A REMARK ON REGULARITY OF AN ELLIPTIC-ELLIPTIC SINGULAR PERTURBATION PROBLEM

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Abstract. This note considers the elliptic-elliptic singular perturbation problem:

$$-\varepsilon^2 \triangle u + u = f \text{ , in } \Omega,$$
$$u = 0 \text{ , on } \partial\Omega.$$

We trace through elliptic regularity theory to find conditions under which solutions are regular uniformly in the singular perturbation parameter. This uniformity result is important for differential filters and approximate deconvolution models.

Key words. uniformity, singular perturbation, shift theorem, p-Laplacian, differential filters

AMS subject classification.

1. Introduction. Let Ω be a bounded, regular domain in $\mathbb{R}^d, d = 1, 2, 3$ with C^{k+2} boundary and $0 < \varepsilon \leq 1$ a small parameter. Consider the elliptic-elliptic singular perturbation problem

$$-\varepsilon^2 \triangle u + u = f \text{ , in } \Omega, \qquad (1.1)$$
$$u = 0 \text{ , on } \partial\Omega.$$

Let $H^k = H^k(\Omega)$ denote the Sobolev space of all functions with derivatives of order $\leq k$ in $L^2(\Omega) = H^0$ with associated norm $||\cdot||_k$ and semi-norm $|\cdot|_k$. If $k = 0$ we drop the subscript 0 in the norm and write simply $||\cdot||$. The Sobolev space $H^1_0(\Omega)$ is $H^1_0 := \{v \in H^1 : v = 0 \text{ on } \partial\Omega\}$. For (1.1) we assume

$$f \in H^k(\Omega) \bigcap H^1_0(\Omega). \qquad (1.2)$$

In particular, we stress that *this implies the important condition*

$$f = 0 \text{ on } \partial\Omega. \qquad (1.3)$$

This condition precludes simple boundary layers in u but does not imply higher derivatives of u are free of layers. From (1.1) it also implies that $\triangle u = 0$ on $\partial\Omega$.

The shift theorem, e.g., Gilbarg and Trudinger [GT], implies that the solution of (1.1) satisfies

$$u \in H^{k+2}(\Omega) \bigcap H^1_0(\Omega) \text{ , and } ||u||_{k+2} \leq C(\varepsilon)||f||_k,$$

where $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$.Herein we investigate the question of *uniform in* ε regularity theorem. Is there a $C = C(k, \Omega)$, independent of ε, such that the solution of (1.1) under (1.2) satisfies

$$||u||_k \leq C||f||_k. \qquad (1.4)$$

This simple question has turned out to be more delicate than it appeared at first. We prove the following in Section 2.

Theorem 1.1. *Suppose* $f \in H^2(\Omega) \bigcap H_0^1(\Omega)$. *Then*

$$||u||_l \leq C||f||_l \text{ , for } l = 0, 1, 2. \tag{1.5}$$

If $f \in H^4(\Omega) \bigcap H_0^1(\Omega), \triangle f \in H_0^1(\Omega)$. *Then*

$$||u||_l \leq C||f||_l \text{ , for } l = 0, 1, 2, 3, 4. \tag{1.6}$$

In general, suppose $f \in H^{2k}(\Omega) \bigcap H_0^1(\Omega), \triangle^j f \in H_0^1(\Omega), j = 1, \cdots, k-1$. *Then for* $l = 1, \cdots, 2k$

$$||u||_l \leq C||f||_l. \tag{1.7}$$

The assumption that powers of $\triangle f$ vanish on the boundary can be weakened in appearance to read that normal derivatives of the same order vanish on the boundary, Section 3. However, examples in section 4 show that some extra condition is necessary. Without it we have the following (Section 3).

