VISCOSITY SUPERSOLUTIONS OF THE EVOLUTIONARY p-LAPLACE EQUATION

PETER LINDQVIST AND JUAN J. MANFREDI

1. Introduction

Often new proofs of old results give additional insight, besides the simplification offered. We hope that the present study of the diffusion equation

$$\frac{\partial v}{\partial t} = \nabla \cdot (|\nabla v|^{p-2} \nabla v) \tag{1.1}$$

has this character. Even obvious results for this equation may require advanced estimates in the proofs. We refer to the books [DB] and [WZYL] about this equation, which is called the "evolutionary p-Laplacian equation," the "p-parabolic equation" or even the "non-Newtonian equation of filtration.".

Our objective is to study the regularity of the *viscosity supersolutions* and their spatial gradients. We give a new proof of the existence of ∇v in Sobolev's sense and of the validity of the equation

$$\iint_{\Omega} \left(-v \frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2} \nabla v,\ \nabla \varphi \rangle \right) dx\ dt \geq 0 \tag{1.2}$$

for all test functions $\varphi \geq 0$. Here Ω is the underlying domain in $\mathbb{R}^{n+1}$ and v is a bounded viscosity supersolution in Ω. The first step of our proof is to establish (1.2) for the so-called infimal convolution v_ϵ, constructed from v through a simple formula. The function v_ϵ has the advantage of being differentiable with respect to all its variables $x_1, x_2, \cdots, x_n$, and t, while the original v is merely lower semicontinuous to begin with. The second step is to pass to the limit as $\epsilon \to 0$. It is clear that $v_\epsilon \to v$ but it is delicate to establish a sufficiently good convergence of the ∇v_ϵ's.

This has earlier been proved in [KL1] for the so-called p-superparabolic functions; according to a theorem in [JLM] they coincide with the viscosity supersolutions. We had better mention that, when it comes to the "supersolutions" several definitions are currently being used. To clarify the concept we mention a few:

- weak supersolutions (test functions under the integral sign);
- viscosity supersolutions (test functions evaluated at points of contact);

Date: December 15, 2006.
1991 *Mathematics Subject Classification.* Primary: 35K85, 35K65; Secondary: 35J60.

- p-superparabolic functions (defined via a comparison principle).

The weak supersolutions are assumed to belong to a Sobolev space; they do not form a good closed class under monotone convergence. The viscosity supersolutions are assumed to be merely lower semicontinuous. So are the p-superparabolic functions. As we mentioned, the viscosity supersolutions and the p-superparabolic functions coincide. This is an important link in our proof. If they, in addition, are bounded, then they are weak supersolutions satisfying (1.2). Our contribution is a new proof of the last fact. Our use of the v_ϵ's replace a technically complicated approximation procedure in the old proof in [KL1].

The present proof is not free of technical complications. The corresponding proof for the stationary equation

$$\nabla \cdot \left(|\nabla v|^{p-2} \nabla v \right) = 0,$$

often called the p-Laplace equation, is much simpler and more transparent. For the benefit of the reader we have written down also this case, although the original proof in [L] is simple enough. See also [KM].

A final remark about unbounded viscosity solutions is appropriate. The truncated functions $v_k = \min(v, k), k = 1, 2, 3, \cdots$, are viscosity supersolutions and the results above apply to them. Then one may proceed from this as in [KL2], [L], and [KM].

2. Preliminaries

We begin with the p-Laplace equation

$$\nabla \cdot \left(|\nabla v|^{p-2} \nabla v \right) = 0$$

in a domain Ω in $\mathbb{R}^n$. This is the stationary case. We say that $v \in W^{1,p}_{\text{loc}}(\Omega)$ is a *weak supersolution* in Ω, if

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v,\ \nabla \varphi \rangle dx \geq 0 \tag{2.1}$$

whenever $\varphi \geq 0$ and $\varphi \in C_0^\infty(\Omega)$. If the integral inequality is reversed, we say that v is a *weak subsolution*. We say that a continuous $h \in W^{1,p}_{\text{loc}}(\Omega)$ is a *p-harmonic function*, if

$$\int_\Omega \langle |\nabla h|^{p-2} \nabla h,\ \nabla \varphi \rangle dx = 0 \tag{2.2}$$

for all $\varphi \in C_0^\infty(\Omega)$. By elliptic regularity theory the continuity is a redundant requirement in the definition.

Definition 1. *We say that the function $v : \Omega \to (-\infty, \infty]$ is p-superharmonic in Ω, if*

(i) $v \not\equiv +\infty$,
(ii) *v is lower semicontinuous,*

(iii) *v obeys the comparison principle in each subdomain $D \subset\subset \Omega$: if $h \in C(\overline{D})$ is p-harmonic in D, then the inequality $v \geq h$ on ∂D implies that $v \geq h$ in D.*

We refer to [L] for this concept. Notice that the definition does not include any hypothesis about ∇v. The next definition is from the modern theory of viscosity solutions.

Definition 2. *Let $p \geq 2$. We say that the function $v : \Omega \to (-\infty, \infty]$ is a viscosity supersolution in Ω, if*

(i) *$v \not\equiv +\infty$,*
(ii) *v is lower semicontinuous, and*
(iii) *whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that*

$$\begin{aligned} v(x_0) &= \varphi(x_0), \text{ and} \\ v(x) &> \varphi(x) \text{ when } x \neq x_0, \end{aligned}$$

we have

$$\nabla \cdot \big(|\nabla\varphi(x_0)|^{p-2}\nabla\varphi(x_0)\big) \leq 0.$$

According to [JLM] (Theorem 2.5), the viscosity supersolutions and the p-superharmonic functions are the same. In other words, Definition 1 and Definition 2 are equivalent.

In [L] the following theorem was proved for the p-superharmonic functions.

