\|\frac{p_{i+1}^h + p_i^h}{2}\| \leq C(\mathbf{u}_0^h, \mathbf{f}, \beta^h),$$

where β^h is the constant from the discrete LBB^h condition (2.1).

Proof. Consider (1.6). Using the Cauchy-Schwarz inequality, the first bound from Lemma 2.1, the discrete LBB^h condition (2.1) and the identity $\frac{3}{2}\mathbf{u}_{n+1}^h - \frac{1}{2}\mathbf{u}_n^h = \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2} + k\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}$, we obtain

$$\begin{aligned}\beta^h\|\frac{p_{n+1}^h + p_n^h}{2}\| \quad \leq \quad & \|\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}\|_{-1} + \nu\|\nabla(\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2})\| \\ & +\alpha hk\|\nabla(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k})\| + C\|\nabla(\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2})\|^2 \\ & +Ck^2\|\nabla(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k})\|^2 + \|\mathbf{f}(t_{n+1/2})\|_{-1}.\end{aligned}$$

Sum over all time levels; the bounds of Lemma 4.1 complete the proof. □

We conclude this section by deriving the pressure error estimate.

Theorem 4.3 (Pressure Convergence). *Let $(\mathbf{u}_n^h, p_n^h)$ satisfy (1.6) for $n \geq 2$. Let $(\mathbf{u}_1^h, p_1^h)$ satisfy the constant extrapolation (1.5) or the linear extrapolation (1.7). Then, under the assumptions of Theorem 4.1,*

$$k\sum_{i=0}^{n-1}\|p(t_{i+1/2}) - p_{i+1/2}^h\| \leq C(\nu, \mathbf{u}, p, T)(h^m + \alpha hk + h^{-3/2}k^4 + k^{3/2+\sigma/2}), \tag{4.18}$$

where $\sigma = 0$ for the constant extrapolation and $\sigma = 1$ for the linear extrapolation.

Proof. Consider (3.10), which holds true for any $\mathbf{v}^h \in \mathbf{X}^h$. Decompose the pressure approximation error into

$$p(t_{n+1}) - p_{n+1}^h = (p(t_{n+1}) - I(p)) - (p_{n+1}^h - I(p)) = \tilde{\eta}_{n+1} - \tilde{\phi}_{n+1}^h, \tag{4.19}$$

where $\tilde{\phi}_{n+1}^h \in Q^h$, $I(p)$ is a projection of $p(t_{n+1})$ into Q^h.

Use the error decomposition (4.19) in (3.10) and apply the discrete LBB^h condition to obtain for any $n \geq 1$

$$\begin{aligned}\beta^h\|\frac{\tilde{\phi}_{n+1}^h + \tilde{\phi}_n^h}{2}\| \quad \leq \quad & \|\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k}\|_{-1} + C\|\nabla(\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2})\| \\ & +C\|\nabla(\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2})\|^2 + Ck^2\|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\|^2 + Ck\|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\| \\ & +\nu\|\nabla(\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2})\| + \alpha hk\|\nabla(\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k})\| \\ & +\|\frac{\tilde{\eta}_{n+1} + \tilde{\eta}_n}{2}\| + C\nu k^2 + Ck^2 + C\alpha hk.\end{aligned} \tag{4.20}$$

Hence from the triangle inequality we get

$$
\begin{aligned}
\beta^h \| \frac{(p(t_{n+1}) - p^h_{n+1}) + (p(t_n) - p^h_n)}{2} \| \quad \leq \quad & \| \frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k} \|_{-1} + C \| \nabla (\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2}) \| \\
& + C \| \nabla (\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2}) \|^2 + C k^2 \| \nabla (\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k}) \|^2 \\
& + C k \| \nabla (\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k}) \| + \nu \| \nabla (\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2}) \| \\
& + \alpha h k \| \nabla (\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k}) \| + C \nu k^2 + C k^2 + C \alpha h k \\
& + \inf_{q^h \in Q^h} \| \frac{p(t_{n+1}) + (p(t_n)}{2} - q^h \|. \qquad (4.21)
\end{aligned}
$$

On the first time level consider the constant extrapolation (1.5). Using the discrete LBB^h condition and (3.36), we obtain the following bound (which can be improved in the case of linear extrapolation):

$$
\beta^h k \| \frac{(p(t_1) - p^h_1) + (p(t_0) - p^h_0)}{2} \| \leq C(k^2 + h^m + \alpha h k). \qquad (4.22)
$$

Add the inequalities (4.21) for all $n \geq 1$, multiply by k and add to (4.22). The proof is concluded by applying the result of Theorem 4.1 □

5 Physical Fidelity: Conservation of Integral Invariants

We begin by proving that CNLEStab exactly conserves a modified kinetic energy.