Corollary 1.2. *Suppose* $f \in H^3(\Omega) \bigcap H_0^1(\Omega)$. *Then*

$$||u||_3 \leq C(||f||_3 + \varepsilon^{-1}||f||_2)$$

It seems likely that all the results herein are in the published literature, explicitly or implicitly, somewhere (e.g., Lions [L73]). *No claim of novelty is made. This report is just a (not to be published) technical note/white paper that presents a result useful for some applications.*

Uniformity in the parameter is critical in some important applications, some of the authors and collaborators papers touching on these applications are given in the reference list; this note is motivated by one such application. Simple examples in $1d$ show that Theorem 1.1 cannot be true if (1.3) does not hold and that local versions of Theorem1.1 cannot hold as well.

2. Regularity by direct estimation of derivatives. We consider first the cases where it may be proven by a direct argument. The estimates in Lemma 1.1 likely appear in every paper on regularity of (1.1). The estimates in Lemma 1.3 appear in Tartar [T93] . All constants are uniform in ε.

Lemma 2.1. *Under (1.2) we have*

$$\varepsilon^2||\triangle u|| + \varepsilon||\nabla u|| + ||u|| \leq C||f||, \tag{2.1}$$

$$\varepsilon||\triangle u|| + ||\nabla u|| \leq C||\nabla f||. \tag{2.2}$$

Proof. Multiply the equation (1.1) by u and integrate over the domain Ω. This gives

$$\varepsilon^2||\nabla u||^2 + ||u||^2 = (f, u) \leq \frac{1}{2}||f||^2 + \frac{1}{2}||u||^2,$$

and the first estimate follows. For the second estimate, the equation (1.1) implies -$\varepsilon^2 \triangle u = f - u$ and thus

$$\varepsilon^2||\triangle u|| \leq ||f|| + ||u|| \leq C||f||.$$

The gradient bound in (2.1) is improvable. Indeed, if $f \in H_0^1(\Omega)$ then

$$u \in H^3(\Omega) \bigcap H_0^1(\Omega).$$

Multiply by $-\triangle u$ and integrate. This gives

$$\varepsilon^2 ||\triangle u||^2 + ||\nabla u||^2 = (\nabla f, \nabla u) \leq \frac{1}{2}||\nabla f||^2 + \frac{1}{2}||\nabla u||^2,$$

and the second estimate (2.2) follows. □

Next we use a reformulation of (1.1) in Tartar [T93]. This reformulation is critical for the next step and the regularity question can be restated for the reformulation. Since $f \in H_0^1(\Omega)$,

$$(u - f) \in H_0^1(\Omega).$$

Subtraction gives the equation

$$\begin{aligned} -\varepsilon^2 \triangle(u - f) + (u - f) &= \varepsilon^2 \triangle f \text{ , in } \Omega, \\ (u - f) &= 0 \text{ , on } \partial\Omega. \end{aligned} \tag{2.3}$$

Lemma 2.2. *There is a C independent of ε such that*

$$||u||_k \leq C||f||_k$$

if and only if

$$||u - f||_k \leq C||f||_k.$$

Proof. This follows from the triangle inequality:

$$\begin{aligned} ||u||_k &\leq ||u - f||_k + ||f||_k, \\ ||u - f||_k &\leq ||f||_k + ||u||_k. \end{aligned}$$

□

Consider therefore problem (2.3).

Lemma 2.3. *Under (1.2) we have*

$$\begin{aligned} ||\nabla(u - f)|| &\leq C||\nabla f|| \\ ||u - f|| &\leq C\varepsilon||\nabla f||, \\ ||\nabla(u - f)|| &\leq C\varepsilon||\triangle f||, \\ ||u - f|| &\leq C\varepsilon^2||\triangle f||, \\ ||\triangle(u - f)|| &\leq C||\triangle f||. \end{aligned}$$

Proof. Multiply (2.3) by $u - f$ and integrate. This gives, after the usual manipulations,

$$\varepsilon^2||\nabla(u - f)||^2 + ||u - f||^2 = (\varepsilon^2 \triangle f, u) \leq \varepsilon^2||\nabla(u - f)||||\nabla f||.$$

