Discontinuous finite elements for solving the Stolz-Adams approximate deconvolution model for turbulent flows

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Abstract

We consider the zeroth order model of the family of approximate deconvolution models of Stolz and Adams. We propose and analyze fully discrete schemes using discontinuous finite elements. Optimal error estimates are derived. The dependence of these estimates with respect to the Reynolds number Re is $\mathcal{O}(\mathrm{Re}\ e^{\mathrm{Re}})$, which is an improvement with respect to the continuous finite element method where the dependence is $\mathcal{O}(\mathrm{Re}\ e^{\mathrm{Re}^3})$.

Key words: differential filter, large eddy simulations, discontinuous Galerkin.

1 Introduction

Turbulence is a phenomena that appears in many processes in the nature and it is connected with many industrial applications because of its richness in scales. Based on the Kolmogorov theory [10], Direct Numerical Simulation (DNS) where all the scales are captured, requires the number of mesh points in space per each time step in to be $\mathcal{O}(Re^{9/4})$ in three-dimensional problems, where Re is the Reynolds number. This is not computational economical and sometimes not even feasible. One promising approach is Large Eddy Simulation (LES) where we are seeking for the large scales, i.e. finding the averaged (filtered) quantities of velocity. A good survey of the spatial filters commonly used in LES is given in [15].

We explore the discontinuous finite element techniques when applied to the zeroth order LES model (introduced below) of local averages of the fluid velocity. First, consider the Navier-Stokes equations under the no-slip boundary condition,

$$\mathbf{u}_t + \nabla\cdot(\mathbf{u}\mathbf{u}) - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad (0,T]\times\Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } [0,T]\times\Omega, \tag{1.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in} \quad [0,T]\times\Gamma, \tag{1.3}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot) \quad \text{in } \Omega, \tag{1.4}$$

where $\Omega\subset\mathbb{R}^2$, is a convex bounded regular domain with boundary Γ, $\mathbf{u}$ is the fluid velocity, p is the fluid pressure and $\mathbf{f}$ is the body force driving the flow. The kinematic viscosity $\nu > 0$ is inversely proportional to the Reynolds number of the flow. The initial velocity is

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given by $\mathbf{u}_0$. A pressure normalization condition $\int_\Omega p = 0$ is also needed for uniqueness of the pressure.

The zeroth-order model is obtained by applying a spatial averaging operator to (1.1)-(1.4) defined by:

$$\overline{\phi} = A^{-1}\phi \quad \text{in } \Omega \tag{1.5}$$
$$\overline{\phi} = \mathbf{0} \quad \text{on } \Gamma \tag{1.6}$$

where $A = -\delta^2\triangle + I$. Here $\delta > 0$ represents the averaging radius, in general, chosen to be of the order of the mesh size [6]. We will assume that the following bound holds:

$$\|\overline{\boldsymbol{\phi}}\|_{2,\Omega} \le C\|\boldsymbol{\phi}\|_{0,\Omega} \tag{1.7}$$

Using the fact that A^{-1} commutes with $\triangle$, we then obtain the following averaged Navier-Stokes equations:

$$\overline{\mathbf{u}}_t + \overline{\nabla\cdot(\mathbf{u}\mathbf{u})} - \nu\Delta\overline{\mathbf{u}} + \overline{\nabla p} = \overline{\mathbf{f}} \quad \text{in} \;\; (0,T]\times\Omega, \tag{1.8}$$
$$\overline{\nabla\cdot\mathbf{u}} = 0 \quad \text{in } [0,T]\times\Omega, \tag{1.9}$$
$$\overline{\mathbf{u}} = 0 \quad \text{on } [0,T]\times\Gamma, \tag{1.10}$$
$$\overline{\mathbf{u}}(0,\cdot) = \overline{\mathbf{u}}_0(\cdot) \quad \text{in } \Omega. \tag{1.11}$$

If we now neglect the error $\overline{\nabla\cdot(\mathbf{u}\mathbf{u})} - \overline{\nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}})}$, which is of order δ^2, and the commutation error $\overline{\nabla\cdot\mathbf{u}} - \nabla\cdot\overline{\mathbf{u}}$, we obtain the zeroth-order model problem satisfied by an approximation $\mathbf{w}$ of the local averages $\overline{\mathbf{u}}$ of the velocity:

$$\mathbf{w}_t + \overline{\nabla\cdot(\mathbf{w}\mathbf{w})} - \nu\Delta\mathbf{w} + \overline{\nabla p} = \overline{\mathbf{f}} \quad \text{in} \;\; (0,T]\times\Omega, \tag{1.12}$$
$$\nabla\cdot\mathbf{w} = 0 \quad \text{in } [0,T]\times\Omega, \tag{1.13}$$
$$\mathbf{w} = 0 \quad \text{on } [0,T]\times\Gamma, \tag{1.14}$$
$$\mathbf{w}(0,\cdot) = \overline{\mathbf{u}}_0(\cdot) \quad \text{in } \Omega. \tag{1.15}$$

The zeroth order model is the lowest order model of a family of approximate deconvolution models introduced by Stolz and Adams [1, 16]. In the case of periodic boundary conditions, existence, uniqueness and regularity of strong solutions of these models is proved in [4]. The particular zeroth order model is considered in [11, 12]. Even though there is a huge amount of papers on the simulation of Stolz-Adams models for incompressible and compressible flows, There is little published work in the literature on the numerical analysis of the models. In [13, 14], two different semi-discrete schemes using conforming finite elements are analyzed.

In this work, we formulate and analyze a class of discontinuous finite element methods for solving the popular lowest order of the Stolz and Adams models. The approximations of the averaged velocity $\mathbf{w}$ and pressure p are discontinuous piecewise polynomials of degree one and zero respectively. Because of the lack of continuity constraint between elements, the Discontinuous Galerkin (DG) methods offer several advantages over the classical continuous finite element methods: (i) local mesh refinement and derefinement are easily implemented (several hanging nodes per edge are allowed); (ii) the incompressibility condition is satisfied locally on each mesh element; and (iii) unstructured meshes and domains with complicated geometries are easily handled. In the case of DNS, DG methods have been applied to the steady-state Navier-Stokes equations in [7] and to the time-dependent Navier-Stokes equations in [8] where they are combined with an operator splitting technique. Another discontinuous Galerkin method for the Navier-Stokes equations based on a mixed formulation are considered in [2]. For high Reynolds numbers, the numerical analysis of a DG scheme combined with a LES turbulence model (subgrid eddy viscosity model) is derived in [9]. This turbulence model involves two grids.

This paper is organized as follows. Section 2 introduces some notation and mathematical properties. In Section 3, the fully discrete schemes are introduced. A priori error estimates are derived in Section 4. Conclusions are given in the last section.

2 Notation and Mathematical Preliminaries

To obtain a discretization of the model we introduce a regular family of triangulations $\mathcal{E}_h$ of $\overline{\Omega}$, consisting of triangles of maximum diameter h. Let h_E denote the diameter of a triangle E and ρ_E the diameter of its inscribed circle. By regular, we mean that there exists a parameter $\zeta > 0$, independent of h, such that

$$\frac{h_E}{\rho_E} = \zeta_E \leq \zeta, \quad \forall E \in \mathcal{E}_h.$$

We shall use this assumption throughout this work. We denote by Γ_h the set of all interior edges of $\mathcal{E}_h$. Let e denote a segment of Γ_h shared by two triangles E^k and E^l $(k < l)$ of $\mathcal{E}_h$; we associate with e a specific unit normal vector $\mathbf{n}_e$ directed from E^k to E^l and we define formally the jump and average of a function ϕ on e by:

$$[\phi] = (\phi|_{E^k})|_e - (\phi|_{E^l})|_e, \quad \{\phi\} = \frac{1}{2}(\phi|_{E^k})|_e + \frac{1}{2}(\phi|_{E^l})|_e.$$

If e belongs to the boundary Γ, then $\mathbf{n}_e$ is the unit normal $\mathbf{n}$ exterior to Ω and the jump and the average of ϕ on e coincide with the trace of ϕ on e. Next, we define the discrete velocity and pressure spaces consisting of discontinuous piecewise polynomials:

$$\mathbf{X}^h = \{\mathbf{v} \in (L^2(\Omega))^2 : \ \forall E \in \mathcal{E}_h, \ \mathbf{v} \in (\mathbb{P}_1(E))^2\}, \tag{2.1}$$

$$Q^h = \{q \in L^2_0(\Omega) : \ \forall E \in \mathcal{E}_h, \ q \in \mathbb{P}_0(E)\}. \tag{2.2}$$

Here, for any domain $\mathcal{O}$, $L^2(\mathcal{O})$ is the classical space of square-integrable functions with inner-product $(f,g)_{\mathcal{O}} = \int_{\mathcal{O}} fg$ and norm $\|\cdot\|_{0,\mathcal{O}}$. The space $L^2_0(\Omega)$ is the subspace of functions of $L^2(\Omega)$ with zero mean value:

$$L^2_0(\Omega) = \{v \in L^2(\Omega) : \ \int_\Omega v = 0\}.$$

We also use the standard Sobolev spaces $H^r(\Omega)$, with norm $\|\cdot\|_{r,\Omega}$ and semi-norm $|\cdot|_{r,\Omega}$. Denoting by $|e|$ the measure of e, we associate with the spaces $\mathbf{X}_h$ and Q^h the following norms

$$||\mathbf{v}||_X = \Big(|||\nabla \mathbf{v}|||^2_{0,\Omega} + \sum_{e\in\Gamma_h\cup\Gamma} \frac{1}{|e|}\|[\mathbf{v}]\|^2_{0,e}\Big)^{1/2}, \tag{2.3}$$

$$||q||_Q = ||q||_{0,\Omega}, \tag{2.4}$$

where $|||\mathbf{v}|||_{0,\Omega}$ is the broken norm defined by:

$$|||\mathbf{v}|||_{0,\Omega} = \Big(\sum_{E\in\mathcal{E}_h} ||\mathbf{v}||^2_{0,E}\Big)^{1/2}.$$