Proposition 5.1. *Let the boundary conditions be periodic; assume also* $\mathbf{f} = \nu = 0$ *. Define*

$$
\textit{Kinetic energy in (1.6)} = KE(t_n) := \frac{1}{2L^3} [|| \mathbf{u}^h_n ||^2 + \alpha k h || \nabla \mathbf{u}^h_n ||^2]
$$

The method exactly conserves kinetic energy. Specifically, for all $t_n > 0$

$$
KE(t_n) = KE(0).
$$

Proof. Set $\mathbf{v}^h = \frac{\mathbf{u}^h_{n+1} + \mathbf{u}^h_n}{2}$ and $\nu = \mathbf{f} = 0$ in (1.5)-(1.6). □

Exact conservation of helicity likely does not hold for CNLEStab. We thus consider approximate helicity conservation experimentally by considering and inviscid ($\nu = 0$) fluid with no forcing term ($\mathbf{f} = 0$). A comparison of the CNLE and CNLEStab under these conditions, in Figure 1, shows that, for a fixed mesh size, CNLEStab nearly conserves helicity, while CNLE does not. Both conserve kinetic energies during these experiments.

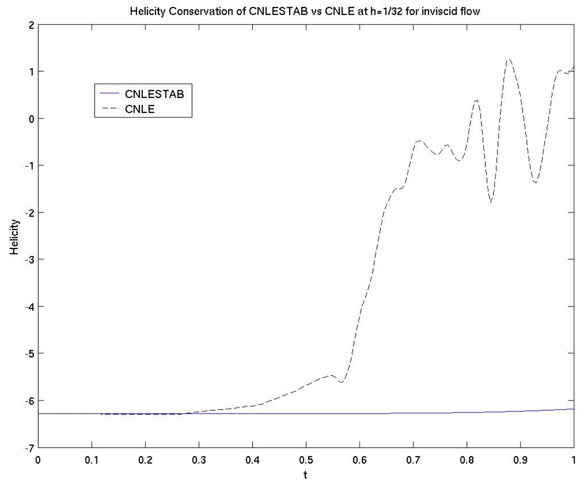

Figure 1: Conservation of helicity, CNLE ($\alpha = 0$) versus CNLEStab ($\alpha = 1$)

A comparison of the CNLEStab performance for different mesh sizes is shown in Figure 2, confirming that as the mesh is refined, conservation of helicity improves.

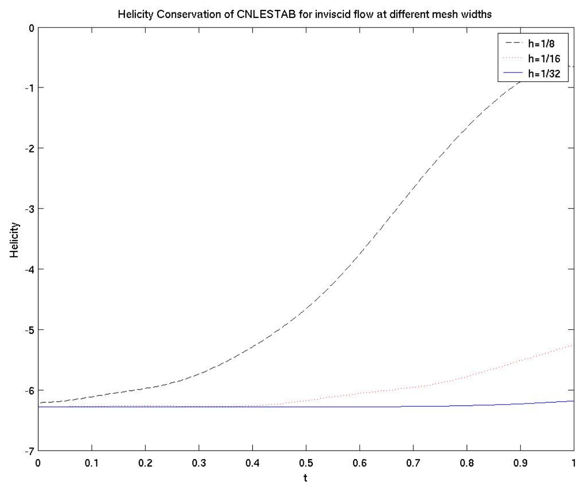

Figure 2: Conservation of helicity for different mesh sizes in CNLESTAB (with $\alpha = 1$): as h gets smaller, helicity is conserved longer.

6 Physical Fidelity: Predictions of the Turbulent Energy Cascade

We consider the energy cascade predicted by (1.1) in the case of homogeneous, isotropic turbulence. Motivated by the consistency error argument, we consider the modified equation of the method (1.1). Since $\alpha h(\nabla \mathbf{u}(t_{n+1}), \nabla \mathbf{v}) - \alpha h(\nabla \mathbf{u}(t_n), \nabla \mathbf{v}) = -\alpha hk(\Delta \mathbf{u}_t, \mathbf{v})$, we postulate a fluid with equations of motion given by: $w : \Omega \times [0,T] \to \mathbb{R}^d$, $q : \Omega \times (0,T] \to \mathbb{R}$ satisfying:

$$\begin{aligned} &[\mathbf{w} - \alpha hk\triangle\mathbf{w}]_t + \mathbf{w}\cdot\nabla\mathbf{w} - \nu\Delta\mathbf{w} + \nabla q = \mathbf{f}, \text{ for } \mathbf{x} \in \Omega,\ 0 < t \leq T \\ &\nabla\cdot\mathbf{w} = 0,\ \mathbf{x} \in \Omega, \text{ for } 0 < t \leq T, \\ &\text{periodic boundary conditions on } \partial\Omega, \text{ for } 0 < t \leq T, \\ &\mathbf{w}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \text{ for } \mathbf{x} \in \Omega, \end{aligned} \tag{6.1}$$

and the usual normalization condition in the periodic case that $\int_\Omega \phi(\mathbf{x},t)\, d\mathbf{x} = 0$ on $\phi = \mathbf{w}, q, \mathbf{f}, \mathbf{u}_0$ for $0 < t \leq T$. Thus we explore more subtle effects of the stabilization in Algorithm 1.1 through its modified equation (6.1).

We multiply (6.1) by $\mathbf{w}$ and integrate over the domain and time to obtain its precise energy balance given by

$$\frac{1}{2}\{||\mathbf{w}(t)||^2 + \alpha hk||\nabla\mathbf{w}(t)||^2\} + \int_0^T \nu||\nabla\mathbf{w}(t)||^2 = \frac{1}{2}\{||\mathbf{w}(0)||^2 + \alpha hk||\nabla\mathbf{w}(0)||^2\} + (\mathbf{f}(t), \mathbf{w}(t)).$$

We can clearly identify three physical quantities of kinetic energy, energy dissipation rate and power input. Let L denote the global length scale, e.g., $L = vol(\Omega)^{1/3}$; then these are given by

$$\text{Modified equations kinetic energy: } E_{model}(\mathbf{w})(t) := \frac{1}{2L^3}\{||\mathbf{w}(t)||^2 + \alpha hk||\nabla\mathbf{w}(t)||^2\}, \tag{6.2}$$

$$\text{Modified equations dissipation rate: } \varepsilon_{model}(\mathbf{w})(t) := \frac{\nu}{L^3}||\nabla\mathbf{w}(t)||^2, \tag{6.3}$$

$$\text{Modified equations power input: } P_{model}(\mathbf{w})(t) := \frac{1}{L^3}(\mathbf{f}(t), \mathbf{w}(t)). \tag{6.4}$$

The kinetic energy has an extra term which reflects extraction of energy from resolved scales. The energy dissipation rate in the model (6.3) is the same as for NSE equations.

Equation (6.1) shares the common features of the Navier-Stokes equations which make existence of an energy cascade likely, e.g. [F95], [P00]. First, (6.1) has the same nonlinearity as the Navier-Stokes equations, which pumps energy from larger to smaller scales. Next, the solution of (6.1) satisfies an energy equality in which its kinetic energy and energy dissipation are readily discernible, and for $\nu = 0$ the kinetic energy is conserved through a large range of scales/wave-numbers. Since both conditions are satisfied we are to proceed to develop a quantitative theory of energy cascade of (6.1).

6.1 Kraichnan's Dynamic Analysis Applied to CNLEStab

The energy cascade will now be investigated more closely using the dynamical argument of Kraichnan, [K71]. Let $\Pi_{model}(\kappa)$ be defined as the total rate of energy transfer from all wave numbers $< \kappa$ to all wave numbers $> \kappa$ (not to confuse the wave number κ with the time step k). Following Kraichnan [K71] we assume that $\Pi_{model}(\kappa)$ is proportional to the total energy ($\kappa E_{model}(\kappa)$) in wave numbers of the order κ and to some effective rate of shear $\sigma(\kappa)$ which acts to distort flow structures of scale $1/\kappa$. That is:

$$\Pi_{model}(\kappa) \simeq \sigma(\kappa)\, \kappa\, E_{model}(\kappa) \tag{6.5}$$

Furthermore, we expect

$$\sigma(\kappa)^2 \simeq \int_0^{\kappa} p^2 E_{model}(p) dp \tag{6.6}$$

The major contribution to (6.6) is from $p \simeq \kappa$, in accord with Kolmogorov's localness assumption, [Kol41]. This is because all wave numbers $\leq \kappa$ should contribute to the effective mean-square shear acting on wave numbers of order κ, while the effects of all wave numbers $\gg \kappa$ can plausibly be expected to average out over the scales of order $1/\kappa$ and over times the order of the characteristic distortion time $\sigma(\kappa)^{-1}$.