This proves the first two estimates. On the above RHS we can also write $(\varepsilon^2\triangle f, u) \leq \frac{\varepsilon^4}{2}||\triangle f||^2 + \frac{1}{2}||u-f||^2$, which proves the third and fourth estimate. Equation (2.3) now implies

$$\varepsilon^2||\triangle(u-f)|| \leq \varepsilon^2||\triangle f|| + ||u-f|| \leq C\varepsilon^2||\triangle f||.$$

Thus,

$$||\triangle(u-f)|| \leq C||\triangle f||.$$

□

Since $\partial\Omega$ is smooth, $||\triangle u||$ and $|u|_2$ are equivalent. Thus we have the following.

COROLLARY 2.4. *For $k = 0, 1, 2$ we have*

$$||u||_k \leq C||f||_k.$$

These simple estimates can be continued and give precise information. If

$$f \in H_0^1(\Omega) \text{ , then } u \in H^3(\Omega) \bigcap H_0^1(\Omega)$$

and from the equation $-\varepsilon^2\triangle u = f - u \in H_0^1(\Omega)$, so

$$\triangle u \in H_0^1(\Omega).$$

Thus $\triangle u$ satisfies

$$-\varepsilon^2\triangle\triangle u + \triangle u = \triangle f \ \in H_0^1(\Omega), \text{ in } \Omega, \qquad (2.4)$$
$$\triangle u = 0 \text{ , on } \partial\Omega.$$

Applying the above estimates to $\triangle u$ gives for $k = 0, 1, 2, 3, 4$

$$||u||_k \leq C||f||_k. \qquad (2.5)$$

Clearly this can be repeated. Repeating this argument proves Theorem 1.1.

THEOREM 2.5. *Suppose $f \in H^{2k}(\Omega) \bigcap H_0^1(\Omega), \triangle^j f \in H_0^1(\Omega), j = 1, \cdots, k-1$. Then for $l = 1, \cdots, 2k$*

$$||u||_l \leq C||f||_l. \qquad (2.6)$$

3. The bootstrap argument. The usual path to regularity is via a bootstrap argument. The section considers how the classical bootstrap argument applies to the regularity issue. We give a different proof of the basic regularity result of the last section. We return to the case:

$$f \in H^k(\Omega) \bigcap H_0^1(\Omega). \qquad (3.1)$$

The case $k = 3$.

The usual procedure, GILBARG AND TRUDINGER [GT] is first to use a partition of unity. Then, change variables to locally flatten the boundary. The sought estimate is first proven for tangential derivatives through tangential difference quotients. Finally, the last derivative in the normal direction is bounded by tangential derivatives through the equation. This section uses an alternate but related strategy (which was suggested to the author by L. TARTAR and used in [T93]) that simplifies the argument considerably[1]. The key is the following observation.

Observation: *Let L be a smooth, first order differential operator which acts tangentially to $\partial\Omega$. Then, for any*

$$v \in H^2(\Omega) \bigcap H^1_0(\Omega)$$

we have

$$\begin{aligned} Lv &\in H^1_0(\Omega) \text{ , and} \\ \triangle Lv &= L\triangle v + Av, \end{aligned}$$

where A is a second order differential operator.

PROPOSITION 3.1. *Suppose L is a smooth, first order differential operator which acts tangentially to $\partial\Omega$. Then,*

$$||\triangle L(u-f)|| \le C||f||_3$$

Proof. Applying L to the equation

$$\begin{aligned} -\varepsilon^2\triangle(u-f) + (u-f) &= \varepsilon^2\triangle f \text{ , in } \Omega, \\ (u-f) &= 0 \text{ , on } \partial\Omega. \end{aligned}$$

gives

$$\begin{aligned} -\varepsilon^2\triangle L(u-f) + L(u-f) = \varepsilon^2 L\triangle f + \varepsilon^2 A(u-f) \text{ , in } \Omega, \\ L(u-f) = 0 \text{ , on } \partial\Omega. \end{aligned} \tag{3.2}$$

