BOUNDS ON ENERGY AND HELICITY DISSIPATION RATES OF APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE

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Abstract. We consider a family of high accuracy, approximate deconvolution models of turbulence. For body force driven turbulence, we prove directly from the models equations of motion the following bounds on the model's time averaged energy dissipation rate, $< \varepsilon_{ADM} >$, and helicity dissipation rate, $< \gamma_{ADM}(w) >$,

$$< \varepsilon_{ADM} > \leq 2\frac{U^3}{L} + \mathrm{Re}^{-1}\frac{U^3}{L}(1+(\frac{\delta}{L})^2)\ , \text{ and}$$
$$| < \gamma_{ADM}(w) > | \leq \frac{U^3}{L^2} + \sqrt{2}\,\mathrm{Re}^{-\frac{1}{2}}(1+\frac{\delta^2}{L^2})^{\frac{1}{2}}\frac{U^3}{L^2} + \mathrm{Re}^{-1}(1+\frac{\delta^2}{L^2})\frac{U^3}{L^2},$$

where U, L are the global velocity scale and length scale and δ is the LES filter radius. We also give a partial result on the helicity dissipation rate of solutions of the Navier-Stokes equations.

Key words. energy dissipation rate, helicity, helicity dissipation rate, large eddy simulation, turbulence, deconvolution

1. Introduction. Turbulent flows consist of complex, interacting three dimensional eddies of various sizes down to the Kolmogorov microscale, $\eta = O(Re^{-3/4})$ in $3d$. A direct numerical simulation of the persistent eddies in a $3d$ flow thus requires roughly $O(Re^{+9/4})$ mesh points in space per time step. Therefore, direct numerical simulation of turbulent flows is often not computationally economical or even feasible. On the other hand, the largest structures in the flow (containing most of the flow's energy) are responsible for much of the mixing and most of the flow's momentum transport.

One promising approach to predicting a flow's large structures is called *Large Eddy Simulation* or *LES*. LES seeks to model and predict the evolution of local, spatial averages over a user-selected length scale δ. If the LES model does not dissipate enough energy, there can be an accumulation of energy around the smallest resolved scales (i.e., wiggles in the computed velocity). The energy dissipation rates in various LES models are adjusted in various ways, such as using mixed models (i.e., adding eddy viscosity) and picking the parameters introduced (e.g., Lilly[L67]) to match the model's time averaged energy dissipation rate to that of homogeneous, isotropic turbulence. Parameter free (not mixed) models have many advantages and understanding their important statistics, such as their energy dissipation rate, is critical to advancing their reliability.

Herein we consider one family of high-accuracy, parameter free models, *Approximate Deconvolution Models* (ADMs), and bound the ADMs time averaged energy and helicity dissipation rates. The bounds derived mirror both the energy dissipation rate of the underlying solution of the NSE (in the limit $\delta \to 0$) and the estimate of it derived in [LN05], [LMNR06] by dimensional analysis (in the limit $\mathrm{Re} \to \infty$).

To begin, consider the Navier-Stokes equations (NSE) in a periodic box in $\mathbb{R}^3$, $\Omega = (0, L_\Omega)^3$:

$$u_t + u\cdot\nabla u - \nu\triangle u + \nabla p = f(x) \quad \text{in } \Omega = (0, L_\Omega)^3,\ t > 0, \tag{1.1}$$
$$\nabla\cdot u = 0 \quad \text{in } (0, L_\Omega)^3,$$

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subject to periodic (with zero mean) conditions

$$u(x + L_\Omega e_j, t) = u(x,t) \quad j = 1,2,3 \quad \text{and,} \qquad (1.2)$$
$$\int_\Omega \phi dx = 0 \ \text{ for } \ \phi = u,\ u_0,\ f,\ p.$$

We suppose throughout that the data $u_0(x), f(x)$ are smooth and satisfy

$$\nabla \cdot u_0 = 0 \text{ , and } \nabla \cdot f = 0.$$

Many averaging operators are used in LES, see, e.g., Sagaut [S01], John [J04], and [BIL06]. Herein we consider a *differential filter*, Germano [Ger86], associated with length-scale $\delta > 0$ related to the Yoshida regularization (and sometimes called a *Helmholz-filter* in the alpha-model literature, e.g., Cheskidov, Holm, Olson and Titi [CHOT05]) defined as follows. Given $\phi(x)$, $\overline{\phi(x)}$ is the unique L-periodic solution of

$$A\overline{\phi} := -\delta^2 \triangle \overline{\phi} + \overline{\phi} = \phi\,, \text{ in } \Omega.$$