Theorem 1. *Suppose that v is a locally bounded p-superharmonic function in Ω. Then the Sobolev derivative*

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \cdots \frac{\partial v}{\partial x_n}\right)$$

exists and $v \in W^{1,p}_{loc}(\Omega)$. Moreover, v is a weak supersolution, i.e.,

$$\int_\Omega \langle|\nabla v|^{p-2}\nabla v,\ \nabla\varphi\rangle dx \geq 0$$

whenever

$$\varphi \in C_0^\infty(\Omega),\ \varphi \geq 0.$$

We aim at giving a new proof of this theorem, using the viscosity theory. The proof for viscosity supersolutions is given in Section 3.

We now proceed to the parabolic equation

$$\frac{\partial v}{\partial t} = \nabla \cdot (|\nabla v|^{p-2}\nabla v)$$

in a domain Ω, this time in $\mathbb{R}^{n+1}$. We use the notation

$$v = v(x,t) = v(x_1, \cdots, x_n, t).$$

We assume that $p \geq 2$. (The case $p < \frac{2n}{n+2}$ is in doubt.) With obvious modifications, we repeat what was written above, but by paying attention to the time variable. We say that v is a *weak supersolution* in Ω, if $v \in L(t_1, t_2; W^{1,p}(D))$ whenever $D \times (t_1, t_2) \subset\subset \Omega$ and

$$\iint_\Omega \left(-v\frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2}\nabla v, \nabla \varphi \rangle \right) dxdt \geq 0 \tag{2.3}$$

for all $\varphi \geq 0, \varphi \in C_0^\infty(\Omega)$. Similarly we define *weak subsolutions*. A continuous function h, belonging to the aforementioned space, is called a *p-parabolic* function, if

$$\iint_\Omega \left(-h\frac{\partial \varphi}{\partial t} + \langle |\nabla h|^{p-2}\nabla h, \nabla \varphi \rangle \right) dxdt = 0 \tag{2.4}$$

for all test functions $\varphi \in C_0^\infty(\Omega)$.

Definition 3. *We say that the function $v : \Omega \to (-\infty, \infty]$ is p-superparabolic in Ω, if*

(i) *v is finite in a dense subset of Ω.*
(ii) *v is lower semicontinuous.*
(iii) *v obeys the comparison principle in each subdomain $D_{t_1,t_2} = D \times (t_1, t_2) \subset\subset \Omega$: if $h \in C(\overline{D_{t_1,t_2}})$ is p-parabolic in D_{t_1,t_2} and if $v \geq h$ on the parabolic boundary of D_{t_1,t_2}, then $v \geq h$ in D_{t_1,t_2}.*

Recall that the parabolic boundary is the union of $\partial D \times [t_1, t_2]$ and $\overline{D} \times \{t_1\}$. Thus $D \times \{t_2\}$ is excluded. See [KL] for some basic facts. Again there is an equivalent definition in terms of the viscosity theory.

Definition 4. *Let $p \geq 2$. Suppose that $v : \Omega \to (-\infty, \infty]$ satisfies (i) and (ii) above. We say that v is a viscosity supersolution, if*

(iii) *whenever $(x_0, t_0) \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $v(x_0, t_0) = \varphi(x_0, t_0)$ and $v(x, t) > \varphi(x, t)$ when $(x, t) \neq (x_0, t_0)$, we have*

$$\frac{\partial \varphi(x_0, t_0)}{\partial t} \geq \nabla \cdot (|\nabla \varphi(x_0, t_0)|^{p-2}\nabla \varphi(x_0, t_0))$$

Again the test function is touching v from below and the differential inequality is evaluated only at the point of contact. According to Theorem 4.4 in [JLM] Definitions 3 and 4 are equivalent. Moreover, one also obtains an equivalent definition by looking only at points (x, t) such that $t < t_0$, see [J]. In [KL] the following theorem was proved for the p-superparabolic functions.

Theorem 2. *Suppose that v is a locally bounded p-superparabolic function in Ω. Then the Sobolev derivative*

$$\nabla v(x, t) = \left(\frac{\partial v(x, t)}{\partial x_1}, \cdots, \frac{\partial v(x, t)}{\partial x_n} \right)$$

exists and $\nabla v \in L^p_{loc}(\Omega)$. Moreover, v is a weak supersolution, i.e.,

$$\iint_\Omega \left(-v\frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2}\nabla v, \nabla \varphi \rangle \right) dx\, dt \geq 0$$

whenever $\varphi \geq 0, \varphi \in C_0^\infty(\Omega)$.

The interpretation of the time derivative requires caution. It is often merely a measure, as the following example shows. Every function of the form $v(x,t) = g(t)$ is p-superparabolic if $g(t)$ is a non-decreasing lower semicontinuous step function. Thus Dirac deltas can appear in v_t.

3. The Stationary Equation

In this section we prove Theorem 1. Aiming at a local result, we may for the proof assume that v is bounded in the whole Ω. By adding a constant, if needed, we have

$$0 \le v(x) \le L, \quad \text{when} \quad x \in \Omega. \tag{3.1}$$

The approximants

$$v_\epsilon(x) = \inf_{y\in\Omega} \left\{ \frac{|x-y|^2}{2\epsilon} + v(y) \right\}, \quad x \in \Omega, \tag{3.2}$$

have many good properties: they are rather smooth, they form an increasing sequence converging to $v(x)$ as $\epsilon \to 0^+$, and from v they inherit the property of being viscosity supersolutions themselves. Some well-known facts are listed below.

1°) At each x in $\Omega, v_\epsilon(x) \nearrow v(x)$ as $\epsilon \to 0^+$.
2°) The function

$$v_\epsilon(x) - \frac{|x|^2}{2\epsilon}$$

is locally concave in Ω.
3°) The Sobolev gradient ∇v_ϵ exists and $\nabla v_\epsilon \in L^\infty_{\text{loc}}(\Omega)$.

In fact, the third assertion follows from the second.