Apply Lemma 2.3 to (3.1). This gives

$$||L(u-f)|| \le C\varepsilon^2\{||L\triangle f|| + ||A(u-f)||\},$$

and since A is second order, Lemma 2.2 implies $||A(u-f)|| \le C||f||_2$. Thus,

$$||L(u-f)|| \le C\varepsilon^2||f||_3.$$

Next, apply the idea in the proof of Lemma 2.2 to equation (3.1). This gives

$$\varepsilon^2||\triangle L(u-f)|| \le \varepsilon^2||f||_3 + ||L(u-f)|| \le C\varepsilon^2||f||_3.$$

Thus we have

$$||\triangle L(u-f)|| \le C||f||_3$$

for any first order differential operator acting tangentially to the boundary. □

There remains only to check the norm of the one third order differential operator acting normal to $\partial\Omega$. This is one additive term in $\nabla\triangle(u-f)$ so that the theorem will hold if $||\triangle(u-f)||_1 \le C||f||_3$.

[1] This considerable shortening of the proof motivated writing this note after checking the much longer argument in Gilbarg and Trudinger [GT].

To verify this estimate, recall that the equation

$$-\varepsilon^2\triangle(u-f)+(u-f)=\varepsilon^2\triangle f \text{ , in } \Omega,$$
$$(u-f)=0 \text{ , on } \partial\Omega.$$

implies

$$\varepsilon^2||\triangle(u-f)||_1 \le \varepsilon^2||\triangle f||_1 + C||\nabla(u-f)||. \tag{3.3}$$

At this point, the best estimate of the last term in Lemma 2.3 is

$$||\nabla(u-f)|| \le C\varepsilon||\triangle f||.$$

This gives the following.

COROLLARY 3.2. *Suppose* $f \in H^3(\Omega)\bigcap H_0^1(\Omega)$. *Then*

$$||u||_3 \le C(||f||_3 + \varepsilon^{-1}||f||_2)$$

To eliminate the ε^{-1} it suffices that $\triangle f = 0$ on $\partial\Omega$.

LEMMA 3.3. *Suppose* $\triangle f \in H_0^1(\Omega)$, *then*

$$||\nabla(u-f)|| \le C\varepsilon^2||\triangle f||_1. \tag{3.4}$$

Proof. Begin with the equation

$$-\varepsilon^2\triangle(u-f)+(u-f)=\varepsilon^2\triangle f \text{ , in } \Omega,$$
$$(u-f)=0 \text{ , on } \partial\Omega.$$

Taking the inner product with $-\triangle(u-f)$ gives

$$\varepsilon^2||\triangle(u-f)||^2 + ||\nabla(u-f)|| = \varepsilon^2(\triangle f, -\triangle(u-f)) \le \varepsilon^2||\triangle f||_1||\triangle(u-f)||_{-1}.$$

The key step depends upon the extra regularity $\triangle f \in H_0^1(\Omega)$. With this we can use the estimate

$$(\triangle f, -\triangle(u-f)) \le ||\triangle f||_1||\triangle(u-f)||_{-1}.$$

Now $(u-f) \in H_0^1(\Omega)$ so $\triangle(u-f) \in H^{-1}(\Omega)$ and

$$||\triangle(u-f)||_{-1} \le C||\nabla(u-f)||.$$

Thus,

$$||\nabla(u-f)||^2 \le C\varepsilon^2||\triangle f||_1||\nabla(u-f)||,$$

completing the proof. □

It is clear that all that is really needed is that the second normal derivative of the RHS be well defined and vanish on the boundary.