Finally, we recall some trace and inverse inequalities, that hold true on each element E in $\mathcal{E}_h$, with diameter h_E. The constant C is independent of h_E.

$$||\mathbf{v}||_{0,e} \leq C(h_E^{-1/2}||\mathbf{v}||_{0,E} + h_E^{1/2}||\nabla\mathbf{v}||_{0,E}), \quad \forall e \in \partial E, \quad \forall \mathbf{v} \in (H^1(E))^2, \tag{2.5}$$

$$||\nabla\mathbf{v}||_{0,e} \leq C(h_E^{-1/2}||\nabla\mathbf{v}||_{0,E} + h_E^{1/2}||\nabla^2\mathbf{v}||_{0,E}), \quad \forall e \in \partial E, \quad \forall \mathbf{v} \in (H^2(E))^2, \tag{2.6}$$

$$||\mathbf{v}||_{0,e} \leq C\, h_E^{-1/2}||\mathbf{v}||_{0,E}, \quad \forall e \in \partial E, \quad \forall \mathbf{v} \in \mathbf{X}^h, \tag{2.7}$$

$$||\nabla\mathbf{v}||_{0,e} \leq C\, h_E^{-1/2}||\nabla\mathbf{v}||_{0,E}, \quad \forall e \in \partial E, \quad \forall \mathbf{v} \in \mathbf{X}^h. \tag{2.8}$$

3 Numerical Methods

In this section, we introduce the DG scheme and show existence of the numerical solution. We first define the bilinear forms $a : \mathbf{X}^h \times \mathbf{X}^h \to \mathbb{R}$, $d : \mathbf{X}^h \times \mathbf{X}^h \to \mathbb{R}$, $J_0 : \mathbf{X}^h \times \mathbf{X}^h \to \mathbb{R}$ and $J_1 : \mathbf{X}^h \times \mathbf{X}^h \to \mathbb{R}$ by

$$a(\mathbf{z},\mathbf{v}) = \sum_{E\in\mathcal{E}_h} \int_E \nabla\mathbf{z} : \nabla\mathbf{v} - \sum_{e\in\Gamma_h\cup\Gamma} \int_e \{\nabla\mathbf{z}\}\mathbf{n}_e \cdot [\mathbf{v}] + \epsilon_a \sum_{e\in\Gamma_h\cup\Gamma} \int_e \{\nabla\mathbf{v}\}\mathbf{n}_e \cdot [\mathbf{z}], \tag{3.1}$$

$$d(\mathbf{z},\mathbf{v}) = \sum_{E\in\mathcal{E}_h} \int_E \nabla\mathbf{z} : \nabla\mathbf{v} + \epsilon_d \sum_{e\in\Gamma_h} \int_e [\nabla\mathbf{z}]\mathbf{n}_e \cdot \{\mathbf{v}\} - \sum_{e\in\Gamma_h} \int_e [\nabla\mathbf{v}]\mathbf{n}_e \cdot \{\mathbf{z}\}, \tag{3.2}$$

$$J_0(\mathbf{z},\mathbf{v}) = \sum_{e\in\Gamma_h\cup\Gamma} \frac{\sigma}{|e|} \int_e [\mathbf{z}] \cdot [\mathbf{v}], \tag{3.3}$$

$$J_1(\mathbf{z},\mathbf{v}) = \sum_{e\in\Gamma_h} \frac{1}{|e|} \int_e [\nabla\mathbf{z}]\mathbf{n}_e \cdot [\nabla\mathbf{v}]\mathbf{n}_e. \tag{3.4}$$

The parameters ϵ_a, ϵ_d take the value $-1, 0$ or 1: this will yield different schemes that are slight variations of each other. We will show that all the resulting schemes are convergent with optimal convergence rate in the energy norm. In the case where $\epsilon_a = \epsilon_d = -1$, the bilinear forms a and d are symmetric; otherwise they are non-symmetric. We remark that the form $a(\mathbf{w},\mathbf{v})$ is the standard primal DG discretization of the operator $-\Delta\mathbf{w}$. The form d is introduced here because of the action of the averaging operator A^{-1}. Finally, we assume that if ϵ_a is either -1 or 0, the jump parameter σ should be chosen sufficiently large to obtain coercivity of a (see Lemma 3.3). If $\epsilon_a = 1$, then the jump parameter σ is taken equal to 1. The choice of ϵ_d does not affect the value of the jump parameter.

The incompressibility condition (1.13) is enforced by means of the bilinear form b : $\mathbf{X}^h \times Q^h \to \mathbb{R}$ defined by

$$b(\mathbf{v},q) = -\sum_{E\in\mathcal{E}_h} \int_E q\nabla\cdot\mathbf{v} + \sum_{e\in\Gamma_h\cup\Gamma} \int_e \{q\}[\mathbf{v}]\cdot\mathbf{n}_e, \tag{3.5}$$

Finally, we recall the DG discretization of the nonlinear convection term $\mathbf{w}\cdot\nabla\mathbf{w}$, which was introduced in [7] and studied extensively in [7, 8].

$$c^{\mathbf{z}}(\mathbf{u};\mathbf{v},\mathbf{t}) = \sum_{E\in\mathcal{E}_h} \Big(\int_E (\mathbf{u}\cdot\nabla\mathbf{v})\cdot\mathbf{t} + \frac{1}{2}\int_E (\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{t} \Big) - \frac{1}{2} \sum_{e\in\Gamma_h\cup\Gamma} \int_e [\mathbf{u}]\cdot\mathbf{n}_e\{\mathbf{v}\cdot\mathbf{t}\}$$
$$+ \sum_{E\in\mathcal{E}_h} \int_{\partial E_-} |\{\mathbf{u}\}\cdot\mathbf{n}_E|(\mathbf{v}^{\text{int}} - \mathbf{v}^{\text{ext}})\cdot\mathbf{t}^{\text{int}}, \tag{3.6}$$

where

$$\partial E_- = \{\mathbf{x} \in \partial E : \ \{\mathbf{z}\} \cdot \mathbf{n}_E < 0\},$$

the superscript $\mathbf{z}$ denotes the dependence of ∂E_- on $\mathbf{z}$ and the superscript int (resp. ext) refers to the trace of the function on a side of E coming from the interior of E (resp. coming from the exterior of E on that side). When the side of E belongs to $\partial\Omega$, the convention is the same as for defining jumps and average, i.e., the jump and average coincide with the trace of the function. Note that the form c is not linear with respect to $\mathbf{z}$, but linear with respect to $\mathbf{u}, \mathbf{v}$ and $\mathbf{t}$.

We can now define the numerical scheme that uses discontinuous finite elements in space and backward Euler in time. For this, we let $\triangle t$ denote the time step such that $M = T/\triangle t$ is a positive integer. We let $0 = t_0 < t_1 < \cdots < t_M = T$ be a subdivision of the interval $(0, T)$. We denote the function ϕ evaluated at the time t_m by ϕ_m. With the above forms, the fully-discrete scheme is : find $(\mathbf{w}_n^h, p_n^h)_{n\geq 0} \in \mathbf{X}^h \times Q^h$ such that:

$$\frac{1}{\triangle t}(\mathbf{w}_{n+1}^h - \mathbf{w}_n^h, \mathbf{v})_\Omega + \frac{\delta^2}{\triangle t} d(\mathbf{w}_{n+1}^h - \mathbf{w}_n^h, \mathbf{v}) + c^{\mathbf{w}_n^h}(\mathbf{w}_n^h; \mathbf{w}_{n+1}^h, \mathbf{v}) + b(\mathbf{v}, p_{n+1}^h)$$
$$+\nu(a(\mathbf{w}_{n+1}^h, \mathbf{v}) + J_0(\mathbf{w}_{n+1}^h, \mathbf{v})) + \delta^2 J_1(\mathbf{w}_{n+1}^h, \mathbf{v}) = (f_{n+1}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{X}^h, \tag{3.7}$$
$$b(\mathbf{w}_{n+1}^h, q) = 0 \quad \forall q \in Q^h, \tag{3.8}$$
$$(\mathbf{w}_0^h, \mathbf{v})_\Omega = (\overline{\mathbf{u}}_0, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{X}^h. \tag{3.9}$$

However, this scheme is not consistent. In order to precisely state the consistency error, we need the following result.

Lemma 3.1. *Let $\boldsymbol{\phi} \in (L^2(\Omega))^2$. For any $\mathbf{v} \in \mathbf{X}^h$, we have:*

$$(\overline{\boldsymbol{\phi}}, A\mathbf{v})_\Omega - (\boldsymbol{\phi}, \mathbf{v})_\Omega = l(\overline{\boldsymbol{\phi}}, \mathbf{v}), \tag{3.10}$$

where

$$l(\overline{\boldsymbol{\phi}}, \mathbf{v}) = \delta^2 \sum_{e \in \Gamma_h \cup \Gamma} \int_e (\nabla \overline{\boldsymbol{\phi}})\mathbf{n}_e \cdot [\mathbf{v}] - \delta^2 \sum_{e \in \Gamma_h} \int_e \overline{\boldsymbol{\phi}} \cdot [\nabla \mathbf{v}]\mathbf{n}_e.$$

Furthermore, the following bound holds:

$$l(\overline{\boldsymbol{\phi}}, \mathbf{v}) \leq C\delta^2 \Big(\|\mathbf{v}\|_X + J_1(\mathbf{v}, \mathbf{v})^{1/2} \Big) \|\boldsymbol{\phi}\|_{0,\Omega}, \tag{3.11}$$

where C only depends on the domain Ω.