Averaging the NSE (i.e., applying A^{-1} to (1.1)) gives the exact space filtered NSE for $\overline{u}$

$$\overline{u}_t + \overline{u \cdot \nabla u} - \nu \triangle \overline{u} + \nabla \overline{p} = \overline{f}(x) \text{ and}$$
$$\nabla \cdot \overline{u} = 0.$$

This is not closed since (noting that $\overline{u \cdot \nabla u} = \nabla \cdot (\overline{uu})$)

$$\overline{u\,u} \ \neq \overline{u}\,\overline{u}.$$

There are many closure models used in LES, see Sagaut[S01], John[J04], Lesieur, Metais and Comte[LMC05] and [BIL06] for surveys. Approximate deconvolution models, studied herein, are used, with success, in many simulations of turbulent flows, e.g., the works of Adams, Kleiser and Stolz [AS01], [AS02], [SA99], [SAK01a], [SAK01b], [SAK02]. They are among the most accurate of turbulence models, and one of the few turbulence models for which a mathematical confirmation of their effectiveness is known, [LL06b] and Dunca and Epshteyn [DE06]. Briefly, an approximate deconvolution operator (constructed in Section 3) denoted by D_N is an operator satisfying

$$\phi = D_N(\overline{\phi}) + O(\delta^{2N+2}) \quad \text{for smooth } \ \phi.$$

Since $D_N \overline{u}$ approximates u to accuracy $O(\delta^{2N+2})$ in the smooth flow regions it is justified to consider the closure approximation:

$$\overline{uu} \simeq \overline{D_N \overline{u} D_N \overline{u}} + O(\delta^{2N+2}). \qquad (1.3)$$

Using this closure approximation, the resulting family of ADMs is given by[1]

$$w_t + \nabla \cdot (\overline{D_N w \ D_N w}) - \nu \triangle w + \nabla q = \overline{f}(x), \qquad (1.4)$$
$$\nabla \cdot w = 0,\ N = 0, 1, 2, \cdots.$$

[1] In practical computations with ADMs an additional time relaxation term, $\chi\,(w - \overline{w})$, has often been added to (1.4). This term can be used as a numerical regularization in any model and is studied in [LN05], [ELN06], Adams and Stolz [AS02], Pruett [P06] and Guenaff [Gue04].

As a special case, $D_0\overline{u} = \overline{u} + O(\delta^2)$ gives the zeroth order ADM:

$$w_t + \nabla \cdot (\overline{w\, w}) - \nu \triangle w + \nabla q = \overline{f}(x) \text{ and } \nabla \cdot w = 0.$$

We consider two important flow statistics: the *time averaged energy* and *helicity dissipation rates*. The energy dissipation rate is a fundamental statistic in experimental and theoretical studies of turbulence, e.g., Sreenivasan [S84], [S98], Bourne and Orszag [BO97], Pope [P00], Frisch [Frisch], Lesieur [Les97]. In the early 90's Constantin and Doering [CD92] (see also Doering and Gibbon [DG95])established a direct link between the phenomenology of energy dissipation and that predicted for shear flows directly from the NSE. This work builds on earlier work of Busse [B78], Howard [H72] (and others) and has developed in many important directions, including Childress, Kerswell and Gilbert [CKG01], Kerswell [K98] and Wang [W97] (shear flows) and Foias [F97], Doering and Foias [DF02] (body force driven flows). Because of the greater difficulties of studying helicity directly from the NSE, this connection remains open for helicity dissipation rates, see Section 5.1.

Let $< \cdot >$ denote long time averaging (defined in Section 2). K41 phenomenology, e.g., Frisch[Frisch], Pope[P00], in [LN05] suggests the scaling of the energy dissipation rate $< \varepsilon_{ADM} >$

$$< \varepsilon_{ADM} > \approx \frac{U^3}{L}(1 + \frac{\delta^2}{L^2}).$$

In Section 4, we prove directly from the equations of motion (1.4) that the energy dissipation rate of the model satisfies

$$< \varepsilon_{ADM} > \leq 2\frac{U^3}{L} + \mathrm{Re}^{-1}\,\frac{U^3}{L}(1 + (\frac{\delta}{L})^2)\ .$$

Here U, L denote natural velocity and length scales associated with the largest scales of the model (1.4), defined precisely in Section 2.1.