COROLLARY 3.4. *Suppose* $f \in H^k(\Omega)\bigcap H_0^1(\Omega)$, *and* $\triangle f \in H_0^1(\Omega)$. *Then, for* $k = 0, 1, 2, 3$ *we have*

$$||u||_k \le C||f||_k.$$

4. Examples. EXAMPLE 4.1 (When $f \neq 0$ on the boundary). *This is an example in which $f \in H^1(\Omega)$ from* LIONS *[L73], page 133. First note that the estimates derived imply that $u \to f$ as $\varepsilon \to 0$ weakly in $L^2(\Omega)$. If the RHS does not vanish on the boundary the gradients cannot converge strongly since $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$.*

The following example illustrates this. Consider the 1d *problem*

$$-\varepsilon^2 u'' + u = e^{-x}, \text{ in } (0,\infty),$$
$$u = 0 \text{ , at } x = 0.$$

The solution is

$$u(x) = \frac{1}{1-\varepsilon^2} e^{-x} - \frac{1}{1-\varepsilon^2} e^{-\frac{x}{\varepsilon}}.$$

It is easy to verify that on subdomains away from $x = 0$ there is no difficulty: $u(x) \to f(x)$. The derivatives do not converge near $x = 0$ due to the layer at $x = 0$.

EXAMPLE 4.2 (Regularity is false in general). *The second example is due to* P. RABIER *and shows that (1.4) cannot hold for all k without extra conditions on the RHS. Consider (1.1) in one dimension, which reduces to*

$$-\varepsilon^2 u'' + u = f \text{ , in } (0,1),$$
$$u = 0 \text{ , at } x = 0, 1.$$

Pick $f \in C^\infty(0,1)$ with $f''(x) \neq 0$ at all x. In the equation let $x \to 0$. This implies

$$|u''(0)| = \varepsilon^{-2}|f(0)| = 0.$$

Differentiate twice and repeat this argument. We have $-\varepsilon^2 u'''' + u'' = f''$, in $(0,1)$. Thus, at $x = 0$

$$|u''''(0)| = \varepsilon^{-2}|f''(0)| \to \infty \text{ , as } \varepsilon \to 0.$$

By the Sobolev theorem we have

$$|u''''(0)| \leq C||u||_5 \nleq C||f||_k$$

for any k (in particular $k = 5$) since the LHS blows up while the RHS is bounded.

EXAMPLE 4.3. *The following* 1d *example, due to* XINFU CHEN, *connects the regularity question to results in asymptotic analysis. Suppose $\Omega = (0,1)$ and $I = [a,b]$ is properly contained in $(0,1)$ so that $0 < a < b < 1$. Let $f(x) \equiv 1$ on I and $\to 0$ smoothly off I. Consider (1.1) in one dimension, which reduces to*

$$-\varepsilon^2 u'' + u = f \text{ , in } (0,1), \tag{4.1}$$
$$u = 0 \text{ , at } x = 0, 1.$$

Differentiating (4.1) and setting $x \in I$, we have

$$u''' = \frac{1}{\varepsilon^2} u',$$
$$u'''' = \varepsilon^{-2} u'' = \varepsilon^{-4} u'$$

and thus

$$u^{(2k)} = \varepsilon^{-2k} u' \text{ on } I.$$

Now, if the Theorem 1.1 holds we must have the apparently impossible relation

$$\varepsilon^{-2k}||u'||_{L^2(I)} = ||u^{(2k)}||_{L^2(I)} \leq ||u^{(2k)}||_{L^2(0,1)} \leq C(k)||f||_{2k}. \tag{4.2}$$

From (4.2) it appears that uniform in ε *regularity is impossible. However, the solution to (1.1) (and thus (4.1)) is an approximation to* $f(x)$ *and thus since this particular function satisfies* $f'(x) = 0$ *on* I *we should have* $u'(x)$ *small there as well. Asymptotic analysis of (3.1) indicates that on* I *,* $u'(x)$ *is, in fact, exponentially close to* $f'(x)$ *and thus is exponentially close to zero, e.g., [B75], [B79], [E79]. Thus,* $\sup_{0<\varepsilon\leq 1} \varepsilon^{-2k}||u'||_{L^2(I)} \leq C(k)$ *, which is consistent with the possibility of uniform regularity. Indeed, if* $f(x)$ *is extended to have compact support then the regularity result does hold while it fails for other smooth extensions by Example 2.*