Proof. By definition of $\overline{\boldsymbol{\phi}}$, we have

$$(\overline{\boldsymbol{\phi}}, A\mathbf{v})_\Omega = (\overline{\boldsymbol{\phi}}, \mathbf{v})_\Omega - \delta^2 \sum_{E \in \mathcal{E}_h} (\overline{\boldsymbol{\phi}}, \Delta \mathbf{v})_E.$$

Using Green's formula and the fact that $\overline{\boldsymbol{\phi}} \in (H^2(\Omega))^2$, we have

$$(\overline{\boldsymbol{\phi}}, A\mathbf{v})_\Omega = (\overline{\boldsymbol{\phi}}, \mathbf{v})_\Omega + \delta^2 \sum_{E \in \mathcal{E}_h} (\nabla \overline{\boldsymbol{\phi}}, \nabla \mathbf{v})_E - \delta^2 \sum_{E \in \mathcal{E}_h} (\nabla \mathbf{v} \mathbf{n}_E, \overline{\boldsymbol{\phi}})_{\partial E}.$$

We again use Green's formula and obtain:

$$(\overline{\boldsymbol{\phi}}, A\mathbf{v})_\Omega = (\overline{\boldsymbol{\phi}}, \mathbf{v})_\Omega - \delta^2 \sum_{E \in \mathcal{E}_h} (\Delta \overline{\boldsymbol{\phi}}, \mathbf{v})_E + \delta^2 \sum_{E \in \mathcal{E}_h} (\nabla \overline{\boldsymbol{\phi}} \mathbf{n}_E, \mathbf{v})_{\partial E} - \delta^2 \sum_{E \in \mathcal{E}_h} (\nabla \mathbf{v} \mathbf{n}_E, \overline{\boldsymbol{\phi}})_{\partial E}$$
$$= (A\overline{\boldsymbol{\phi}}, \mathbf{v})_\Omega + \delta^2 \sum_{e \in \Gamma_h \cup \Gamma} \int_e (\nabla \overline{\boldsymbol{\phi}})\mathbf{n}_e \cdot [\mathbf{v}] - \delta^2 \sum_{e \in \Gamma_h \cup \Gamma} \int_e [\nabla \mathbf{v}]\mathbf{n}_e \cdot \overline{\boldsymbol{\phi}}.$$

Using the boundary condition (1.6), we then have (3.10). In order to prove (3.11), we use Cauchy-Schwarz's inequality, trace inequalities (2.5), (2.6) and the bound (1.7):

$$
\begin{aligned}
l(\overline{\boldsymbol{\phi}}, \mathbf{v}) &\leq \delta^2 \sum_{e \in \Gamma_h \cup \Gamma} \|(\nabla \overline{\boldsymbol{\phi}}) \mathbf{n}_e\|_{0,e} \|[\mathbf{v}]\|_{0,e} + \delta^2 \sum_{e \in \Gamma_h} \|\overline{\boldsymbol{\phi}}\|_{0,e} \|[\nabla \mathbf{v}] \cdot \mathbf{n}_e\|_{0,e} \\
&\leq C\delta^2 \big(\sum_{e \in \Gamma_h \cup \Gamma} \frac{1}{|e|} \|[\mathbf{v}]\|_{0,e}^2 \big)^{1/2} \|\overline{\boldsymbol{\phi}}\|_{2,\Omega} + C\delta^2 J_1(\mathbf{v}, \mathbf{v})^{1/2} \|\overline{\boldsymbol{\phi}}\|_{1,\Omega} \\
&\leq C\delta^2 \|\mathbf{v}\|_X \|\boldsymbol{\phi}\|_{0,\Omega} + C\delta^2 J_1(\mathbf{v}, \mathbf{v})^{1/2} \|\boldsymbol{\phi}\|_{0,\Omega}.
\end{aligned}
$$

In the inequality above and throughout the paper, the constant C is a generic constant that is independent of h, ν and Δt, and that takes different values at different places. □

Lemma 3.2. *Consistency. Let* $(\mathbf{w}, p)$ *be the solution to (1.12)-(1.15). Using the notation of Lemma 3.1, define*

$$
E_c(\mathbf{w}, p, \mathbf{f}; \mathbf{v}) = l(\overline{\nabla \cdot (\mathbf{w}\mathbf{w})}, \mathbf{v}) + l(\overline{\nabla p}, \mathbf{v}) + l(\overline{\mathbf{f}}, \mathbf{v}).
$$

Then $(\mathbf{w}, p)$ *satisfies*

$$
\begin{aligned}
(\mathbf{w}_t, \mathbf{v})_\Omega + \delta^2 d(\mathbf{w}_t, \mathbf{v}) + c^{\mathbf{w}}(\mathbf{w}; \mathbf{w}, \mathbf{v}) + b(\mathbf{v}, p) + \nu(a(\mathbf{w}, \mathbf{v}) + J_0(\mathbf{w}, \mathbf{v}))& \\
+\delta^2 J_1(\mathbf{w}, \mathbf{v}) = (f, \mathbf{v})_\Omega - E_c(\mathbf{w}, p, \mathbf{f}; \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}^h, \forall t > 0,& \qquad (3.12) \\
b(\mathbf{w}, q) = 0 \quad \forall q \in Q^h, \forall t > 0,& \qquad (3.13) \\
(\mathbf{w}_0, \mathbf{v})_\Omega = (\overline{\mathbf{u}}_0, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{X}^h.& \qquad (3.14)
\end{aligned}
$$

Proof. Equations (3.13) and (3.14) are clearly satisfied because of (1.13), (1.14), (1.15) and the regularity of $\mathbf{w}$. Next, we multiply (1.12) by $A\mathbf{v}$ and integrate over one mesh element E:

$$
(\mathbf{w}_t, A\mathbf{v})_E + (\overline{\nabla \cdot (\mathbf{w}\mathbf{w})}, A\mathbf{v})_E - \nu(\Delta \mathbf{w}, A\mathbf{v})_E + (\overline{\nabla p}, A\mathbf{v})_E = (\overline{\mathbf{f}}, A\mathbf{v})_E.
$$

Summing over all elements E, using Lemma 3.1 and the fact that $\nabla \cdot (\mathbf{w}\mathbf{w})$, ∇p and $\mathbf{f}$ belong to $(L^2(\Omega))^2$, we have:

$$
\begin{aligned}
\sum_{E \in \mathcal{E}_h} (\mathbf{w}_t, A\mathbf{v})_E + (\nabla \cdot (\mathbf{w}\mathbf{w}), \mathbf{v})_\Omega - \nu \sum_{E \in \mathcal{E}_h} (\Delta \mathbf{w}, A\mathbf{v})_E& \\
+(\nabla p, \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega - E_c(\mathbf{w}, p, \mathbf{f}; \mathbf{v}).&
\end{aligned}
$$

Next, using the definition of A, Green's formula and the fact that $\mathbf{w}_t = \mathbf{0}$ on the boundary, we have:

$$
\begin{aligned}
\sum_{E \in \mathcal{E}_h} (\mathbf{w}_t, A\mathbf{v})_E &= (\mathbf{w}_t, \mathbf{v})_\Omega - \delta^2 \sum_{E \in \mathcal{E}_h} (\mathbf{w}_t, \Delta \mathbf{v})_E \\
&= (\mathbf{w}_t, \mathbf{v})_\Omega + \delta^2 \sum_{E \in \mathcal{E}_h} (\nabla \mathbf{w}_t, \nabla \mathbf{v})_E - \delta^2 \sum_{e \in \Gamma_h} \int_e [\nabla \mathbf{v}] \mathbf{n}_e \cdot \{\mathbf{w}_t\}.
\end{aligned}
$$

The regularity of $\mathbf{w}_t$ then gives:

$$
\sum_{E \in \mathcal{E}_h} (\mathbf{w}_t, A\mathbf{v})_E = (\mathbf{w}_t, \mathbf{v})_\Omega + \delta^2 d(\mathbf{w}_t, \mathbf{v}).
$$

Similarly, we have by the definition of A and Green's formula:

$$-\nu \sum_{E\in\mathcal{E}_h} (\Delta\mathbf{w}, A\mathbf{v})_E = -\nu(\Delta\mathbf{w},\mathbf{v})_\Omega + \nu\delta^2 \sum_{E\in\mathcal{E}_h} (\Delta\mathbf{w},\Delta\mathbf{v})_E$$

$$= \nu \sum_{E\in\mathcal{E}_h} (\nabla\mathbf{w},\nabla\mathbf{v})_E - \nu \sum_{e\in\Gamma_h\cup\Gamma} \int_e (\nabla\mathbf{w})\mathbf{n}_e\cdot[\mathbf{v}] + \nu\delta^2 \sum_{E\in\mathcal{E}_h} (\Delta\mathbf{w},\Delta\mathbf{v})_E.$$

The regularity of $\mathbf{w}$ and the fact that $\Delta\mathbf{v}=0$ then yield:

$$-\nu \sum_{E\in\mathcal{E}_h} (\Delta\mathbf{w}, A\mathbf{v})_E = \nu(a+J_0)(\mathbf{w},\mathbf{v}) + \delta^2 J_1(\mathbf{w},\mathbf{v}).$$

Therefore, we obtain the following equation for $\mathbf{w}$:

$$(\mathbf{w}_t,\mathbf{v})_\Omega + \delta^2 d(\mathbf{w}_t,\mathbf{v}) + \nu(a+J_0)(\mathbf{w},\mathbf{v})$$
$$+\delta^2 J_1(\mathbf{w},\mathbf{v}) + (\nabla\cdot(\mathbf{w}\mathbf{w}),\mathbf{v})_\Omega + (\nabla p,\mathbf{v})_\Omega = (\mathbf{f},\mathbf{v})_\Omega - E_c(\mathbf{w},p,\mathbf{f};\mathbf{v}).$$

The final result is obtained by noting that Green's formula yields

$$(\nabla p,\mathbf{v})_\Omega = b(\mathbf{v},p),$$

and that the incompressibility condition with the regularity of $\mathbf{w}$ yield

$$(\nabla\cdot(\mathbf{w}\mathbf{w}),\mathbf{v})_\Omega = (\mathbf{w}\cdot\nabla\mathbf{w},\mathbf{v})_\Omega = c^{\mathbf{w}}(\mathbf{w};\mathbf{w},\mathbf{v}).$$

□

We now recall important properties satisfied by the forms a, b, c ([17, 7, 8]):

Lemma 3.3. *Coercivity. If $\epsilon_a = 1$, assume that $\sigma = 1$. If $\epsilon_a \in \{-1,0\}$, assume that σ is sufficiently large enough. Then, there is a constant $\kappa > 0$, independent of h, such that*

$$a(\mathbf{v},\mathbf{v}) + J_0(\mathbf{v},\mathbf{v}) \geq \kappa||\mathbf{v}||_X^2, \qquad \forall\mathbf{v}\in\mathbf{X}^h. \tag{3.15}$$

It is clear that $\kappa = 1$ if $\epsilon_a = 1$. Otherwise, κ is a constant that depends on the polynomial degree of $\mathbf{v}$ and of the smallest angle in the mesh. A precise lower bound for σ is given in [5].