Proposition 1. *The approximant v_ϵ is a viscosity supersolution in the open subset of Ω where*

$$\textit{dist } (x, \partial\Omega) > \sqrt{2L\epsilon}.$$

Proof. Choose x in Ω as required above. Then the infimum in (3.2) is attained at some point y in Ω, say $y = x^*$. Formally, the possibility that x^* escapes to $\partial\Omega$ is prohibited by the inequalities

$$\frac{|x-x^*|^2}{2\epsilon} \le \frac{|x-x^*|^2}{2\epsilon} + v(x^*) = v_\epsilon(x) \le v(x) \le L$$

and

$$|x - x^*| \le \sqrt{2L\epsilon} < \text{dist}(x, \partial\Omega).$$

Fix a point x_0 so that $x_0^* \in \Omega$. Assume that the test function φ touches v_ϵ from below at x_0. We have

$$\varphi(x_0) = v_\epsilon(x_0) = \frac{|x_0 - x_0^*|^2}{2\epsilon} + v(x_0^*)$$

and

$$\varphi(x) \le v_\epsilon(x) \le \frac{|x-y|^2}{2\epsilon} + v(y)$$

for all x and y in Ω. Using this one can verify that the function

$$\psi(x) = \varphi(x + x_0 - x_0^*) - \frac{|x_0 - x_0^*|^2}{2\epsilon} \tag{3.3}$$

touches the original v from below at the point x_0^*. By assumption the inequality

$$\nabla \cdot \big(|\nabla\psi(x_0^*)|^{p-2}\nabla\psi(x_0^*)\big) \geq 0$$

holds since x_0^* is an interior point. Because

$$\nabla\psi(x_0^*) = \nabla\varphi(x_0),\ D^2\psi(x_0^*) = D^2\varphi(x_0),$$

we also have that

$$\nabla \cdot \big(|\nabla\varphi(x_0)|^{p-2}\nabla\varphi(x_0)\big) \geq 0 \tag{3.4}$$

at the original point x_0. □

Write

$$\Omega_\epsilon = \Big\{x \in \Omega\colon \text{dist }(x, \partial\Omega) > \sqrt{2\epsilon L}\Big\}.$$

Theorem 3. *The approximant v_ϵ obeys the comparison principle in Ω_ϵ. In other words, given a domain $D \subset\subset \Omega_\epsilon$ and a p-harmonic function $h \in C(\overline{D})$, then the implication*

$$v_\epsilon \geq h \text{ on } \partial D \Rightarrow\ v_\epsilon \geq h \text{ in } D$$

holds.

Proof. This is Theorem 2.5 in [JLM]. □

The comparison principle implies that v_ϵ is a weak supersolution with test functions under the integral sign. The proof is based on an obstacle[1]problem in the calculus of variations.

Theorem 4. *The approximant v_ϵ is a weak supersolution in Ω_ϵ, i.e.,*

$$\int_\Omega \langle|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla\varphi\rangle dx \geq 0 \tag{3.5}$$

whenever $\varphi \in C_0^\infty(\Omega_\epsilon)$ and $\varphi \geq 0$.

Proof. Let $D \subset\subset \Omega_\epsilon$ be a regular domain. We regard v_ϵ as an obstacle and consider the class consisting of all functions w such that

$$\begin{cases} w \in C(\bar{D}) \cap W^{1,p}(D), \\ w \geq v_\epsilon \text{ in } D, \text{ and} \\ w = v_\epsilon \text{ on } \partial D. \end{cases}$$

The problem of minimizing the variational integral $\int |\nabla w|^p dx$ has a unique solution w_ϵ in this class. In other words,

$$\int_D |\nabla w_\epsilon|^p dx \leq \int_D |\nabla w|^p dx$$

[1]It is not clear, whether the obstacle problem can be totally avoided in the passage to (3.5).

for all w in the aforementioned class. We refer to [MZ] for the continuity. By a standard argument, the minimizer is weak supersolution, i.e.,

$$\int_D \langle |\nabla w_\epsilon|^{p-2}\nabla w_\epsilon, \nabla\varphi\rangle dx \geq 0$$

whenever

$$\varphi \in C_0^\infty(D),\ \varphi \geq 0.$$

The theorem follows from the claim $w_\epsilon = v_\epsilon$ in D. To prove the claim, we notice that $w_\epsilon \geq v_\epsilon$. In the open set $A_\epsilon = \{w_\epsilon > v_\epsilon\}$ one knows that w_ϵ is p-harmonic. On the boundary ∂A_ϵ we have $w_\epsilon = v_\epsilon$. The comparison principle (Definition 1) implies that $v_\epsilon \geq w_\epsilon$ in A_ϵ. It follows that A_ϵ is empty and $w_\epsilon = v_\epsilon$. This was the claim. □

The next lemma contains a bound that is independent of ϵ.

Lemma 1. (Caccioppoli) *We have*

$$\int_\Omega \zeta^p |\nabla v_\epsilon|^p dx \leq p^p\, L^p \int_\Omega |\nabla\zeta|^p dx \tag{3.6}$$

whenever $\zeta \in C_0^\infty(\Omega_\epsilon)$ *and* $\zeta \geq 0$.

Proof. Use the test function

$$\varphi = (L - v_\epsilon)\zeta^p$$

in (3.5) to obtain this well-known estimate. □

Corollary 1. *The Sobolev derivative* ∇v *exists and* $\nabla v \in L^p_{loc}(\Omega)$.

Proof. Use Lemma 1 and a standard compactness argument. □

In order to proceed to the limit under the integral sign in (3.5) we need more than the weak convergence:

$$\nabla v_\epsilon \to \nabla v$$

locally weakly in $L^p(\Omega)$. Actually, the convergence is strong.

Lemma 2. *We have that* $\nabla v_\epsilon \to \nabla v$ *strongly in* $L^p_{loc}(\Omega)$.