The helicity dissipation rate is defined several ways in the literature on the phenomenology of helicity cascades due to possible coupling with energy dissipation rates. Section 5 uses a definition which is natural from the point of view of the equations of motion for the model (1.4). We prove that the helicity dissipation rate, $| < \gamma_{ADM}(w) > |$, satisfies

$$| < \gamma_{ADM}(w) > | \leq \frac{U^3}{L^2} + \sqrt{2}\,\mathrm{Re}^{-\frac{1}{2}}(1 + \frac{\delta^2}{L^2})^{\frac{1}{2}}\frac{U^3}{L^2} + \mathrm{Re}^{-1}(1 + \frac{\delta^2}{L^2})\frac{U^3}{L^2}.$$

This estimate of $| < \gamma_{ADM}(w) > |$ is consistent with the dimensional analysis estimate of $\frac{U^3}{L^2}$.

The cases of the zeroth order model and the entire family of ADMs are closely related. Proofs will be given for the zeroth order model and the corresponding results (whose proofs involve only additional subscripts) for the N^{th} ADM stated. Aside from the family of ADMs, we also believe that the techniques used herein (beginning with their similar kinetic energy balances) can be used to prove parallel estimates of energy and helicity disssipation rates for the alpha-model.

2. Notation and preliminaries. The time average of a function $\phi(t)$ is defined by

$$< \phi > := \limsup_{T \to \infty} \frac{1}{T} \int_0^T \phi(t) dt.$$

First consider the zeroth order model. With $|\Omega|$ the volume of the flow domain, the scale of the body force and large scale velocity are defined by

$$F := (\frac{1}{|\Omega|}\int_\Omega |f(x)|^2 dx)^{\frac{1}{2}} \text{ , and}$$
$$U := < \frac{1}{|\Omega|}\int_\Omega |w(x,t)|^2 dx >^{\frac{1}{2}}$$

Let $||\cdot||, (\cdot,\cdot)$ denote the usual $L^2(\Omega)$ norm and inner product (other norms are explicitly indicated by a subscript). The global length scale associated with the power input the large scales, i.e., with $f(x)$, is

$$L := \min\{L_\Omega, \frac{F}{||\nabla f||_{L^\infty(\Omega)}}, \frac{F}{(\frac{1}{|\Omega|}||\nabla f||^2)^{\frac{1}{2}}}, \frac{F}{(\frac{1}{|\Omega|}||\nabla\times f||^2)^{\frac{1}{2}}}, \frac{F^{\frac{1}{2}}}{(\frac{1}{|\Omega|}||\triangle f||^2)^{\frac{1}{4}}}\}.$$

It is easy to check that L has units of length and satisfies the inequalities:

$$\begin{aligned} ||\nabla f||_{L^\infty} &\le \frac{F}{L}, \\ \frac{1}{|\Omega|}\int_\Omega |\nabla f(x)|^2 dx &\le \frac{F^2}{L^2}, \text{ and } \frac{1}{|\Omega|}\int_\Omega |\nabla\times f(x)|^2 dx \le \frac{F^2}{L^2} \\ \frac{1}{|\Omega|}\int_\Omega |\triangle f(x)|^2 dx &\le \frac{F^2}{L^4}. \end{aligned} \tag{2.1}$$

The kinetic viscosity is denoted ν and the associated global Reynolds number is $\text{Re} := \frac{LU}{\nu}$.

The energy dissipation rate induced by the model depends on the precise form of the model's kinetic energy balance. Let w denote the solution of the zeroth order model. The appropriate definitions (see Proposition 3.1 and Remark 3.1, (3.2) through (3.5) as well as [LL03], [LL06a], [LL06b]) for the zeroth order model are

$$\varepsilon_{ADM-0}(w)(t) = \frac{\nu}{L^3}\{||\nabla w(t)||^2 + \delta^2||\triangle w(t)||^2\} \text{ and}$$
$$< \varepsilon_{ADM-0} > := < \varepsilon_{ADM-0}(w)(t) > .$$

Before introducing the notation for the general case we must first define the van Cittert approximate deconvolution operators.