Lemma 3.4. *Inf-sup condition. There exists a positive constant β, independent of h such that*

$$\inf_{q\in Q^h}\sup_{\mathbf{v}\in\mathbf{X}^h} \frac{b(\mathbf{v},q)}{||\mathbf{v}||_X\,||q||_{0,\Omega}} \geq \beta. \tag{3.16}$$

Lemma 3.5. *Positivity*

$$c^{\mathbf{v}}(\mathbf{v},\mathbf{z},\mathbf{z}) \geq 0, \quad \forall\mathbf{v},\mathbf{z}\in\{\mathbf{t}\in(L^2(\Omega))^2 : \mathbf{t}|_E\in(H^2(E))^2\,\forall E\in\mathcal{E}_h\}. \tag{3.17}$$

We can now state the existence and uniqueness of the discrete solution.

Proposition 3.1. *Assume that Lemma 3.3 holds. Assume that δ and Δt are of the order h. In addition, if $\epsilon_d \in \{-1,0\}$, assume that Δt is sufficiently small. Then, there exists a unique solution to (3.7)-(3.9).*

Proof. The existence of $\mathbf{w}_0^h$ is trivial. Given $\mathbf{w}_n^h$, the problem of finding a unique $\mathbf{w}_{n+1}^h$ satisfying (3.7)-(3.8) is linear and finite-dimensional. Therefore, it suffices to show uniqueness of the solution. We first consider the problem restricted to the subspace $\mathbf{V}^h$ defined by

$$\mathbf{V}^h = \{\mathbf{v} \in \mathbf{X}^h : b(\mathbf{v}, q) = 0 \quad \forall q \in Q^h\}.$$

Let $\mathbf{w}_{n+1}^h$ and $\hat{\mathbf{w}}_{n+1}^h$ be two solutions and let $\boldsymbol{\chi}_{n+1} = \mathbf{w}_{n+1}^h - \hat{\mathbf{w}}_{n+1}^h$. Then, $\boldsymbol{\chi}_{n+1}$ satisfies:

$$\frac{1}{\Delta t}(\boldsymbol{\chi}_{n+1}, \mathbf{v})_\Omega + \frac{\delta^2}{\Delta t} d(\boldsymbol{\chi}_{n+1}, \mathbf{v}) + c^{\mathbf{w}_n^h}(\mathbf{w}_n^h; \boldsymbol{\chi}_{n+1}, \mathbf{v})$$
$$+\nu(a(\boldsymbol{\chi}_{n+1}, \mathbf{v}) + J_0(\boldsymbol{\chi}_{n+1}, \mathbf{v})) + \delta^2 J_1(\boldsymbol{\chi}_{n+1}^h, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}^h.$$

Choosing $\mathbf{v} = \boldsymbol{\chi}_{n+1}$ and using the coercivity and positivity results (3.15), (3.17) gives:

$$\frac{1}{\Delta t}\|\boldsymbol{\chi}_{n+1}\|_{0,\Omega}^2 + \frac{\delta^2}{\Delta t} d(\boldsymbol{\chi}_{n+1}, \boldsymbol{\chi}_{n+1}) + \nu\kappa\|\boldsymbol{\chi}_{n+1}\|_X^2 + \delta^2 J_1(\boldsymbol{\chi}_{n+1}, \boldsymbol{\chi}_{n+1}) \leq 0. \tag{3.18}$$

We now expand the term $\frac{\delta^2}{\Delta t} d(\boldsymbol{\chi}_{n+1}, \boldsymbol{\chi}_{n+1})$.

$$\frac{\delta^2}{\Delta t} d(\boldsymbol{\chi}_{n+1}, \boldsymbol{\chi}_{n+1}) = \frac{\delta^2}{\Delta t} |||\nabla \boldsymbol{\chi}_{n+1}|||_{0,\Omega}^2 + \frac{\delta^2}{\Delta t}(\epsilon_d - 1) \sum_{e \in \Gamma_h} \int_e [\nabla \boldsymbol{\chi}_{n+1}]\mathbf{n}_e \cdot \{\boldsymbol{\chi}_{n+1}\}. \tag{3.19}$$

In the case where $\epsilon_d = 1$, all the terms in (3.18) are non-negative and we easily conclude that $\boldsymbol{\chi}_{n+1} = \mathbf{0}$. Otherwise, if $\epsilon_d \in \{-1, 0\}$, we bound the second term in (3.19) by using the fact that δ and Δt are of order h, Cauchy-Schwarz's inequality and trace inequality (2.7):

$$\frac{\delta^2}{\Delta t}(\epsilon_d - 1) \sum_{e \in \Gamma_h} \int_e [\nabla \boldsymbol{\chi}_{n+1}]\mathbf{n}_e \cdot \{\boldsymbol{\chi}_{n+1}\} \leq \frac{\delta^2}{2} J_1(\boldsymbol{\chi}_{n+1}, \boldsymbol{\chi}_{n+1}) + C\|\boldsymbol{\chi}_{n+1}\|_{0,\Omega}^2.$$

Thus, we obtain

$$(\frac{1}{\Delta t} - C)\|\boldsymbol{\chi}_{n+1}\|_{0,\Omega}^2 + \frac{\delta^2}{\Delta t}|||\nabla \boldsymbol{\chi}_{n+1}|||_{0,\Omega}^2 + \nu\kappa\|\boldsymbol{\chi}_{n+1}\|_X^2 + \frac{\delta^2}{2} J_1(\boldsymbol{\chi}_{n+1}, \boldsymbol{\chi}_{n+1}) \leq 0,$$

which yields that $\boldsymbol{\chi}_{n+1} = \mathbf{0}$ if Δt is sufficiently small enough. The existence and uniqueness of the pressure p_{n+1}^h is then obtained from the inf-sup condition (3.16). □

We end this section by recalling some approximation properties of the spaces $\mathbf{X}^h$ and Q^h. From [3, 7], for any $\mathbf{v} \in (H_0^1(\Omega))^2$, there is a unique discrete velocity $\tilde{\mathbf{v}} \in \mathbf{X}^h$ such that

$$b(\mathbf{v} - \tilde{\mathbf{v}}, q) = 0 \quad \forall q \in Q^h. \tag{3.20}$$

Furthermore, if $\mathbf{v} \in (H_0^1(\Omega))^2 \cap (H^2(\Omega))^2$, there is a constant C independent of h such that

$$||\mathbf{v} - \tilde{\mathbf{v}}||_X \leq Ch|\mathbf{v}|_{2,\Omega}, \tag{3.21}$$
$$|\mathbf{v} - \tilde{\mathbf{v}}|_{m,\Omega} \leq Ch^{2-m}|\mathbf{v}|_{2,\Omega}, \quad m = 0, 1. \tag{3.22}$$

We will apply these error bounds to both $\mathbf{w}$ and $\mathbf{w}_t$.

For the pressure space, we use the approximation given by the L^2 projection. For any $q \in L_0^2(\Omega)$, there exists a unique discrete pressure $\tilde{q} \in Q^h$ such that

$$(q - \tilde{q}, z)_\Omega = 0 \quad \forall z \in Q^h. \tag{3.23}$$

In addition, if $q \in H^1(\Omega)$, then

$$||q - \tilde{q}||_{m,E} \leq Ch^{1-m}|q|_{1,E}, \quad \forall E \in \mathcal{E}_h, \quad m = 0, 1, 2. \tag{3.24}$$

4 A Priori Error Estimates

In this section, convergence of the scheme (3.7)-(3.9) is proved. Optimal error estimates in the energy norm are obtained.

Theorem 4.1. *Assume that* $\mathbf{w} \in l^2(0,T;(H^2(\Omega))^2)$, $\mathbf{w}_t \in l^2(0,T;(H^2(\Omega))^2) \cap L^\infty((0,T)\times\Omega)$, $\mathbf{w}_{tt} \in L^2(0,T;(H^1(\Omega))^2)$ *and* $p \in l^2(0,T;H^1(\Omega))$. *Assume that* $\overline{\mathbf{u}}_0 \in (H^2(\Omega))^2$ *and* $\mathbf{f} \in l^2(0,T;(L^2(\Omega))^2)$. *Assume also that the coercivity Lemma 3.3 holds. If* δ *and* Δt *are chosen of the order of* h, *and if* Δt *is chosen sufficiently small, there exists a constant* C, *independent of* h *and* Δt *but dependent on* ν^{-1} *such that the following error bounds holds, for any* $1 \le m \le M$:

$$||\mathbf{w}_m - \mathbf{w}^h_m||^2_{0,\Omega} + \nu\kappa\Delta t \sum_{n=1}^{m} ||\mathbf{w}_n - \mathbf{w}^h_n||^2_X \le Ch^2(\nu^{-1} + \nu + 1).$$