Proof. Let $\theta \in C_0^\infty(\Omega)$ and $\theta \geq 0$. Use the test function $\varphi = (v - v_\epsilon)\theta$ in (3.5). The inequality can be written as

$$\int_\Omega \theta\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon,\ \nabla v - \nabla v_\epsilon\rangle\, dx$$

$$+ \int_\Omega (v - v_\epsilon)\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla\theta\rangle\, dx$$

$$\leq \int_\Omega \langle |\nabla v|^{p-2}\nabla v,\ \nabla((v - v_\epsilon)\theta)\rangle\, dx$$

The last integral approaches zero as $\epsilon \to 0^+$, because of the weak convergence. We obtain

$$\left|\int_\Omega (v-v_\epsilon)\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla\theta\rangle\, dx\right|$$

$$\leq \left(\int_\Omega (v-v_\epsilon)^p dx\right)^{\frac{1}{p}} \|\nabla\theta\|_{L^\infty} \left\{ \left(\int_{\theta\neq 0} |\nabla v|^p\, dx\right)^{\frac{p-1}{p}} + \left(\int_{\theta\neq 0} |\nabla v_\epsilon|^p\, dx\right)^{\frac{p-1}{p}} \right\}$$

$$\to 0 \text{ as } \epsilon \to 0^+.$$

We conclude that

$$\limsup_{\epsilon\to 0} \int_\Omega \theta\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla v - \nabla v_\epsilon\rangle\, dx = 0.$$

The integrand is non-negative. For $p \geq 2$ the elementary inequality

$$2^{2-p}|b-a|^p \leq \langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle$$

yields the desired result. □

Now we can take the limit under the integral sign in (3.5). Thus (2.1) follows. This concludes our proof of Theorem 1.

4. The Parabolic Case

For the proof of Theorem 2 we may assume that the viscosity supersolution v of the evolutionary p-Laplacian equation is bounded in the domain Ω in $\mathbb{R}^{n+1}$. Suppose that

$$0 \leq v(x,t) \leq L \text{ when } (x,t) \in \Omega. \tag{4.1}$$

The approximants

$$v_\epsilon(x,t) = \inf_{(y,\tau)\in\Omega} \left\{ \frac{|x-y|^2 + (t-\tau)^2}{2\epsilon} + v(y,\tau) \right\}, \quad \epsilon > 0, \tag{4.2}$$

play a central role in our study. Some useful properties are

1°) At each point (x,t) in $\Omega, v_\epsilon(x,t) \nearrow v(x,t)$ as $\epsilon \to 0^+$.

2°) The function

$$v_\epsilon(x,t) - \frac{|x|^2 + t^2}{2\epsilon}$$

is locally concave in Ω.

3°) The Sobolev derivatives $\frac{\partial v_\epsilon}{\partial t}$ and ∇v_ϵ exist and belong to $L^\infty_{\text{loc}}(\Omega)$.

Given a point (x,t) in Ω, the infimum in (4.2) is attained at some point (x^*,t^*) in Ω provided that

$$\operatorname{dist}((x,t), \partial\Omega) > \sqrt{2L\epsilon}. \tag{4.3}$$

Formally, the inequalities

$$\begin{aligned} \frac{|t-t^*|^2 + |x-x^*|^2}{2\epsilon} &\leq \frac{|t-t^*|^2 + |x-x^*|^2}{2\epsilon} + v(x^*,t^*) \qquad (4.4) \\ &= v_\epsilon(x,t) \leq v(x,t) \leq L, \end{aligned}$$

and

$$\sqrt{(t-t^*)^2+|x-x^*|^2} \leq \sqrt{2L\epsilon} < \operatorname{dist}\left((x,t),\partial\Omega\right),$$

and the semincontinuity guarantee this. For simplicity, we denote the open set defined by (4.3) as Ω_ϵ. We then have $\Omega_\epsilon \subset\subset \Omega$ and $\lim_{\epsilon\to 0^+}\Omega_\epsilon = \Omega$.

Proposition 2. *The approximant v_ϵ is a viscosity supersolution in Ω_ϵ.*

Proof. Fix a point (x_0,t_0) in Ω_ϵ. Then the infimum (4.2) is attained at some *interior* point (x_0^*,t_0^*) in Ω. Select an arbitrary test function φ that touches v from below at (x_0,t_0). The inequalities

$$\varphi(x_0,t_0) = v_\epsilon(x_0,t_0) = \frac{(t_0-t_0^*)^2+|x_0-x_0^*|^2}{2\epsilon} + v(x_0^*,t_0^*),$$

$$\varphi(x,t) \leq v_\epsilon(x,t) \leq \frac{(t-\tau)^2+|x-y|^2}{2\epsilon} + v(y,\tau)$$

are at our disposal for all (x,t) and (y,τ) in Ω. Manipulating these inequalities, one can verify that the function

$$\psi(x,t) = \varphi(x+x_0-x_0^*, t+t_0-t_0^*) - \frac{(t_0-t_0^*)^2+|x_0-x_0^*|^2}{2\epsilon}$$

touches v from below at the point (x_0^*,t_0^*). It will do as a test function. Because v is a viscosity supersolution, the inequality

$$\frac{\partial\psi}{\partial t} \leq \nabla\cdot\left(|\nabla\psi|^{p-2}\nabla\psi\right)$$

holds at the point (x_0^*,t_0^*). The partial derivatives of ψ evaluated at (x_0^*,t_0^*) coincide with those of φ evaluated at the original point (x_0,t_0):

$$\psi_t(x_0^*,t_0^*) = \varphi_t(x_0,t_0), \nabla\psi(x_0^*,t_0^*) = \nabla\varphi(x_0,t_0), \ldots$$

Hence the desired inequality

$$\frac{\partial\varphi}{\partial t} \leq \nabla\cdot\left(|\nabla\varphi|^{p-2}\nabla\varphi\right)$$

holds at (x_0,t_0). □

Theorem 5. *The approximant v_ϵ obeys the comparison principle in Ω_ϵ. In other words, given a domain $D_{t_1,t_2} = D\times(t_1,t_2) \subset\subset \Omega_\epsilon$ and a p-parabolic function $h \in C(\overline{D_{t_1,t_2}})$ then $v_\epsilon \geq h$ on the parabolic boundary of D_{t_1,t_2} implies that $v_\epsilon \geq h$ in D_{t_1,t_2}.*

Proof. This was proved for viscosity supersolutions in Theorem 4.4, p. 712 of [JLM] □

The *parabolic* comparison principle allows comparison in space-time cylinders. We need domains of a more general shape but we do not need to distinguish the parabolic boundary. It turns out that parabolic comparison implies the following *elliptic* comparison principle:

Proposition 3. *Given a domain $\Upsilon \subset\subset \Omega$ and a p-parabolic function $h \in C(\overline{\Upsilon})$, then $v_\epsilon \geq h$ on $\partial\Upsilon$ implies that $v_\epsilon \geq h$ in Υ.*

Now Υ does not have to be a space-time cylinder and $\partial\Upsilon$ is the total boundary in $\mathbb{R}^{n+1}$.