Let $E(\kappa) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T E(\kappa, t)$ is the distribution of the time averaged kinetic energy by wave number. Here, we have $E(\kappa, t) = \frac{L}{2\pi}\sum_{|\mathbf{k}|=\kappa} \frac{1}{2}|\hat{\mathbf{u}}(\mathbf{k}, t)|^2$ where L - the reference length, $\mathbf{k}, \kappa$ - the wave number vector and the wave number respectively, and $\hat{\mathbf{u}}(\mathbf{k}, t)$ - the Fourier modes of the Navier-Stokes velocity.

We shall say that there is an energy cascade if in some "inertial" range, $\Pi_{model}(\kappa)$ is independent of the wave number, i.e., $\Pi_{model}(\kappa) = \varepsilon_{model}$. Using the equations (6.5) and (6.6) we get

$$E_{model}(\kappa) \simeq \varepsilon_{model}^{2/3}\kappa^{-5/3}$$

We have the relation

$$E_{\ model}(\kappa) \simeq (1 + \alpha h k \kappa^2) E(\kappa). \tag{6.7}$$

Using (6.7) we obtain:

$$\begin{aligned} \text{Model's cutoff lengthscale} \quad &: \quad \sqrt{\alpha h k}, \\ E(\kappa) \quad &\simeq \quad \varepsilon_{model}^{2/3}\kappa^{-5/3}, \text{ for } \kappa \ll \frac{1}{(\alpha h k)^{1/2}}, \\ E(\kappa) \quad &\simeq \quad \varepsilon_{model}^{2/3}(\alpha h k)^{-1}\kappa^{-11/3}\ , \text{for } \kappa \gg \frac{1}{(\alpha h k)^{1/2}}\ . \end{aligned}$$

Therefore, (6.1) possesses an energy cascade with an enhanced kinetic energy. The extra term in (6.1) triggers an accelerated energy decay of $O(\kappa^{-11/3})$ beyond the cutoff length scale. Above the cutoff length scale (6.1) predicts the correct energy cascade of $O(\kappa^{-5/3})$.

7 Computational Tests

We first test convergence rates for a problem with a known exact solution. The example is one for which the true solution is known,

$$\mathbf{u} = \begin{pmatrix} \cos(2\pi(z+t)) \\ \sin(2\pi(z+t)) \\ \sin(2\pi(x+t)) \end{pmatrix}, \tag{7.1}$$

and then the right-hand side $\mathbf{f}$ and initial condition $\mathbf{u}_0$ are computed such that (7.1) satisfies (1.1). We selected this test problem because it is simple but already possesses complex rotational structures.

For $\alpha = 1$, $\nu = 1$ and final time $T = 0.5$, the calculated convergence rates in Table 1 confirm what is predicted by Theorem 3.1 for (P_2, P_1) discretization in space.

h	$\|\| \mathbf{u} - \mathbf{u}^h \|\|_{H^1(\Omega)}$	ratio	rate
1/8	0.6910	-	-
1/16	0.1772	3.8995	1.9633
1/32	0.0447	3.9642	1.9870

Table 1: Experimental convergence rates.

Next we give a simple test of the positive effects of the stabilization on the methods complexity. The linear solver used in the simulations was (unpreconditioned) Conjugate Gradient Squared (CGS). On a $h = 1/16$ mesh in $\mathbb{R}^3$, with $\nu = \frac{1}{500}$ and the same true solution (7.1), the number of CGS iterates needed for the first 8 solves of Crank-Nicolson with Linear Extrapolation (CNLE), i.e. $\alpha = 0$, and CNLE with stabilization (CNLEStab, $\alpha > 0$) are compared in Table 2.

time level	CNLE	CNLEStab
1	349	193
2	350	199
3	347	200
4	372	212
5	348	206
6	347	206
7	351	205
8	365	192

Table 2: Number of CGS iterations for CNLE versus CNLEStab.

The linear system to be solved at each time step is also better conditioned when $\alpha > 0$.

8 Conclusions

A simple second order time stepping algorithm for the Navier-Stokes equations was analyzed. It is a modification (by introduction of artificial viscosity stabilization and correction for the associated loss of accuracy) of the commonly used Crank-Nicolson scheme that requires the solution of only one linear system per time step. We not only proved that it is unconditionally stable and investigated how the rates of convergence for velocity and pressure behave, but we also went beyond error analysis. We showed that this scheme conserves kinetic energy exactly, and provided experimental numerical evidence that it nearly conserves helicity, an important integral invariant in three dimensional rotational flows. Dynamic analysis applied to their algorithm reveals the existence of an energy cascade with the correct statistics up to a cutoff length scale and with an accelerated energy decay above the cutoff length scale. Lastly, we presented more computational tests. The first confirms the velocity convergence rates obtained in the analysis in Section 3, and the second shows that even with a simple, unpreconditioned iterative method the linear system to be solved at each time step is better conditioned than the corresponding system without stabilization.