Proof. Defining $\mathbf{e}_n = \mathbf{w}(t^n) - \mathbf{w}^h(t^n)$ and subtracting (3.7) from (3.2), we have:

$$\begin{aligned}(\mathbf{w}_t(t_{n+1}),\mathbf{v})_\Omega &+ \frac{1}{\Delta t}(\mathbf{e}_{n+1}-\mathbf{e}_n,\mathbf{v})_\Omega + \delta^2 d(\mathbf{w}_t(t_{n+1}),\mathbf{v}) + \frac{\delta^2}{\Delta t} d(\mathbf{e}_{n+1}-\mathbf{e}_n,\mathbf{v})\\ &+\nu(a+J_0)(\mathbf{e}_{n+1},\mathbf{v}) + c^{\mathbf{w}_{n+1}}(\mathbf{w}_{n+1};\mathbf{w}_{n+1},\mathbf{v}) - c^{\mathbf{w}^h_n}(\mathbf{w}^h_n;\mathbf{w}^h_{n+1},\mathbf{v}) + b(\mathbf{v},p_{n+1}-p^h_{n+1})\\ &+\delta^2 J_1(\mathbf{e}_{n+1},\mathbf{v}) = \frac{1}{\Delta t}(\mathbf{w}_{n+1}-\mathbf{w}_n,\mathbf{v})_\Omega + \frac{\delta^2}{\Delta t} d(\mathbf{w}_{n+1}-\mathbf{w}_n,\mathbf{v}) - E_c(\mathbf{w}_{n+1},p_{n+1},\mathbf{f}_{n+1};\mathbf{v}) \qquad \forall \mathbf{v}\in\mathbf{X}^h.\end{aligned}$$

We now decompose the error $\mathbf{e}_n = \boldsymbol{\eta}_n - \boldsymbol{\phi}_n$, where $\boldsymbol{\phi}_n = \mathbf{w}^h_n - \tilde{\mathbf{w}}_n$ and $\boldsymbol{\eta}_n$ is the interpolation error $\boldsymbol{\eta}_n = \mathbf{w}_n - \tilde{\mathbf{w}}_n$. Choosing $\mathbf{v} = \boldsymbol{\phi}_{n+1}$ in the equation above and using the coercivity result (3.15), we obtain:

$$\begin{aligned}&\frac{1}{2\Delta t}(||\boldsymbol{\phi}_{n+1}||^2_{0,\Omega} - ||\boldsymbol{\phi}_n||^2_{0,\Omega}) + \frac{\delta^2}{\Delta t} d(\boldsymbol{\phi}_{n+1}-\boldsymbol{\phi}_n,\boldsymbol{\phi}_{n+1}) + \nu\kappa||\boldsymbol{\phi}_{n+1}||^2_X\\ &-c^{\mathbf{w}_{n+1}}(\mathbf{w}_{n+1};\mathbf{w}_{n+1},\boldsymbol{\phi}_{n+1}) + c^{\mathbf{w}^h_n}(\mathbf{w}^h_n;\mathbf{w}^h_{n+1},\boldsymbol{\phi}_{n+1}) + \delta^2 J_1(\boldsymbol{\phi}_{n+1},\boldsymbol{\phi}_{n+1}) \le (\boldsymbol{\eta}_t(t_{n+1}),\boldsymbol{\phi}_{n+1})_\Omega\\ &+\delta^2 d(\boldsymbol{\eta}_t(t^{n+1}),\boldsymbol{\phi}_{n+1}) + \nu(a+J_0)(\boldsymbol{\eta}_{n+1},\boldsymbol{\phi}_{n+1}) + (\tilde{\mathbf{w}}_t(t_{n+1}) - \frac{1}{\Delta t}(\tilde{\mathbf{w}}_{n+1}-\tilde{\mathbf{w}}_n),\boldsymbol{\phi}_{n+1})_\Omega\\ &+\delta^2 d(\tilde{\mathbf{w}}_t(t_{n+1}) - \frac{1}{\Delta t}(\tilde{\mathbf{w}}_{n+1}-\tilde{\mathbf{w}}_n),\boldsymbol{\phi}_{n+1}) + b(\boldsymbol{\phi}_{n+1},p_{n+1}-p^h_{n+1})\\ &+\delta^2 J_1(\boldsymbol{\eta}_{n+1},\boldsymbol{\phi}_{n+1}) - E_c(\mathbf{w}_{n+1},p_{n+1},\mathbf{f}_{n+1};\boldsymbol{\phi}_{n+1}).\end{aligned} \tag{4.1}$$

Consider now the nonlinear terms from the above equation. We first note that since $\mathbf{w}$ is continuous, we can rewrite

$$c^{\mathbf{w}_{n+1}}(\mathbf{w}_{n+1};\mathbf{w}_{n+1},\boldsymbol{\phi}_{n+1}) = c^{\mathbf{w}^h_n}(\mathbf{w}_{n+1};\mathbf{w}_{n+1},\boldsymbol{\phi}_{n+1}),$$

so, for readibility, we can drop the superscript $\mathbf{w}^h_n$ in the c form. Therefore, adding and subtracting the interpolant $\tilde{\mathbf{w}}_{n+1}$ yields

$$\begin{aligned}&c^{\mathbf{w}^h_n}(\mathbf{w}^h_n,\mathbf{w}^h_{n+1},\boldsymbol{\phi}_{n+1}) - c^{\mathbf{w}^h_n}(\mathbf{w}_{n+1},\mathbf{w}_{n+1},\boldsymbol{\phi}_{n+1})\\ &= c(\mathbf{w}^h_n,\boldsymbol{\phi}_{n+1},\boldsymbol{\phi}_{n+1}) - c(\boldsymbol{\phi}_n,\boldsymbol{\eta}_{n+1},\boldsymbol{\phi}_{n+1}) + c(\boldsymbol{\phi}_n,\mathbf{w}_{n+1},\boldsymbol{\phi}_{n+1})\\ &-c(\boldsymbol{\eta}_n,\tilde{\mathbf{w}}_{n+1},\boldsymbol{\phi}_{n+1}) - c(\mathbf{w}_n,\boldsymbol{\eta}_{n+1},\boldsymbol{\phi}_{n+1}) - c(\mathbf{w}_{n+1}-\mathbf{w}_n,\mathbf{w}_{n+1},\boldsymbol{\phi}_{n+1}).\end{aligned}$$

Thus, we rewrite the error equation (4.1) as

$$\frac{1}{2\triangle t}(||\boldsymbol{\phi}_{n+1}||^2_{0,\Omega} - ||\boldsymbol{\phi}_n||^2_{0,\Omega}) + \frac{\delta^2}{\triangle t} d(\boldsymbol{\phi}_{n+1} - \boldsymbol{\phi}_n, \boldsymbol{\phi}_{n+1}) + \nu\kappa ||\boldsymbol{\phi}_{n+1}||^2_X$$
$$+c(\mathbf{w}^h_n; \boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) + \delta^2 J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) \leq |c(\boldsymbol{\phi}_n, \boldsymbol{\eta}_{n+1}, \boldsymbol{\phi}_{n+1})| + |c(\boldsymbol{\phi}_n, \mathbf{w}_{n+1}, \boldsymbol{\phi}_{n+1})|$$
$$+|c(\boldsymbol{\eta}_n, \tilde{\mathbf{w}}_{n+1}, \boldsymbol{\phi}_{n+1})| + |c(\mathbf{w}_n, \boldsymbol{\eta}_{n+1}, \boldsymbol{\phi}_{n+1})| + |c(\mathbf{w}_{n+1} - \mathbf{w}_n, \mathbf{w}_{n+1}, \boldsymbol{\phi}_{n+1})| + |(\boldsymbol{\eta}_t(t_{n+1}), \boldsymbol{\phi}_{n+1})_\Omega|$$
$$+\delta^2 |d(\boldsymbol{\eta}_t(t^{n+1}), \boldsymbol{\phi}_{n+1})| + \nu |(a + J_0)(\boldsymbol{\eta}_{n+1}, \boldsymbol{\phi}_{n+1})| + |(\tilde{\mathbf{w}}_t(t_{n+1}) - \frac{1}{\triangle t}(\tilde{\mathbf{w}}_{n+1} - \tilde{\mathbf{w}}_n), \boldsymbol{\phi}_{n+1})_\Omega|$$
$$+\delta^2 |d(\tilde{\mathbf{w}}_t(t_{n+1}) - \frac{1}{\triangle t}(\tilde{\mathbf{w}}_{n+1} - \tilde{\mathbf{w}}_n), \boldsymbol{\phi}_{n+1})| + |b(\boldsymbol{\phi}_{n+1}, p_{n+1} - p^h_{n+1})|$$
$$+|\delta^2 J_1(\boldsymbol{\eta}_{n+1}, \boldsymbol{\phi}_{n+1})| + |E_c(\mathbf{w}_{n+1}, p_{n+1}, \mathbf{f}_{n+1}; \boldsymbol{\phi}_{n+1})|$$
$$\leq |T_0| + |T_1| + \cdots + |T_{12}|. \tag{4.2}$$

From property (3.17), the term $c(\mathbf{w}^h_n; \boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1})$ in the left-hand side of (4.2) is positive and therefore it will be dropped. For the other terms of the form $c(\cdot, \cdot, \cdot)$ that appear on the right-hand side of the above error equation we obtain bounds, exactly as in the proof of Theorem 5.2 in [9]. We recall that the constant C is a generic constant that is independent of h, ν and Δt, and that takes different values at different places.

$$\begin{aligned}
|T_0| &= |c(\boldsymbol{\phi}_n, \boldsymbol{\eta}_{n+1}, \boldsymbol{\phi}_{n+1})| \leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||^2_X + \frac{C}{\nu}||\boldsymbol{\phi}_n||^2_{0,\Omega}, \\
|T_1| &= |c(\boldsymbol{\phi}_n, \mathbf{w}_{n+1}, \boldsymbol{\phi}_{n+1})| \leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||^2_X + \frac{C}{\nu}||\boldsymbol{\phi}_n||^2_{0,\Omega}, \\
|T_2| &= |c(\boldsymbol{\eta}_n, \tilde{\mathbf{w}}_{n+1}, \boldsymbol{\phi}_{n+1})| \leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||^2_X + \frac{C}{\nu}h^2|\mathbf{w}_n|^2_{2,\Omega}, \\
|T_3| &= |c(\mathbf{w}_n, \boldsymbol{\eta}_{n+1}, \boldsymbol{\phi}_{n+1})| \leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||^2_X + \frac{C}{\nu}h^2|\mathbf{w}_n|^2_{2,\Omega}, \\
|T_4| &= |c(\mathbf{w}_{n+1} - \mathbf{w}_n, \mathbf{w}_{n+1}, \boldsymbol{\phi}_{n+1})| \leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||^2_X + \frac{C}{\nu}\triangle t^2||\mathbf{w}_t||^2_{L^\infty([t_n,t_{n+1}]\times\Omega)}.
\end{aligned}$$

