Preliminary Exam in Analysis, January 2023

Problem 1. Let $\{f_n\}$ be a sequence of C^∞ functions on a compact interval I such that for each $k \geq 0$ there exists M_k such that

$$|f_n^{(k)}(x)| \leq M_k \qquad \text{for all } n \text{ and } x \in I.$$

Prove that there exists a subsequence converging uniformly, together with the derivatives of all orders, to a C^∞ function.

Hint: A function f is C^∞ means that $f \in C^k$ for all k. You may consider using a diagonalization argument.

Problem 2. Compute the surface integral

$$I = \iint_\Sigma \frac{x\,dydz + y\,dzdx + z\,dxdy}{(x^2+y^2+z^2)^{3/2}}$$

for each of the following cases:

(1) $\Sigma = \{(x,y,z) : x^2+y^2+z^2 = t^2\}$;

(2) $\Sigma = \partial V$ where V is a bounded smooth closed region that does not include the origin;

(3) $\Sigma = \partial V$ where V is a bounded smooth closed region that contains the origin.

Problem 3. Assume $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Given that for all $x, y \in \mathbb{R}^n$ we have

$$f\Big((x+y)/2\Big) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

show that actually for any $\lambda \in [0,1]$, and any $x, y \in \mathbb{R}^n$ we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

You don't need to use any convexity properties, but if you do: you must prove all of them. You can not *assume that f is differentiable.*

Hint: The result is obviously true for $\lambda = 1/2$. You may try proving it for $\lambda = 1/4$ and $3/4$, and from there to try to find a pattern.

Problem 4.

(a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. Assume that $U \subset \mathbb{R}^n$ is connected. Show that $f(U) \subset \mathbb{R}^m$ is connected.

(b) Show that there is *no* $f : \mathbb{R}^2 \to \mathbb{R}$ such that

- f is continuous
- f is injective
- for any open set $U \subset \mathbb{R}^2$ we have $f(U)$ is open

Problem 5. Let $M_{n\times n}$ denote the vector space of $n \times n$ real matrices. Prove that there are neighborhoods U and V of the identity matrix I_n such that for every $A \in U$ there is a unique $X \in V$ such that $X^4 = A$, where here X^4 is a matrix power.

Hint: Implicit or inverse function theorem.

Problem 6. Let $F = (F_1, \ldots, F_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a differentiable mapping satisfying $F(0) = 0$. Suppose that

$$\sum_{i,j=1}^{n} \left| \frac{\partial F_i}{\partial x_j}(0) \right|^2 = c < 1.$$

Prove that there is a ball B in $\mathbb{R}^n$ centered at 0 such that

$$f(B) \subset B.$$

Hint: Note that here F may NOT be C^1, and hence no implicit/inverse function theorem. Try using differentiability.