Proof. For the proof of the necessity, it is enough to realize that the proof is immediate when Υ is a finite union of space-time cylinders $D_j \times (a_j, b_j)$. To verify this, just start with the earliest cylinder(s). Then the general case follows by exhausting Υ with such unions. Indeed, given $\alpha > 0$ the compact set $\{h(x,t) \geq v_\epsilon(x,t) + \alpha\}$ is contained in an open finite union

$$\bigcup D_j \times (a_j, b_j)$$

comprised in Ω so that $h < v_\epsilon + \alpha$ on the (Euclidean) boundary of the union. It follows that $h \leq v_\epsilon + \alpha$ in the union. Since α was arbitrary, we conclude that $v_\epsilon \geq h$ in Υ. □

The above *elliptic* comparison principle does not acknowledge the parabolic boundary. The reasoning can easily be slightly modified so that the latest boundary part is exempted.[2] Suppose that $t < T$ for all $(x,t) \in \Upsilon$. (In this case $\partial\Upsilon$ may have a plane portion with $t = T$.) It is sufficient to verify that

$$v_\epsilon \geq h \text{ on } \partial\Upsilon \text{ when } t < T$$

in order to conclude that $v_\epsilon \geq h$ in Υ.

This variant of the comparison principle is convenient for the following conclusion.

Lemma 3. *The approximant v_ϵ is a weak supersolution in Ω_ϵ. That is, we have*

$$\iint_\Omega \left(-v_\epsilon \frac{\partial \varphi}{\partial t} + \langle |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon, \nabla \varphi \rangle \right) dxdt \geq 0 \tag{4.5}$$

for all $\varphi \in C_0^\infty(\Omega_\epsilon)$, $\varphi \geq 0$.

Proof. We show that in a given domain $D_{t_1,t_2} = D \times (t_1, t_2) \subset\subset \Omega_\epsilon$ our v_ϵ coincides with the solution of an obstacle problem. The solutions of the obstacle problem are *per se* weak supersolutions. Hence, so is v_ϵ. Consider the class of all functions

$$\begin{cases} w \in C(\overline{D_{t_1,t_2}}) \cap L^p(t_1, t_2, W^{1,p}(D)), \\ w \geq v_\epsilon \text{ in } D_{t_1,t_2}, \text{ and} \\ w = v_\epsilon \text{ on the parabolic boundary of } D_{t_1,t_2}. \end{cases}$$

The function v_ϵ itself acts as an obstacle and induces the boundary values. There exists a (unique) weak supersolution w_ϵ in this class satisfying the variational inequality

$$\int_{t_1}^{t_2}\int_D \left[(\psi - w_\epsilon) \, \frac{\partial \psi}{\partial t} + \langle |\nabla w_\epsilon|^{p-2} \nabla w_\epsilon, \nabla(\psi - w_\epsilon) \rangle \right] dxdt$$

[2]Another way to see this is to use $v_\epsilon(x,t) + \alpha/(T-t)$ in the place of v_ϵ and then let $\alpha \to 0^+$.

$$\geq \frac{1}{2}\int_D (\psi(x,t_2) - w_\epsilon(x,t_2))^2 dx$$

for all smooth ψ in the aforementioned class. Moreover, w_ϵ is p-parabolic in the open set $A_\epsilon = \{w_\epsilon > v_\epsilon\}$. We refer to [C].

On the boundary ∂A_ϵ we know that $w_\epsilon = v_\epsilon$ except possibly when $t = t_2$. By the "elliptic" comparison principle we have $v_\epsilon \geq w_\epsilon$ in A_ϵ. On the other hand $w_\epsilon \geq v_\epsilon$. Hence $w_\epsilon = v_\epsilon$.

Let $\varphi \in C_0^\infty(D_{t_1,t_2}),\ \varphi \geq 0$, and choose $\psi = w_\epsilon + \varphi = v_\epsilon + \varphi$ above. An easy manipulation yields (4.5.) □

Recall that $0 \leq v \leq L$. Then also $0 \leq v_\epsilon \leq L$. An estimate for ∇v_ϵ is provided in the well-known lemma below.

Lemma 4. (Caccioppoli) *We have*

$$\begin{aligned}\iint_\Omega \zeta^p |\nabla v_\epsilon|^p \,dxdt \quad &\leq \quad CL^2 \iint_\Omega \left|\frac{\partial \zeta^p}{\partial t}\right| dxdt \\ &+ \quad CL^p \iint_\Omega |\nabla \zeta|^p \,dxdt\end{aligned} \tag{4.6}$$

whenever $\zeta \in C_0^\infty(\Omega_\epsilon), \zeta \geq 0$. *Here* C *depends only on* p.

Proof. The test function

$$\varphi(x,t) = (L - v_\epsilon(x_1,t))\zeta(x,t)$$

leads to this estimate. □

Keeping $0 \leq v \leq L$, we can conclude from the Caccioppoli estimate that ∇v exists and $\nabla v \in L^p_{\mathrm{loc}}(\Omega)$. Moreover, we have

$$\nabla v_\epsilon \to \nabla v \text{ weakly in } L^p_{\mathrm{loc}}(\Omega),$$

at least for a subsequence.[1] This proves the first part of the main theorem. The second part follows, if we can pass to the limit under the integral sign in

$$\iint_\Omega \Big(- v_\epsilon \, \frac{\partial \varphi}{\partial t} + \langle |\nabla v_\epsilon^{p-2} \nabla v_\epsilon, \nabla \varphi\rangle \Big)\, dxdt \geq 0 \tag{4.7}$$

as $\epsilon \to 0+$. When $p \neq 2$ the weak convergence alone does not directly justify such a procedure. Strong local convergence in L^p is, as it were, difficult to achieve. The difficulty is that no good bound on $\frac{\partial v_\epsilon}{\partial t}$ is available. In fact, calculations with the example

$$v(x,t) = \begin{cases} 1 & \text{when } t > 0 \\ 0, & \text{when } t \leq 0 \end{cases}$$

reveal that simple adaptations of the proof given in the stationary case fail. However, the elementary vector inequality

$$\Big| |b|^{p-2}b - |a|^{p-2}a \Big| \leq (p-1)|b-a|(|b|+|a|)^{p-2}$$

[1] In fact, one does not have to extract a subsequence.