Indeed, this last example cannot be a counterexample because the same local argument leading to (4.2) would apply to the same problem under periodic boundary conditions (where the RHS is extended periodically). In the periodic case we can verify uniform regularity by direct calculation. Indeed, we calculate

$$u(x) = \sum_{j\in\mathbb{Z}} \frac{1}{1+\varepsilon^2(\frac{j}{2\pi})^2} f_j e^{ijx/2\pi},$$

where f_j are the Fourier coefficients of $f(x)$. We calculate further that in the periodic case uniform regularity holds with $C(k) = 1$ trivially since:

$$||u^{(k)}||^2 = \sum_{j\in\mathbb{Z}} \left[\frac{1}{1+\varepsilon^2(\frac{j}{2\pi})^2}\right]^2 (\frac{j}{2\pi})^{2k}|f_j|^2 \leq$$
$$\leq \sum_{j\in\mathbb{Z}} (\frac{j}{2\pi})^{2k}|f_j|^2 = ||f^{(k)}||^2.$$

5. A nonlinear singular perturbation problem. We consider some simple extensions to a nonlinear elliptic-elliptic singular perturbation problem involving the p-Laplacian. It is well known that the p-Laplacian operator is strongly monotone and locally Lipschitz and thus many global estimates extend from the linear case to the p-Laplacian with very little modification. To begin, consider the elliptic-elliptic singular perturbation problem

$$\begin{aligned} -\varepsilon^p\nabla\cdot(|\nabla u|^{p-2}\nabla u) + u = f \text{ , in } \Omega, \\ u = 0 \text{ , on } \partial\Omega. \end{aligned} \tag{5.1}$$

We assume that f is as smooth as desired (and will be specific as such assumptions occur) and $f = 0$ on $\partial\Omega$. The scaling of the small parameter is changed for dimensional reasons. Indeed, if the dimensions of length is L and of u is denoted by $[u] = U$. Then in (1.1) $[\varepsilon] = L$. In the above nonlinear problem we calculate

$$[\varepsilon^p\nabla\cdot(|\nabla u|^{p-2}\nabla u)] = [\varepsilon]^p \cdot \left[\nabla\cdot(|\nabla u|^{p-2}\nabla u)\right] =$$
$$= [\varepsilon]^p \frac{1}{L}(\frac{U}{L})^{p-2}\frac{U}{L},$$

so that scaling by ε^p instead of ε^2 retains the dimensions $[\varepsilon] = L$ and thus the meaning of the small parameter.

Proceeding informally, the first estimate of Section 2 is proven by multiplication by u and integration. Following the same path, we have

$$\varepsilon^{2p}||\nabla\cdot(|\nabla u|^{p-2}\nabla u)||_{L^2}^2+\varepsilon^p||\nabla u||_{L^p}^p+||u||_{L^2}^2\leq C||f||_{L^2}^2.$$

For the second estimate, since f is smooth and vanishes on the boundary, we may multiply by $\varepsilon^p\nabla\cdot(|\nabla u|^{p-2}\nabla u)$. This gives

$$\begin{aligned}\varepsilon^p||\nabla\cdot(|\nabla u|^{p-2}\nabla u)||_{L^2}^2+||\nabla u||_{L^p}^p &\leq \int_\Omega \nabla f\cdot|\nabla u|^{p-2}\nabla u dx\leq\\ &\leq ||\nabla f||_{L^p}||\nabla u||_{L^p}^{p-1}.\end{aligned}$$