Therefore, we have

$$|T_0|+\cdots+|T_4| \leq \frac{5\nu\kappa}{26}\|\boldsymbol{\phi}_{n+1}\|^2_X + C\nu^{-1}\|\boldsymbol{\phi}_n\|^2_{0,\Omega} + C\nu^{-1}h^2|\mathbf{w}_n|^2_{2,\Omega} + C\nu^{-1}\triangle t^2||\mathbf{w}_t||^2_{L^\infty([t_n,t_{n+1}]\times\Omega)}. \tag{4.3}$$

We now consider the term $D = \frac{\delta^2}{\triangle t} d(\boldsymbol{\phi}_{n+1} - \boldsymbol{\phi}_n, \boldsymbol{\phi}_{n+1})$ in the left-hand side of (4.2). We first decompose it into two parts:

$$D = \frac{\delta^2}{\triangle t} d(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) - \frac{\delta^2}{\triangle t} d(\boldsymbol{\phi}_n, \boldsymbol{\phi}_{n+1}) = D_1 + D_2.$$

Then, by the definition of the bilinear form $d(\cdot, \cdot)$ we have

$$\begin{aligned}
D_1 &= \frac{\delta^2}{\triangle t} \sum_{E\in\mathcal{E}_h} \int_E \nabla\boldsymbol{\phi}_{n+1} : \nabla\boldsymbol{\phi}_{n+1} + (\epsilon_d - 1)\frac{\delta^2}{\triangle t} \sum_{e\in\Gamma_h} \int_e \{\boldsymbol{\phi}_{n+1}\} \cdot [\nabla\boldsymbol{\phi}_{n+1}]\mathbf{n}_e \\
&= D_{11} + D_{12}.
\end{aligned}$$

The term D_{11} is positive and stays in the left-hand side of the error equation. In the case where $\epsilon_d = 1$, the other term D_{12} vanishes. In the case where $\epsilon_d \in \{-1, 0\}$, we need to bound D_{12}. Using the definition of J_1 term, Cauchy-Schwarz's inequality, trace inequality (2.7) and the fact that δ and Δt are of the order of h, we have

$$\begin{aligned} D_{12} &\leq \frac{\delta^2}{12} J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) + C(1-\epsilon_d)^2 \frac{\delta^2}{\Delta t^2} \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2 \\ &\leq \frac{\delta^2}{12} J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) + C(1-\epsilon_d)^2 \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2. \end{aligned} \tag{4.4}$$

Nex, we expand the term D_2:

$$\begin{aligned} D_2 &= -\frac{\delta^2}{\Delta t} \sum_{E\in\mathcal{E}_h} \int_E \nabla\boldsymbol{\phi}_{n+1} : \nabla\boldsymbol{\phi}_n + \frac{\delta^2}{\Delta t} \sum_{e\in\Gamma_h} \int_e \{\boldsymbol{\phi}_n\} \cdot [\nabla\boldsymbol{\phi}_{n+1}]\mathbf{n}_e - \epsilon_d \frac{\delta^2}{\Delta t} \sum_{e\in\Gamma_h} \int_e \{\boldsymbol{\phi}_{n+1}\} \cdot [\nabla\boldsymbol{\phi}_n]\mathbf{n}_e \\ &= D_{21} + D_{22} + D_{23}. \end{aligned}$$

To bound D_{21} we simply use Cauchy-Schwarz inequality and Young's inequality:

$$D_{21} \leq \frac{\delta^2}{2\Delta t} |||\nabla\boldsymbol{\phi}_{n+1}|||_{0,\Omega}^2 + \frac{\delta^2}{2\Delta t} |||\nabla\boldsymbol{\phi}_n|||_{0,\Omega}^2. \tag{4.5}$$

To bound D_{22} we use Cauchy-Schwarz inequality and Young's inequality together with the definition of the jump J_1, trace inequality (2.7) and the fact that δ and Δt are of the order of h:

$$\begin{aligned} D_{22} &\leq C\frac{\delta^2}{\Delta t} J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1})^{1/2} \|\boldsymbol{\phi}_n\|_{0,\Omega} \\ &\leq \frac{\delta^2}{12} J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) + C\frac{\delta^2}{\Delta t^2} \|\boldsymbol{\phi}_n\|_{0,\Omega}^2 \\ &\leq \frac{\delta^2}{12} J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) + C\|\boldsymbol{\phi}_n\|_{0,\Omega}^2. \end{aligned} \tag{4.6}$$

We bound D_{23} in the same way as D_{22}:

$$\begin{aligned} D_{23} &\leq \frac{\delta^2}{\Delta t} \sum_{e\in\Gamma_h} ||\{\boldsymbol{\phi}_{n+1}\}||_{0,e} ||[\nabla\boldsymbol{\phi}_n]\mathbf{n}_e||_{0,e} \\ &\leq \frac{\delta^2}{2} J_1(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n) + C\frac{\delta^2}{\Delta t^2} \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2 \\ &\leq \frac{\delta^2}{2} J_1(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n) + C\|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2. \end{aligned}$$

We now bound the rest of the terms on the right hand side of equation (4.2). To bound T_5 we use Cauchy-Schwarz's inequality, Young's inequality and the approximation result (3.22) applied to $\mathbf{w}_t$.

$$\begin{aligned} |T_5| &\leq \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega} \|\boldsymbol{\eta}_t(t_{n+1})\|_{0,\Omega} \\ &\leq \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2 + Ch^4 |\mathbf{w}_t(t^{n+1})|_{2,\Omega}^2. \end{aligned} \tag{4.7}$$

We expand the term T_6:

$$\begin{aligned} |T_6| &\leq |\delta^2 \sum_{E\in\mathcal{E}_h} \int_E \nabla\boldsymbol{\phi}_{n+1} : \nabla\boldsymbol{\eta}_t(t^{n+1})| \\ &\quad + |\delta^2 \sum_{e\in\Gamma_h} \int_e \{\boldsymbol{\eta}_t(t^{n+1})\} \cdot [\nabla\boldsymbol{\phi}_{n+1}]\mathbf{n}_e| + |\epsilon_d \delta^2 \sum_{e\in\Gamma_h} \int_e \{\boldsymbol{\phi}_{n+1}\} \cdot [\nabla\boldsymbol{\eta}_t(t^{n+1})]\mathbf{n}_e| \\ &= |T_{61}| + |T_{62}| + |T_{63}|. \end{aligned}$$

We bound T_{61} using Cauchy-Schwarz's inequality, Young's inequality, and the approximation result (3.21).

$$\begin{aligned}|T_{61}| &\leq \delta^2||\boldsymbol{\phi}_{n+1}||_X||\boldsymbol{\eta}_t(t^{n+1})||_X \\ &\leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||_X^2 + C\nu^{-1}\delta^4||\boldsymbol{\eta}_t(t^{n+1})||_X^2 \\ &\leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||_X^2 + C\nu^{-1}\delta^4 h^2|\mathbf{w}_t(t^{n+1})|_{2,\Omega}^2. \end{aligned} \tag{4.8}$$

Using the definitions of the jump J_1, trace inequality (2.5), and the approximation result (3.22) we have

$$|T_{62}| \leq \frac{\delta^2}{12}J_1(\boldsymbol{\phi}_{n+1},\boldsymbol{\phi}_{n+1}) + C\delta^2h^4|\mathbf{w}_t(t^{n+1})|_{2,\Omega}^2. \tag{4.9}$$

The term T_{63} vanishes if $\epsilon_d = 0$. Otherwise, we bound it using trace inequalities (2.6), (2.7) and approximation result (3.22) and the fact that δ is of the order of h:

$$\begin{aligned}|T_{63}| &\leq \delta^2\sum_{e\in\Gamma_h}||\{\boldsymbol{\phi}_{n+1}\}||_{0,e}||[\nabla\boldsymbol{\eta}_t(t^{n+1})]\cdot\mathbf{n}_e||_{0,e} \\ &\leq \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2 + C\delta^2h^2|\mathbf{w}_t(t^{n+1})|_{2,\Omega}^2. \end{aligned} \tag{4.10}$$

From the above bounds (4.8), (4.9) and (4.10), we have

$$|T_6| \leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||_X^2 + \frac{\delta^2}{12}J_1(\boldsymbol{\phi}_{n+1},\boldsymbol{\phi}_{n+1}) + C(\nu^{-1}+1)\delta^2h^2|\mathbf{w}_t(t^{n+1})|_{2,\Omega}^2 + \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2. \tag{4.11}$$

We also expand the term T_7:

$$\begin{aligned}|T_7| &\leq |\nu\sum_{E\in\mathcal{E}_h}\int_E\nabla\boldsymbol{\eta}_{n+1}:\nabla\boldsymbol{\phi}_{n+1}| + |\nu\sum_{e\in\Gamma_h\cup\Gamma}\int_e\{\nabla\boldsymbol{\eta}_{n+1}\}\mathbf{n}_e\cdot[\boldsymbol{\phi}_{n+1}]| \\ &\quad +|\nu\epsilon_a\sum_{e\in\Gamma_h\cup\Gamma}\int_e\{\nabla\boldsymbol{\phi}_{n+1}\}\mathbf{n}_e\cdot[\boldsymbol{\eta}_{n+1}]| + |\nu J_0(\boldsymbol{\eta}_{n+1},\boldsymbol{\phi}_{n+1})| \\ &= |T_{71}| + |T_{72}| + |T_{73}| + |T_{74}|. \end{aligned} \tag{4.12}$$