valid for $p \geq 2$, implies that strong convergence in L^{p-1}_{loc} is sufficient for the passage to the limit. This is more accessible. Thus the theorem follows from

Lemma 5. *We have that* $\nabla v_\epsilon \to \nabla v$ *strongly in* $L^{p-1}_{loc}(\Omega)$*, when* $p \geq 2$.

Remark: The same proof yields strong convergence in $L^{q}_{\text{loc}}(\Omega)$, where $q < p$. The method fails for $q = p$, except when the original v is continuous.

Proof. For the proof of the lemma we may assume that

$$Q_T = Q \times (0, T) \subset\subset \Omega$$

represents a *general* subdomain. The mollified function

$$\frac{1}{\sigma}\int_0^t e^{-(t-\tau)/\sigma} v(x,\tau)d\tau + e^{-t/\sigma} v(x,0),$$

where $\sigma > 0$, is expedient in bypassing some problems caused by the "forbidden quantity" v_t. It is here convenient to abandon the last term and so we use only

$$v^*(x,t) = \frac{1}{\sigma}\int_0^t e^{-(t-\tau)/\sigma} v(x,\tau)d\tau$$

for $0 \leq t \leq T$ and $x \in Q$. The notation hides the dependence on $\sigma > 0$. We mention that

$$v^* \to v, \ \nabla v^* \to \nabla v \ \text{ strongly in } \ L^p(Q_T)$$

as $\sigma \to 0^+$. The rule

$$\frac{\partial v^*}{\partial t} \ = \ \frac{v - v^*}{\sigma} \tag{4.8}$$

will be used to conclude that

$$(v - v^*)\ \frac{\partial v^*}{\partial t} \geq 0$$

a. e. in Q_T. We refer to [N, p. 36] and Lemma 2.2 in [KL1] for these properties.

Next we need a suitable test function. Let $\theta \in C_0^\infty(Q_T), 0 \leq \theta \leq 1$. In passing, we remark that, under the presence of discontinuities, $(v - v_\epsilon)\theta$ does not work as in the elliptic case. We now use the test function[1]

$$\varphi = (v^* - v_\epsilon + \delta)_+\theta$$

where $\delta > 0$ is a small number to be adjusted. The plus sign indicates that the positive part is taken. At the end the parameters δ, ϵ, σ will vanish, but it is decisive that ϵ is the one that first approaches zero. Given $\alpha > 0$, there exists according to Egorov's theorem a set E_α with $(n+1)$-dimensional measure $|E_\alpha| < \alpha$, such that

$$v^* \to v \text{ uniformly in } F_\alpha = Q_T \setminus E_\alpha,$$

as $\sigma \to 0$.

[1]We seize the opportunity to mention that the parameter δ is missing from the test function $(v^* - v_k)\theta$ in [KL1], which should be $(v^* - v_k + \delta)_+\theta$. To correct the error there the Egorov theorem is convenient.

Remark: If v is continuous we do not need E_α, since $v^*(x,t)+e^{-t/\sigma}v(x,0)$ converges uniformly in the whole Q_T in this favorable case. This allows us to skip the plus sign in φ.

We thus have $v^*-v+\delta\ge 0$ in F_α, when $\sigma<\sigma(\alpha,\delta)$. Then we also have

$$v^*-v_\epsilon+\delta\ge v^*-v+\delta\ge 0 \text{ in } F_\alpha$$

when σ is small enough.

Inserting the selected test function into (4.5) we obtain

$$\begin{aligned}\int_0^T\!\!\int_Q \langle |\nabla v^*|^{p-2}\nabla v^* - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla(v^*-v_\epsilon+\delta)_+\theta\rangle\,dxdt \\ \le \int_0^T\!\!\int_Q \langle |\nabla v^*|^{p-2}\nabla v^*, \nabla(v^*-v_\epsilon+\delta)_+\theta\rangle\,dxdt \\ -\int_0^T\!\!\int_Q v_\epsilon\frac{\partial}{\partial t}((v^*-v_\epsilon+\delta)_+\theta)\,dxdt.\end{aligned}$$

We rearrange this as

$$\begin{aligned}\int_0^T\!\!\int_Q \theta\langle |\nabla v^*|^{p-2}\nabla v^* - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla(v^*-v_\epsilon+\delta)_+\rangle\,dxdt \\ \le \int_0^T\!\!\int_Q (v^*-v_\epsilon+\delta)_+\langle |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla\theta\rangle\,dxdt \qquad (4.9)\\ +\int_0^T\!\!\int_Q \theta\langle |\nabla v^*|^{p-2}\nabla v^*, \nabla(v^*-v_\epsilon+\delta)_+\rangle\,dxdt - \int_0^T\!\!\int_Q v_\epsilon\frac{\partial}{\partial t}(v^*-v_\epsilon+\delta)_+\theta\,dxdt \\ = I_\epsilon + II_\epsilon + III_\epsilon.\end{aligned}$$

The procedure is the following. First we prove that the three terms on the right-hand side can be made as small as we please, as $\epsilon\to 0$. Because of its structure the term on the left-hand side controls the norm $\|\theta(\nabla v^*-\nabla v_\epsilon)\|_p$ taken over the set F_α. The triangle inequality will then show that also $\|\theta(\nabla v-\nabla v_\epsilon)\|_p$ is under control. The exceptional set E_α requires an extra consideration, yielding

$$\lim_{\epsilon\to 0}\|\theta(\nabla v-\nabla v_\epsilon)\|_{L^{p-1}(E_\alpha)}=0$$

where we have $p-1$ instead of p. This weakens the final result.