Thus,

$$||\nabla u||_{L^p}\leq||\nabla f||_{L^p}.$$

The estimates in Lemma 2.3 can also be extended directly. Indeed, for compactness, let

$$N(u):=-\nabla\cdot(|\nabla u|^{p-2}\nabla u).$$

With this $u-f$ satisfies the equation

$$\varepsilon^p\{N(u)-N(f)\}+(u-f)=\varepsilon^pN(f).$$

Multiplying by $u-f$, integrating and using strong monotonicity for the first term gives

$$\begin{aligned}C(p)\varepsilon^p||\nabla(u-f)||_{L^p}^p+||u-f||_{L^2}^2 &\leq(\varepsilon^pN(f),u-f)\\ &\leq C\varepsilon^p\int_\Omega|\nabla f|^{p-1}|\nabla(u-f)|dx\\ &\leq C\varepsilon^p||\nabla f||_{L^p}^{p-1}||\nabla(u-f)||_{L^p}.\end{aligned}$$

This gives an alternate proof of the above estimate:

$$||\nabla(u-f)||_{L^p}\leq C||\nabla f||_{L^p}.$$

6. Application to differential filters. As noted in the introduction, this note was motivated by an application to Germano's idea of using differential filters as a basic for large eddy simulation , LES. In LES almost universally the cutoff length scale is denoted δ rather than ε (while in singular perturbations it is denoted ε also almost universally). Thus, for this section we shift notation slightly with this replacement. Given (typically a fluid velocity) $\phi\in H^k(\Omega)\bigcap H_0^1(\Omega)$, its differential filter $\overline{\phi}$ is the unique solution of

$$\begin{aligned}A\overline{\phi}\ :\ &=-\delta^2\triangle\overline{\phi}+\overline{\phi}=\phi\ ,\text{ in }\Omega,\\ \overline{\phi}&=0\ ,\text{ on }\partial\Omega.\end{aligned}\tag{6.1}$$

The following question occurs in the analysis of the accuracy of approximate deconvolution operators:

For what n and k do we have

$$||A^{-n}\phi||_k\leq C||\phi||_k$$

uniformly in δ?

We trace through now some answers provided by Theorem 1.1. For $n = 1$ Theorem 1.1 implies

$$||\overline{\phi}||_k \leq C||\phi||_k \text{ , for } k = 0, 1, 2 \text{ and that}$$
$$\triangle\overline{\phi} = 0 \text{ on } \partial\Omega.$$

Since $\triangle\overline{\phi} = 0$ on $\partial\Omega$, we can apply a higher estimate to the case $n = 2$. Indeed, $A^{-2}\phi = A^{-1}\overline{\phi} = \overline{\overline{\phi}}$ so that

$$||\overline{\overline{\phi}}||_k \leq C||\overline{\phi}||_k \text{ , for } k = 0, 1, 2, 3, 4 \text{ and that}$$
$$\triangle\overline{\overline{\phi}} = \triangle\overline{\phi} = 0 \text{ on } \partial\Omega.$$

Further, the equation for $\overline{\overline{\phi}}$ is

$$-\delta^2\triangle\overline{\overline{\phi}} + \overline{\overline{\phi}} = \overline{\phi} \text{ , in } \Omega. \tag{6.2}$$

Taking the Laplacian of this equation gives

$$-\delta^2\triangle^2\overline{\overline{\phi}} + \triangle\overline{\overline{\phi}} = \triangle\overline{\phi} \text{ , in } \Omega. \tag{6.3}$$

Now, let $x \to \partial\Omega$ and use $\triangle\overline{\overline{\phi}} = \triangle\overline{\phi} = 0$ on $\partial\Omega$. This implies

$$\triangle^2\overline{\overline{\phi}} = \triangle\overline{\overline{\phi}} = \overline{\overline{\phi}} = 0 \text{ on } \partial\Omega$$

so that even higher uniform regularity can be inferred for $\overline{\overline{\overline{\phi}}}$.

$$||\overline{\overline{\overline{\phi}}}||_k \leq C||\overline{\overline{\phi}}||_k \text{ , for } k = 0, 1, 2, 3, 4, 5, 6.$$

This argument can be continued.

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