2.1. Approximate Deconvolution Operators. The filtering or convolution operator $u \to \overline{u}$ is a bounded map: $L^2(\Omega) \to L^2(\Omega)$. If (as in the case we study) it is smoothing, its inverse cannot be bounded due to small divisor problems. An approximate deconvolution operator D_N is an approximate inverse $\overline{u} \to D_N(\overline{u}) \approx u$ which

- is a bounded operator on $L^2(\Omega)$,
- approximates u in some useful (typically asymptotic) sense, and
- satisfies other conditions necessary for the application at hand.

The de-convolution operator we consider was studied by van Cittert in 1931, e.g., Bertero and Boccacci[BB98], and its use in LES pioneered by Adams, Kleiser and Stolz [AS01], [SA99], [AS02], [SAK01a], [SAK01b], [SAK02]. The Nth van Cittert approximate deconvolution operator D_N is defined by N steps of Picard iteration, [BB98], for the fixed point problem:

$$\text{given } \overline{u} \text{ solve } u = u + \{\overline{u} - A^{-1}u\} \text{ for } u.$$

ALGORITHM 2.1 (van Cittert Approximate Deconvolution Operator). $u_0 = \overline{u}$,
for n=1,2,...,N-1, perform
$u_{n+1} = u_n + \{\overline{u} - A^{-1}u_n\}$
Define $D_N\overline{u} := u_N$.

By eliminating the intermediate steps, the N^{th} de-convolution operator D_N is given explicitly by

$$D_N\phi := \sum_{n=0}^{N}(I - A^{-1})^n\phi. \tag{2.2}$$

For example, the approximate de-convolution operator corresponding to $N = 0, 1, 2$ are:

$$D_0\overline{u} = \overline{u},$$
$$D_1\overline{u} = 2\overline{u} - \overline{\overline{u}},$$
$$D_2\overline{u} = 3\overline{u} - 3\overline{\overline{u}} + \overline{\overline{\overline{u}}}.$$

DEFINITION 2.1. *The deconvolution weighted inner product and norm,* $(\cdot, \cdot)_N$ *and* $|| \cdot ||_N$ *are*

$$(u, v)_N := (u, D_N v)\,, \quad ||u||_N := (u, u)_N^{\frac{1}{2}}$$

LEMMA 2.2. *Consider the approximate deconvolution operator*

$$D_N : L^2(\Omega) \to L^2(\Omega)$$

D_N *is a bounded, self-adjoint, positive-definite operator and satisfies*

$$||\phi||^2 \ \leq \ ||\phi||_N \ \leq \ (N + 1)||\phi||^2, \, \forall \phi \in L^2(\Omega) \ .$$

Proof. D_N is a function of the bounded, self-adjoint operator A^{-1} and is thus bounded and self-adjoint. By the spectral mapping theorem we have

$$\lambda(D_N) = \sum_{n=0}^{N}\lambda(I - A^{-1})^n = \sum_{n=0}^{N}(1 - \lambda(A^{-1}))^n, \text{ and}$$
$$0 < \lambda(A^{-1}) \leq 1 \text{ by the definition of operator A.}$$

Thus, $1 \leq \lambda(D_N) \leq N + 1$. Since D_N is a self-adjoint operator, this proves positive definiteness and the above equivalence of norms. □

3. Kinetic energy balance of ADM turbulence models. To see the mathematical key to the estimates of energy and helicity dissipation rates we first recall from [LL03], [DE06] (see also [LL06a], and [MM06] for the difficult case of no-slip boundary conditions) the energy equality for the ADM (1.4).

PROPOSITION 3.1. *If w is a weak or strong solution*[2] *of (1.4), w satisfies*

$$\frac{1}{2}[||w(T)||_N^2 + \delta^2||\nabla w(T)||_N^2] + \int_0^T \nu||\nabla w(t)||_N^2 + \nu\delta^2||\triangle w(t)||_N^2 dt =$$
$$= \frac{1}{2}[||\overline{u}_0||_N^2 + \delta^2||\nabla \overline{u}_0||_N^2] + \int_0^T (f, w(t))_N \, dt.$$

Proof. (Sketch) Let (w, q) denote a periodic solution of the ADM (1.4). Multiplying (1.4) by $AD_N w$ and integrating over Ω gives

$$\int_\Omega w_t \cdot AD_N w + \nabla \cdot (\overline{D_N w \; D_N w}) \cdot AD_N w - \nu\triangle w \cdot AD_N w + \nabla q \cdot AD_N w d\mathbf{x} =$$
$$= \int_\Omega \overline{f} \cdot AD_N w d\mathbf{x}.$$