To this end, let us proceed to estimate the three terms. We begin with the crucial term involving the time derivative. Integrations by part yield

$$\begin{aligned}III_\epsilon &= -\iint v_\epsilon\frac{\partial}{\partial t}(v-v_\epsilon+\delta)_+\theta\,dxdt\\ &= \iint (v^*-v_\epsilon+\delta)\frac{\partial}{\partial t}(v^*-v_\epsilon+\delta)_+\theta\,dxdt - \iint (v^*+\delta)\frac{\partial}{\partial t}(v^*-v_\epsilon+\delta)_+\theta\,dxdt\\ &= \frac{1}{2}\iint (v^*-v_\epsilon+\delta)_+^2\frac{\partial\theta}{\partial t}\,dxdt + \iint \theta(v^*-v_\epsilon+\delta)_+\frac{\partial v^*}{\partial t}\,dxdt.\end{aligned}$$

This expression has a limit as $\epsilon \to 0$. Hence

$$\begin{aligned}\lim_{\epsilon\to 0} III_\epsilon &\leq \|v^*-v\|_2^2\,\|\theta_t\|_\infty\, T|Q| + \delta^2\|\theta_t\|_1\\ &+ \iint \theta(v^*-v+\delta)_+\frac{\partial v^*}{\partial t}\,dxdt,\end{aligned}$$

where the last integral has to be estimated. In the set where $v^*-v+\delta>0$ we reason as follows:

$$\begin{aligned}\theta(v^*-v+\delta)_+\frac{\partial v^*}{\partial t} &= \theta(v^*-v+\delta)\cdot\frac{v-v^*}{\sigma}\\ &\leq \delta\theta\frac{v-v^*}{\sigma}\\ &= \delta\theta\frac{\partial v^*}{\partial t}\cdot\end{aligned}$$

This is the place where we have taken advantage of the structure of v^*, see (4.8). We are left with the term

$$\delta\iint_{v^*-v+\delta>0}\theta\frac{\partial v^*}{\partial t}\,dxdt.$$

In the formula

$$\delta\int_0^T\!\!\int_Q\theta\frac{\partial v^*}{\partial t}\,dxdt = \delta\iint_{v^*-v+\delta>0}\theta\frac{\partial v^*}{\partial t}\,dxdt + \delta\iint_{v^*-v+\delta\leq 0}\theta\frac{\partial v^*}{\partial t}\,dxdt$$

the last integral is positive, because

$$\theta\frac{\partial v^*}{\partial t} = \theta\frac{v-v^*}{\sigma} \geq \frac{\theta\delta}{\sigma} \geq 0,\ \text{when } v^*-v+\delta\leq 0.$$

It follows that

$$\begin{aligned}\delta\iint_{v^*-v+\delta>0}\theta\frac{\partial v^*}{\partial t}\,dxdt &\leq \delta\int_0^T\!\!\int_Q\theta\frac{\partial v^*}{\partial t}\,dxdt\\ &= -\delta\int_0^T\!\!\int_Q v^*\frac{\partial\theta}{\partial t}\,dxdt\\ &\leq \delta L\|\theta_t\|_1.\end{aligned}$$

Collecting terms, we record the result

$$\lim_{\epsilon\to 0} III_\epsilon \leq c_1\|v^*-v\|_2^2 + c_2\delta^2 + c_3 L\delta. \tag{4.10}$$

This majorant can be made as small as we please.

Now we turn our attention to the first term on the right-hand side of (4.9). An easy estimate is

$$\begin{aligned}I_\epsilon &\leq \|\nabla\theta\|_\infty\,\|v^*-v_\epsilon+\delta\|_p\;\|\nabla v_\epsilon\|_p^{p-1}\\ &\leq c_4(\|v^*-v_\epsilon\|_p+\delta),\end{aligned} \tag{4.11}$$

since the norms $\|\nabla v_\epsilon\|_p$ are uniformly bounded because of the weak convergence.

The second term II_ϵ is delicate, since by taking the positive part we risk to destroy cancellations, vital to weak convergence. We split the integral over Q_T in two parts, depending on the sign of $v^* - v_\epsilon + \delta$:

$$\int_0^T\int_Q \theta\langle|\nabla v^*|^{p-2}\nabla v^*, \nabla(v^* - v_\epsilon + \delta)\rangle\, dxdt = II_\epsilon$$
$$+ \iint_{v^*-v_\epsilon+\delta<0} \theta\langle|\nabla v^*|^{p-2}\nabla v^*, \nabla(v^* - v_\epsilon + \delta)\rangle\, dxdt.$$

As $\epsilon \to 0$, the weak convergence implies that the left-hand side is majorized in magnitude by

$$\left|\int_0^T\int_Q \theta\langle|\nabla v^*|^{p-2}\nabla v^*, \nabla(v^* - v)\rangle dx\ dt\right|$$
$$\leq \|\nabla v^*\|_p^{p-1}\|\nabla(v^* - v)\|_p \leq \|\nabla v\|_p^{p-1}\|\nabla(v^* - v)\|_p,$$

where a contraction property was used at the last step (it can be avoided). For the integral over the set $\{v^* - v_\epsilon - \delta < 0\} \subset E_\alpha$ it is decisive that the set is small. We obtain

$$\left|\iint_{v^*-v_\epsilon+\delta<0} \theta\langle|\nabla v^*|^{p-2}\nabla v^*, \nabla(v^* - v_\epsilon + \delta)\rangle\ dx\ dt\right|$$
$$\leq \left(\iint_{E_\alpha} |\nabla v^*|^p\ dxdt\right)^{1-\frac{1}{p}} \|\nabla(v^* - v_\epsilon)\|_p$$
$$\leq (\|\nabla v^*\|_p + \|\nabla v_\epsilon\|_p)\, \|\nabla v^*\|_{L^p(E_\alpha)}^{p-1} \leq c_6||\nabla v^*||_{L^p(E_\alpha)}^{p-1}.$$