We bound T_{71} using Cauchy-Schwarz inequality, Young's inequality and the approximation result (3.21)

$$\begin{aligned}|T_{71}| &\leq \nu||\boldsymbol{\phi}_{n+1}||_X||\boldsymbol{\eta}_{n+1}||_X \\ &\leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||_X^2 + C\nu||\boldsymbol{\eta}_{n+1}||_X^2 \\ &\leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||_X^2 + C\nu h^2|\mathbf{w}_{n+1}|_{2,\Omega}^2. \end{aligned} \tag{4.13}$$

Using Cauchy-Schwarz's inequality, trace inequality (2.6) and approximation result (3.22) we have

$$\begin{aligned}|T_{72}| &\leq \nu\sum_{e\in\Gamma_h\cup\Gamma}||\{\nabla\boldsymbol{\eta}_{n+1}\}\mathbf{n}_e||_{0,e}\sum_{e\in\Gamma_h\cup\Gamma}||[\boldsymbol{\phi}_{n+1}]||_{0,e} \\ &\leq C\nu(\sum_{e\in\Gamma_h\cup\Gamma}\frac{1}{|e|}\|[\boldsymbol{\phi}_{n+1}]\|_{0,e}^2)^{1/2}(|||\nabla\boldsymbol{\eta}_{n+1}|||_{0,\Omega} + h|||\nabla^2\boldsymbol{\eta}_{n+1}|||_{0,\Omega}) \\ &\leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||_X^2 + C\nu h^2|\mathbf{w}_{n+1}|_{2,\Omega}^2. \end{aligned} \tag{4.14}$$

Using Cauchy-Schwarz's inequality, trace inequality (2.8), and approximation result (3.21), we have

$$\begin{aligned} |T_{73}| &\leq \nu(\sum_{e\in\Gamma_h\cup\Gamma} ||\{\nabla\boldsymbol{\phi}_{n+1}\}\mathbf{n}_e||_{0,e}^2)^{1/2}(\sum_{e\in\Gamma_h\cup\Gamma} ||[\boldsymbol{\eta}_{n+1}]||_{0,e}^2)^{1/2} \\ &\leq C\nu||\boldsymbol{\phi}_{n+1}||_X(\sum_{e\in\Gamma_h\cup\Gamma}\frac{1}{|e|}||[\boldsymbol{\eta}_{n+1}]||_{0,e}^2)^{1/2} \\ &\leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||_X^2 + C\nu h^2|\mathbf{w}_{n+1}|_{2,\Omega}^2. \end{aligned} \quad (4.15)$$

Using the approximation result (3.21) we have

$$\begin{aligned} |T_{74}| &\leq \nu(\sum_{e\in\Gamma_h\cup\Gamma}\frac{\sigma}{|e|}||[\boldsymbol{\phi}_{n+1}]||_{0,e}^2)^{1/2}(\sum_{e\in\Gamma_h\cup\Gamma}\frac{\sigma}{|e|}||[\boldsymbol{\eta}_{n+1}]||_{0,e}^2)^{1/2} \\ &\leq C\nu||\boldsymbol{\phi}_{n+1}||_X||\boldsymbol{\eta}_{n+1}||_X \\ &\leq \frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||_X^2 + C\nu h^2|\mathbf{w}_{n+1}|_{2,\Omega}^2. \end{aligned} \quad (4.16)$$

Putting together the bounds (4.13), (4.14), (4.15) and (4.16), we obtain

$$|T_7| \leq 4\frac{\nu\kappa}{26}||\boldsymbol{\phi}_{n+1}||_X^2 + C\nu h^2|\mathbf{w}_{n+1}|_{2,\Omega}^2. \quad (4.17)$$

To bound the term T_8, we first use a Taylor expansion with integral remainder.

$$\tilde{\mathbf{w}}_n = \tilde{\mathbf{w}}_{n+1} - \Delta t\tilde{\mathbf{w}}_t(t_{n+1}) + \frac{1}{2}\int_{t_n}^{t_{n+1}}(s-t_n)\tilde{\mathbf{w}}_{tt}(s)ds. \quad (4.18)$$

This implies that

$$||\tilde{\mathbf{w}}_t(t_{n+1}) - \frac{\tilde{\mathbf{w}}_{n+1}-\tilde{\mathbf{w}}_n}{\Delta t}||_{0,\Omega}^2 \leq \frac{\Delta t}{6}\int_{t_n}^{t_{n+1}}\|\tilde{\mathbf{w}}_{tt}(s)\|_{0,\Omega}^2 ds.$$

Thus, with (3.22), we have

$$\begin{aligned} |T_8| &\leq \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2 + C\Delta t\int_{t_n}^{t_{n+1}}\|\tilde{\mathbf{w}}_{tt}(s)\|_{0,\Omega}^2 ds \\ &\leq \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2 + C\Delta t\int_{t_n}^{t_{n+1}}\|\mathbf{w}_{tt}(s)\|_{0,\Omega}^2 ds. \end{aligned} \quad (4.19)$$

Using the Taylor expansion (4.18) and defining $\boldsymbol{\theta} = \frac{1}{2}\int_{t_n}^{t_{n+1}}(s-t_n)\tilde{\mathbf{w}}_{tt}(s)ds$, we can rewrite the term T_9 as:

$$\begin{aligned} |T_9| &= |\delta^2 d(\boldsymbol{\theta},\boldsymbol{\phi}_{n+1})| \\ &\leq |\delta^2\sum_{E\in\mathcal{E}_h}\int_E\nabla\boldsymbol{\phi}_{n+1}:\nabla\boldsymbol{\theta}| + |\delta^2\sum_{e\in\Gamma_h}\int_e\{\boldsymbol{\theta}\}\cdot[\nabla\boldsymbol{\phi}_{n+1}]\mathbf{n}_e| \\ &\quad +|\epsilon_d\delta^2\sum_{e\in\Gamma_h}\int_e\{\boldsymbol{\phi}_{n+1}\}\cdot[\nabla\boldsymbol{\theta}]\mathbf{n}_e| \\ &= |T_{91}| + |T_{92}| + |T_{93}|. \end{aligned} \quad (4.20)$$

We bound T_{101} using Cauchy-Schwarz inequality, Young's inequality and (3.22).

$$
\begin{aligned}
|T_{91}| &\leq \delta^2 |||\nabla \boldsymbol{\phi}_{n+1}|||_{0,\Omega} |||\nabla \boldsymbol{\theta}|||_{0,\Omega} \\
&\leq \frac{\nu\kappa}{26}\|\boldsymbol{\phi}_{n+1}\|_X^2 + C\nu^{-1}\delta^4 |||\nabla \boldsymbol{\theta}|||_{0,\Omega}^2 \\
&\leq \frac{\nu\kappa}{26}\|\boldsymbol{\phi}_{n+1}\|_X^2 + C\nu^{-1}\delta^4 \Delta t \int_{t_n}^{t_{n+1}} |||\nabla \tilde{\mathbf{w}}_{tt}(s)|||_{0,\Omega}^2 ds \\
&\leq \frac{\nu\kappa}{26}\|\boldsymbol{\phi}_{n+1}\|_X^2 + C\nu^{-1}\delta^4 \Delta t \int_{t_n}^{t_{n+1}} |||\nabla \mathbf{w}_{tt}(s)|||_{0,\Omega}^2 ds. \qquad (4.21)
\end{aligned}
$$

Using the definition of J_1, trace inequality (2.7), approximation result (3.22) and the fact that δ is of the order of h, we have

$$
\begin{aligned}
|T_{92}| &\leq C\delta^2 J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1})^{1/2} \Big(\sum_{e\in\Gamma_h} \|\{\boldsymbol{\theta}\}\|_{0,e}^2\Big)^{1/2} \\
&\leq \frac{\delta^2}{12} J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) + C\delta^2 \Delta t \int_{t_n}^{t_{n+1}} \sum_{e\in\Gamma_h} \|\{\tilde{\mathbf{w}}_{tt}(s)\}\|_{0,e}^2 ds \\
&\leq \frac{\delta^2}{12} J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) + C\delta \Delta t \int_{t_n}^{t_{n+1}} \|\tilde{\mathbf{w}}_{tt}(s)\|_{0,\Omega}^2 ds, \\
&\leq \frac{\delta^2}{12} J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) + C\delta \Delta t \int_{t_n}^{t_{n+1}} \|\mathbf{w}_{tt}(s)\|_{0,\Omega}^2 ds. \qquad (4.22)
\end{aligned}
$$

The term T_{93} vanishes if $\epsilon_d = 0$. Otherwise, we bound it using trace inequalities (2.7), (2.8), approximation result (3.22) and the fact that δ is of the order of h.

$$
\begin{aligned}
|T_{93}| &\leq \delta^2 \sum_{e\in\Gamma_h} \|\{\boldsymbol{\phi}_{n+1}\}\|_{0,e} \|[\nabla \boldsymbol{\theta}]\cdot \mathbf{n}_e\|_{0,e} \\
&\leq \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2 + C\delta^2 \Delta t \int_{t_n}^{t_{n+1}} |||\nabla \tilde{\mathbf{w}}_{tt}(s)|||_{0,\Omega}^2 ds, \\
&\leq \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2 + C\delta^2 \Delta t \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{w}_{tt}(s)\|_{0,\Omega}^2 ds. \qquad (4.23)
\end{aligned}
$$

Putting together the three estimates (4.21), (4.22) and (4.23), we have

$$
|T_9| \leq \frac{\nu\kappa}{26}\|\boldsymbol{\phi}_{n+1}\|_X^2 + \frac{\delta^2}{12} J_1(\boldsymbol{\phi}_{n+1}, \boldsymbol{\phi}_{n+1}) + \|\boldsymbol{\phi}_{n+1}\|_{0,\Omega}^2 + C(\nu^{-1}+1)\delta \Delta t \int_{t_n}^{t_{n+1}} \|\mathbf{w}_{tt}(s)\|_{1,\Omega}^2 ds. \qquad (4.24)
$$