The nonlinear term exactly vanishes exactly because

$$\int_\Omega \nabla \cdot (\overline{D_N w \; D_N w}) \cdot AD_N w d\mathbf{x} = \int_\Omega A^{-1}(\nabla \cdot (D_N w \; D_N w)) \cdot AD_N w d\mathbf{x} =$$
$$= \int_\Omega \nabla \cdot (D_N w \; D_N w) \cdot D_N w d\mathbf{x} = 0.$$

Integrating by parts the remaining terms gives

$$\frac{d}{dt}\frac{1}{2}\{||w(t)||_N^2 + \delta^2||\nabla w(t)||_N^2\} + \nu\{||\nabla w(t)||_N^2 + \delta^2||\triangle w(t)||_N^2\} = (f, w(t))_N \,. \quad (3.1)$$

The results follows by integrating this from 0 to t. $\square$

From Proposition 3.1, the ADMs kinetic energy, energy dissipation rate and power input are clearly identified.

ADM energy:

$$E_{ADM-N}(w)(t) := \frac{1}{2|\Omega|}\{||w(t)||_N^2 + \delta^2||\nabla w(t)||_N^2\}, \quad (3.2)$$

ADM dissipation rate:

$$\varepsilon_{ADM-N}(w)(t) := \frac{\nu}{|\Omega|}\{||\nabla w(t)||_N^2 + \delta^2||\triangle w(t)||_N^2\}, \quad (3.3)$$

Time averaged dissipation rate:

$$< \varepsilon_{ADM-N} > := < \varepsilon_{ADM-N}(w)(t) >, \quad (3.4)$$

ADM power input:

$$P_{ADM-N}(w)(t) := \frac{1}{|\Omega|}(f, w(t))_N. \quad (3.5)$$

Let $||\cdot||_N$ denote the deconvolution-weighted $L^2(\Omega)$ norm (Definition 2.1). The deconvolution weighted scales of the body force and large scale velocity are defined by

$$F_N := (\frac{1}{|\Omega|}||f||_N^2)^{\frac{1}{2}} \text{ , and } U_N := < \frac{1}{|\Omega|}||w||_N^2 >^{\frac{1}{2}}$$

[2] Unlike the NSE case, it is known that weak=strong for the ADM and both exist and are unique.

Note that these are related to F and U by

$$F \leq F_N \leq (N+1)^{\frac{1}{2}} F \ , \ U \leq U_N \leq (N+1)^{\frac{1}{2}} U \ , \text{ and also}$$
$$< \varepsilon_{ADM-0}(w) > \leq < \varepsilon_{ADM-N}(w) > \leq (N+1)^{\frac{1}{2}} < \varepsilon_{ADM-0}(w) > .$$

For $N = 1, 2, 3, \cdots$ the deconvolution-weighted global length scale associated with the power input to the large scales, i.e., with $f(x)$, is defined to be

$$L_N := \min\{L_\Omega, \frac{F_N}{||D_N^{\frac{1}{2}} \nabla f||_{L^\infty(\Omega)}}, \frac{F_N}{(\frac{1}{|\Omega|}||\nabla f||_N^2)^{\frac{1}{2}}}, \frac{F_N}{(\frac{1}{|\Omega|}||\nabla \times f||_N^2)^{\frac{1}{2}}}, \frac{F_N^{\frac{1}{2}}}{(\frac{1}{|\Omega|}||\triangle f||_N^2)^{\frac{1}{4}}}\}.$$

It is easy to check that L_N has units of length and satisfies the deconvolution-weighted form of the inequalities (2.1) above.

LEMMA 3.2. *As $\delta \to 0$, for $N = 0, 1, 2, \cdots$*

$$E_{ADM-N}(w)(t) \to E(w)(t) = \frac{1}{2|\Omega|}||w(t)||^2,$$
$$\varepsilon_{ADM-N}(w)(t) \to \varepsilon(w)(t) = \frac{\nu}{2|\Omega|}||\nabla w(t)||^2, \text{ and}$$
$$P_{ADM-N}(w)(t) \to P(w)(t) = \frac{1}{|\Omega|}(f(t), w(t)).$$

Proof. As $\delta \to 0$ all the δ^2 terms drop out in the definitions above, $D_N \to I$ and $||\phi||_N \to ||\phi||$. □