References

[ALP04] M. Anitescu, W. Layton and F. Pahlevani, *Implicit for local effects and explicit for nonlocal effects is unconditionally stable*, ETNA 18 (2004), 174-187.

[AS96] F. Armero and J.C. Simo, *Long-term dissipativity of time-stepping algorithms for an abstract evolution equation*, CMAME 131(1996),41-90.

[Bak83] A. J. Baker, *Finite element computational fluid mechanics*, Hemisphere, Washington, 1983.

[B76] G.A. Baker, *Galerkin approximations for the Navier-Stokes equations*, Technical Report,1976.

[BDK82] G.A. Baker, V. Dougalis and O. Karakashian, *On a higher order accurate, fully discrete Galerkin approximation to the Navier-Stokes equations*, Math. Comp. 39(1982),339-375.

[BS94] S.C. Brenner and L.R. Scott, *The mathematical theory of finite element methods*, Springer, Berlin, 1994.

[BGP87] M.O. Bristeau, R. Glowinski, J. Periaux, *Numerical methods for the Navier-Stokes equations: Applications to the simulation of compressible and incompressible flows*, Comput. Phys. Reports, 6(1987) 73-187.

[ESW05] H. C. Elman, D. J. Silvester, A. J. Wathen, *Finite Elements and Fast Iterative Solvers: with applications in incompressible fluid dynamics*, OUP, Oxford, 2005.

[GS00] P.M. Gresho and R.L. Sani, *Incompressible flows and the finite element method, volume 2*, John Wiley and Sons, Chichester, 2000.

[Kol41] A. V. Kolmogorov, *The local structure of turbulence in incompressible viscous fluids for very large Reynolds number*, Dokl. Akad. Nauk SSR, 30:9-13, 1941.

[He03] Yinnian He, *Two-Level Method Based on Finite Element and Crank-Nicolson Extrapolation for the Time-Dependent Navier-Stokes Equations*, SIAM J. Numer. Anal. 41(2003), 1263.

[HR90] J. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier Stokes equations, IV: Error analysis for second order time discretizations*, SIAM JNA 27, (1990), 353-384.

[F95] U. Frisch, *Turbulence*, Cambridge, 1995.

[G89] M. D. Gunzburger, *Finite element methods for viscous incompressible flows: A guide to theory, practice, and algorithms*, San Diego, CA, Academic Press, Inc., 1989.

[GR79] V. Girault, P.A. Raviart, *Finite element approximation of the Navier-Stokes equations*, Lecture notes in mathematics, no. 749, Springer-Verlag, 1979.

[GR86] V. Girault, P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations - Theory and Algorithms*, Springer Verlag, Berlin, 1986.

[KT07] V. K. Kalantarov and E. S. Titi, *Global attractors and determining modes for the 3D Navier-Stokes-Voight equations*, preprint 2007.

[K71] R. H. Kraichnan, *Inertial-range transfer in two- and three- dimensiona turbulence*, J. Fluid Mech., 47, 525 (1971).

[LT98] W. Layton, L. Tobiska, *A Two-Level Method with Backtracking for the Navier-Stokes Equations*, SIAM JNA, 35, (1998), 2035-2054.

[O80] A. P. Oskolkov, *On the theory of Voight fluids*, Zap. Nauchn. Sem. LOMI, 96, (1980), 233-236.

[P83] O. Pironneau, *The Finite Element Methods for Fluids*, John Wiley, Chichester, 1983.

[P00] S. Pope, *Turbulent Flows*, Cambridge Univ. Press, 2000.

[P06] F. Pahlevani, *Sensitivity computations of eddy viscosity models with an application in drag computation*, International Journal for Numerical Methods in Fluids, 52, 381-392 (2006)

[Q93] L. Quartapelle ,*Numerical solution of the incompressible Navier-Stokes equations*: Birkhauser Verlag, 1993.

[SAT95] J.C. Simo, F. Armero and C.A. Taylor, *Stable and time-dissipative finite element methods for incompressible Navier-Stokes equations in advection dominated flows*, IJNME 39(1995),1475-1506.

[Tom81] F.-C. Thomasset, *Implementation of Finite Element Methods for Navier-Stokes Equations*, Springer Verlag, Berlin, 1981.

[T84] R. Temam,*Navier-Stokes equations: theory and numerical analysis*, Elsevier North-Holland, 1984.

[EL01] Weinan E, J.-G. Liu, *Simple finite element method in vorticity formulation for incompressible flows*, Math. Comp, 70, 1997, pp. 579-593.