Together, the previous estimates yield the majorant

$$\limsup_{\epsilon\to 0} II_\epsilon \leq c_5\|\nabla(v^* - v)\|_p + c_6\|\nabla v^*\|_{L^p(E_\alpha)}^{p-1}. \tag{4.12}$$

Adding up the estimates (4.10), (4.11) and (4.12) we have a majorant for the right-hand side of (4.9). The elementary vector inequality

$$2^{2-p}|b-a|^p \leq \langle|b|^{p-2}b - |a|^{p-2}a, b-a\rangle,$$

$p \geq 2$, yields a minorant for the left-hand member. We arrive at

$$\begin{aligned}\limsup_{\epsilon\to 0} 2^{2-p}\iint_{F_\alpha} \theta|\nabla(v^* - v_\epsilon)|^p\ dx\ dt &\leq \limsup_{\epsilon\to 0}(I_\epsilon + II_\epsilon + III_\epsilon)\\ &\leq a\delta + c_2\delta^2 + c_4\|v^* - v\|_p + c_1\|v^* - v\|_2^2 \quad (4.13)\\ &\quad +c_5||\nabla v^* - \nabla v||_p + c_6||\nabla v^*\|_{L^p(E_\alpha)}^{p-1}.\end{aligned}$$

This controls the norm $\|\theta\nabla(v^* - v_\epsilon\|_p$ over F_α. An estimation over the exceptional set E_α is yet missing. In order to utilize the small measure of E_α, we take a smaller exponent than p, say $p-1$, and use Hölder's inequality to achieve

$$\iint_{E_\alpha} \theta|\nabla(v^* - v_\epsilon)|^{p-1}\, dxdt \leq |E_\alpha|^{\frac{1}{p}}(\|\nabla v^*\|_p + \|\nabla v_\epsilon\|_p)^{p-1} \leq c_7\alpha^{1/p} \tag{4.14}$$

We have assumed that $\theta \leq 1$. Together (4.13) and (4.14) yield an estimate over the entire Q_T. Thus, we have an estimate for

$$\limsup \|\theta(\nabla v^* - \nabla v_\epsilon)\|_{L^{p-1}(Q_T)}$$

as $\epsilon \to 0$.

Finally, we use

$$\limsup_{\epsilon \to 0} \|\theta(\nabla v - \nabla v_\epsilon)\|_{p-1} \leq \|\theta(\nabla v - \nabla v^*)\|_{p-1} + \limsup_{\epsilon \to 0} \|\theta(\nabla v^* - \nabla v_2)\|_{p-1}.$$

Here we let $\sigma \to 0$. Recall that $\sigma < \sigma(\alpha, \delta)$. The first term on the right-hand side vanishes. The result is a majorant for

$$\limsup_{\epsilon \to 0} \|\theta(\nabla v - \nabla v_\epsilon)\|_{p-1}$$

that vanishes together with the quantities

$$\delta, \alpha \text{ and } \|\nabla v\|^{p-1}_{L^p(E_\alpha)}.$$

It can be made as small as we please, by adjusting δ and α in advance. It follows that

$$\limsup_{\epsilon \to 0} \|\theta(\nabla v - \nabla v_\epsilon)\|_{p-1} = 0.$$

We are free to choose θ. This proves the strong L^{p-1}_{loc}-convergence. □

Remark: We have locally that $\nabla v_\epsilon \to \nabla v$ strongly in each fixed L^q-norm with $q < p$. The claim in [KL1] that this convergence also holds for $q = p$ has not been rigorously proved, so far as we know. (The error is described in the footnote on page 12 of this manuscript).

References

[C] H.-J. CHOE, *A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities,* Differential and Integral Equations **5**, 1992, pp. 915-944.

[DB] E. DiBENEDETTO, *Degenerate Parabolic Equations*, Springer-Verlag, New York, 1933.

[J] P. JUUTINEN, *On the definition of viscosity solutions for parabolic equations*, Proceedings of the American Mathematical Society **129**, 10, pp, 2907–2911, 2001.

[JLM] P. JUUTINEN, P. LINDQVIST, J. MANFREDI, *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation*, SIAM Journal on Mathematical Analysis **33**, 2001 , pp. 699-717.

[KL] T. KILPELÄINEN, P. LINDQVIST, *On the Dirichlet boundary value problem for a degenerate parabolic equation*, SIAM Journal on Mathematical Analysis **27**, 1996, pp. 661-683.

[KL1] J. KINNUNEN, P. LINDQVIST, *Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation*, Annali di Matematica Pura ed Applicata (4) **185**, 2006, pp. 411-435.

[KL2] J. KINNUNEN, P. LINDQVIST, *Summability of semicontinuous supersolutions to a quasilinear parabolic equation*, Annali della Scuola Normale Superiore di Pisa (Serie V) **4**, 2005, pp. 59-78.

[KM] T. KILPELÄINEN, J. MALÝ, *Degenerate elliptic equations with measure data and nonlinear potentials,* Annali della Scuola Normale Superiore di Pisa (Serie IV) **19**, 1992, pp. 591-613.

[L] P. LINDQVIST, *On the definition and properties of p-superharmonic functions*, Journal für die Reine und Angewandte Mathematik **365**, 1986, pp. 67-79.

[MZ] J. MICHEL, P. ZIEMER, *Interior regularity for solutions to obstacle problems*, Nonlinear Analysis **10**, 1986, pp. 1427-1448.

[N] J. NAUMANN, *Einführung in die Theorie parabolischer Variationsungleichungen*, Teubner-Texte zur Mathematik 64, Leipzig, 1984.

[WZYL] Z. WU, J. ZHAO, J. YIN, H. LI, *Nonlinear Diffusion Equations*, World Scientific, Singapore, 2001.

Department of Mathematics, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

E-mail address: peter.lindqvist@math.ntnu.no

URL: http://www.math.ntnu.no/~lqvist

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260,USA

E-mail address: manfredi@pitt.edu

URL: http://www.pitt.edu/~manfredi