COROLLARY 3.3. *Let $f = f(x) \in L^2(\Omega)$ and w be a solution of the ADM turbulence model (1.4) then*

$$\sup_{t \in (0,\infty)} E_{ADM-N}(w)(t) \leq C(data) < \infty,$$
$$\frac{1}{T}\int_0^T \varepsilon_{ADM-N}(w)(t)dt \leq C(data) < \infty.$$

Proof. We begin with (3.1) from the proof of the Proposition 3.1. Using the Poincaré and Cauchy-Schwarz inequalities we have from (3.1)

$$\frac{d}{dt}E_{ADM-N}(t) + \alpha E_{ADM-N}(t) \leq ||f||_N^2,$$

for some $\alpha > 0$ which implies $E_{ADM-N}(w)(t)$ is uniformly bounded in time. For boundedness of the time averaged dissipation rate, divide the energy estimate of the ADM turbulence model energy equality from Proposition 3.1 by T :

$$\frac{1}{T}E_{ADM-N}(w)(T) + \frac{1}{T}\int_0^T \varepsilon_{ADM-N}(w)(t)dt =$$
$$= \frac{1}{T}E_{ADM-N}(w)(0) + \frac{1}{T}\int_0^T (f, w(t))_N dt \leq$$
$$\leq \frac{1}{T}E_{ADM-N}(w)(0) + ||f||_N[\frac{1}{T}\int_0^T ||w(t)||_N^2 dt]^{\frac{1}{2}} \leq \tag{3.6}$$
$$\leq C(data). \tag{3.7}$$

the result follows by letting $T \to \infty$. □

4. Bounds on energy dissipation rates. We prove the following estimate on the model's time averaged energy dissipation rates.

PROPOSITION 4.1. *For all cases* $N = 0, 1, 2, 3, \cdots$

$$< \varepsilon_{ADM-N}(w) > \leq 2\frac{U_N^3}{L_N} + \mathrm{Re}^{-1}\frac{U_N^3}{L_N}(1 + \frac{\delta^2}{L_N^2})$$

The proof in the general case follows the zeroth order case by adding subscripts as appropriate. The mathematics driving the proof in both cases is the precise estimate of the model's energy balance. In the general case, the model's kinetic energy balance satisfies the analog of the zeroth order's energy balance with norms replaced by deconvolution weighted norms. We shall give the proof in detail for the notationally clearest, $N = 0$, case and prove:

PROPOSITION 4.2. *For the case* $N = 0$,

$$< \varepsilon_{ADM-0}(w) > \leq 2\frac{U^3}{L} + \mathrm{Re}^{-1}\frac{U^3}{L}(1 + \frac{\delta^2}{L^2}) \ .$$

The proof of this estimate combines the energy estimate for the ADM in Proposition 3.1 with the breakthrough arguments of Foias[F97] and Doering and Foias[DF02] from the NSE case. The first of two key bounds is obtained by time averaging the energy inequality of Proposition 3.1. Using Corollary 3.3 we have for $N = 0, 1, 2, 3, \cdots$

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \nu||\nabla w(t)||_N^2 + \nu\delta^2||\triangle w(t)||_N^2 dt \leq \limsup_{T\to\infty}\frac{1}{T}\int_0^T (f, w(t))_N\, dt.$$

For the general case the same holds with norms and inner products replaced by their deconvolution-weighted versions (by adding subscripts N). The Cauchy-Schwarz inequality and Corollary 3.3 imply

$$< \ \varepsilon_{ADM-0} > \leq FU \ , \text{ and } < \ \varepsilon_{ADM-N} > \leq F_N U_N \tag{4.1}$$

Time averaging the ADM turbulence model (1.4) gives for $N = 0, 1, 2, \cdots$

$$\begin{aligned} &< w >_t + \nabla \cdot (\overline{< D_N w \ D_N w >}) - \nu\triangle < w > + \nabla < q > = < \overline{f} > (x), \quad (4.2) \\ &\nabla \cdot < w > = 0. \end{aligned}$$

Set $N = 0$ and recall that $D_0 = I$. Take the inner product of the time averaged model (4.2) with Af . Note that $f = f(x), (\overline{f}, Af) = (A^{-1}f, Af) = ||f||^2$, analogously for the nonlinear term, and that $\nabla \cdot f = 0$ so the pressure term vanishes. This gives

$$\frac{1}{|\Omega|}||f||^2 = \frac{1}{|\Omega|}(Af, < w >_t) - \frac{1}{|\Omega|}(\nabla f, < D_0 w \ D_0 w >) + \frac{\nu}{|\Omega|}(A\nabla f, \nabla < w >).$$