Because of (3.20) and (3.23), the pressure term T_{10} is reduced to

$$
\begin{aligned}
|T_{10}| &= |b(\boldsymbol{\phi}_{n+1}, p_{n+1} - \tilde{p}_{n+1}) + b(\boldsymbol{\phi}_{n+1}, \tilde{p}_{n+1} - p_{n+1}^h)| \\
&= |b(\boldsymbol{\phi}_{n+1}, p_{n+1} - \tilde{p}_{n+1})| \\
&= |\sum_{e\in\Gamma_h} \int_e \{p_{n+1} - \tilde{p}_{n+1}\}[\boldsymbol{\phi}_{n+1}]\cdot \mathbf{n}_e|,
\end{aligned}
$$

which is bounded by using Cauchy-Schwarz's inequality, Young's inequality, trace inequality (2.5) and approximation result (3.24)

$$
\begin{aligned}
|T_{10}| &\leq C\Big(\sum_{e\in\Gamma_h\cup\Gamma} \frac{1}{|e|} \|[\boldsymbol{\phi}_{n+1}]\|_{0,e}\Big)^{1/2} (\|p_{n+1} - \tilde{p}_{n+1}\|_{0,\Omega} + h|||\nabla p_{n+1} - \nabla \tilde{p}_{n+1}|||_{0,\Omega}) \\
&\leq \frac{\nu\kappa}{26} ||\boldsymbol{\phi}_{n+1}||_X^2 + C\nu^{-1}h^2 |p_{n+1}|_{1,\Omega}^2. \qquad (4.25)
\end{aligned}
$$

The term T_{11} is simply bounded using Cauchy-Schwarz, approximation result (3.21) and the fact that δ is of the order of h.

$$\begin{aligned}|T_{11}| &\leq \frac{\delta^2}{12}J_1(\boldsymbol{\phi}_{n+1},\boldsymbol{\phi}_{n+1}) + C\delta^2 J_1(\boldsymbol{\eta}_{n+1},\boldsymbol{\eta}_{n+1}) \\ &\leq \frac{\delta^2}{12}J_1(\boldsymbol{\phi}_{n+1},\boldsymbol{\phi}_{n+1}) + Ch^2|\mathbf{w}_{n+1}|^2_{2,\Omega}. \end{aligned}\tag{4.26}$$

We finally need to bound the consistency error term $E_c(\mathbf{w}_{n+1}, p_{n+1}, \mathbf{f}_{n+1}; \boldsymbol{\phi}_{n+1})$. Using the bound (3.11), we have

$$\begin{aligned}&|E_c(\mathbf{w}_{n+1}, p_{n+1}, \mathbf{f}_{n+1}; \boldsymbol{\phi}_{n+1}| \leq \frac{\delta^2}{12}J_1(\boldsymbol{\phi}_{n+1},\boldsymbol{\phi}_{n+1}) + \frac{\nu\kappa}{26}\|\boldsymbol{\phi}_{n+1}\|^2_X \\ &+C\delta^2(1+\nu^{-1})(\|(\nabla\cdot(\mathbf{w}\mathbf{w}))_{n+1}\|^2_{0,\Omega} + \|\nabla p_{n+1}\|^2_{0,\Omega} + \|\mathbf{f}_{n+1}\|^2_{0,\Omega}).\end{aligned}\tag{4.27}$$

With the bounds (4.3), (4.4), (4.5), (4.6), (4.7), (4.7), (4.11), (4.17), (4.19), (4.24), (4.25), (4.26), and (4.27), the error equation becomes:

$$\begin{aligned}&\frac{1}{2\Delta t}(||\boldsymbol{\phi}_{n+1}||^2_{0,\Omega} - ||\boldsymbol{\phi}_n||^2_{0,\Omega}) + \frac{\nu\kappa}{2}||\boldsymbol{\phi}_{n+1}||^2_X + \frac{\delta^2}{2}(J_1(\boldsymbol{\phi}_{n+1},\boldsymbol{\phi}_{n+1}) - J_1(\boldsymbol{\phi}_n,\boldsymbol{\phi}_n)) \\ &+\frac{\delta^2}{2\Delta t}(|||\nabla\boldsymbol{\phi}_{n+1}|||^2_{0,\Omega} - |||\nabla\boldsymbol{\phi}_n|||^2_{0,\Omega}) \leq C(\nu^{-1}+1)\|\boldsymbol{\phi}_n\|^2_{0,\Omega} + \tilde{C}\|\boldsymbol{\phi}_{n+1}\|^2_{0,\Omega} \\ &+Ch^2(\nu^{-1}|\mathbf{w}_n|^2_{2,\Omega} + (\nu+1)|\mathbf{w}_{n+1}|^2_{2,\Omega} + (\nu^{-1}+1)|\mathbf{w}_t(t_{n+1})|^2_{2,\Omega}) + Ch^2\nu^{-1}|p_{n+1}|^2_{1,\Omega} \\ &+C\Delta t(\nu^{-1}+1)\int_{t_n}^{t_{n+1}}\|\mathbf{w}_{tt}(s)\|^2_{1,\Omega}ds + C\Delta t^2\nu^{-1}\|\mathbf{w}_t\|^2_{L^\infty([t_n,t_{n+1}]\times\Omega)} \\ &+C\delta^2(1+\nu^{-1})(\|(\nabla\cdot(\mathbf{w}\mathbf{w}))_{n+1}\|^2_{0,\Omega} + \|\nabla p_{n+1}\|^2_{0,\Omega} + \|\mathbf{f}_{n+1}\|^2_{0,\Omega}).\end{aligned}$$

where C and $\tilde{C}$ are constants independent of h, ν and Δt. We now multiply the equation by $2\Delta t$ and sum from $n = 0$ to $n = m-1$:

$$\begin{aligned}&(1-2\Delta t\tilde{C})||\boldsymbol{\phi}_m||^2_{0,\Omega} + \nu\kappa\Delta t\sum_{n=0}^{m-1}||\boldsymbol{\phi}_{n+1}||^2_X + \Delta t\delta^2 J_1(\boldsymbol{\phi}_m,\boldsymbol{\phi}_m) + \delta^2|||\nabla\boldsymbol{\phi}_m|||^2_{0,\Omega} \\ &\leq ||\boldsymbol{\phi}_0||^2_{0,\Omega} + \Delta t\delta^2 J_1(\boldsymbol{\phi}_0,\boldsymbol{\phi}_0) + \delta^2|||\nabla\boldsymbol{\phi}_0|||^2_{0,\Omega} + C(\nu^{-1}+1)\sum_{n=0}^{m-1}\|\boldsymbol{\phi}_n\|^2_{0,\Omega} \\ &+Ch^2\Delta t\sum_{n=0}^{m-1}(\nu^{-1}|\mathbf{w}_n|^2_{2,\Omega} + (\nu+1)|\mathbf{w}_{n+1}|^2_{2,\Omega} + (\nu^{-1}+1)|\mathbf{w}_t(t_{n+1})|^2_{2,\Omega}) + Ch^2\nu^{-1}\Delta t\sum_{n=0}^{m-1}|p_{n+1}|^2_{1,\Omega} \\ &+C\Delta t^2(\nu^{-1}+1)\int_0^T\|\mathbf{w}_{tt}(s)\|^2_{1,\Omega}ds + C\Delta t^2\nu^{-1}\|\mathbf{w}_t\|^2_{L^\infty([0,T]\times\Omega)} \\ &+C\delta^2(1+\nu^{-1})\Delta t\sum_{n=0}^{m-1}(\|(\nabla\cdot(\mathbf{w}\mathbf{w}))_{n+1}\|^2_{0,\Omega} + \|\nabla p_{n+1}\|^2_{0,\Omega} + \|\mathbf{f}_{n+1}\|^2_{0,\Omega}).\end{aligned}$$

Thus, if Δt is small enough, using Gronwall's lemma, we conclude that there is a constant C independent of h and Δt, but dependent on ν^{-1}, such that:

$$\begin{aligned}&||\boldsymbol{\phi}_m||^2_{0,\Omega} + \nu\kappa\Delta t\sum_{n=1}^{m}||\boldsymbol{\phi}_n||^2_X \leq ||\boldsymbol{\phi}_0||^2_{0,\Omega} + \Delta t\delta^2 J_1(\boldsymbol{\phi}_0,\boldsymbol{\phi}_0) + \delta^2|||\nabla\boldsymbol{\phi}_0|||^2_{0,\Omega} \\ &+Ch^2(\nu^{-1}+\nu+1) + C\delta^2(1+\nu^{-1}).\end{aligned}$$

The final result is then obtained by noting that the term $||\boldsymbol{\phi}_0||^2_{0,\Omega} + \Delta t \delta^2 J_1(\boldsymbol{\phi}_0, \boldsymbol{\phi}_0)$ is of order h^2 and by using triangle inequality and approximation results. □

5 Conclusion

In this paper, we formulated and analyzed a numerical scheme for solving the Stolz-Adams approximate deconvolution problem for turbulent flows. The proposed method is convergent with optimal convergence rates with respect to the mesh size. The approximations of the average velocity and pressure are discontinuous piecewise polynomials. One benefit of using discontinuous elements is that the error estimates depend on the Reynolds number as $\mathcal{O}(\mathrm{Re}\ e^{\mathrm{Re}})$, whereas the dependence is $\mathcal{O}(\mathrm{Re}\ e^{\mathrm{Re}^3})$ for continuous finite elements [13].

In this work, since the time discretization technique is backward Euler, we limited the order of approximation to linear and constant for the velocity $\mathbf{w}$ and pressure p respectively. If we use a second order in time approach, such as Crank-Nicolson, we can increase the order of spatial approximation to quadratic and linear for $\mathbf{w}$ and p. However, it does not make sense to go to higher order since the consistency error is of second order only.

Finally, we point out that our proposed scheme contain parameters $\epsilon_a, \epsilon_d \in \{-1, 0, 1\}$ that yield different but similar numerical approximations. Only numerical simulations of benchmark problems for high Reynolds numbers, will help determine which choices of ϵ_a and ϵ_d are preferred for a given mesh size. This is the object of a future paper.

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