The time derivative term vanishes in the limit as $T \to \infty$ by the Cauchy-Schwarz inequality and Corollary 3.2. The remaining terms on the RHS are integrated by parts (as in the derivation of the energy equality):

$$\begin{aligned} \frac{1}{|\Omega|}||f||^2 = -\frac{1}{|\Omega|}(\nabla f, < D_0 w \ D_0 w >) + \\ + \frac{\nu}{|\Omega|}\{(\nabla f, \nabla < w >) + \delta^2(\triangle f, \triangle < w >)\}. \end{aligned}$$

Thus, using the Cauchy-Schwarz inequality,

$$\frac{1}{|\Omega|}||f||^2 \leq -\frac{1}{|\Omega|}(\nabla f, < D_0 w\ D_0 w >) + \varepsilon_{ADM-0}(f)^{\frac{1}{2}} \varepsilon_{ADM-0}(< w >)^{\frac{1}{2}}. \qquad (4.3)$$

Next, consider the nonlinear term on the above RHS. By the definitions of L, F, U we have (recall $D_0 w = w$)

$$\frac{1}{|\Omega|}(\nabla f, < w\ w >) \leq ||\nabla f||_{L^\infty} < \frac{1}{|\Omega|}||w||^2 > \leq \frac{FU^2}{L}. \qquad (4.4)$$

By the triangle inequality we have

$$||\nabla < w > ||^2 \leq < ||\nabla w||^2 > \text{ , and } ||\triangle < w > ||^2 \leq < ||\triangle w||^2 > . \qquad (4.5)$$

(This step is not sharp.) This implies, by the definitions of F, L,

$$\varepsilon_{ADM-0}(< w >)^{\frac{1}{2}} \leq < \varepsilon_{ADM-0}(w) >^{\frac{1}{2}} , \qquad (4.6)$$
$$\frac{\nu}{|\Omega|}||\nabla f||^2 \leq \nu \frac{F^2}{L^2} \text{ , and } \frac{\nu\delta^2}{|\Omega|}||\triangle f||^2 \leq \nu\delta^2 \frac{F^2}{L^4}.$$

Using the bounds (4.4), (4.6) and (4.7) in (4.3) gives

$$F^2 \leq \frac{FU^2}{L} + (\frac{\nu F^2}{L^2} + \frac{\nu\delta^2 F^2}{L^4})^{\frac{1}{2}} < \varepsilon_{ADM-0}(w) >^{\frac{1}{2}} \qquad (4.7)$$

From the first basic estimate $< \varepsilon_{ADM-0}(w) > \leq FU$. Inserting this in the RHS and cancelling the obvious terms gives

$$< \varepsilon_{ADM-0}(w) > \leq FU \leq \frac{U^3}{L} + U(\frac{\nu}{L^2} + \frac{\nu\delta^2}{L^4})^{\frac{1}{2}} < \varepsilon_{ADM-0}(w) >^{\frac{1}{2}} . \qquad (4.8)$$

Thus, by Young's inequality

$$< \varepsilon_{ADM-0}(w) > \leq 2\frac{U^3}{L} + \frac{\nu U^2}{L^2}(1 + \frac{\delta^2}{L^2}) ,$$

and Proposition 4.2 is proven:

$$< \varepsilon_{ADM-0}(w) > \leq 2\frac{U^3}{L} + \mathrm{Re}^{-1}\frac{U^3}{L}(1 + \frac{\delta^2}{L^2}) .$$

REMARK 4.1. *Exactly as in the NSE case, Foias [F97] and Doering and Foias [DF02], the estimate can be improved by more careful treatment of the quadratic equation. The result is elimination of the multiplier 2 on the RHS and a slight modification of the second term.*

5. Bounds on Helicity dissipation rates. Kinetic energy is a fundamental integral invariant of the Euler equations. The other fundamental integral invariant in 3d, discovered only in 1961 by Moreau[M61] (see also Moffatt[M84], Moffatt and Tsoniber[MT92]), is *helicity* or streamwise vorticity:

$$H(u)(t) := \frac{1}{|\Omega|}\int_\Omega u \cdot \nabla \times u dx. \qquad (5.1)$$

For the NSE, it is known[3] that the helicity satisfies the balance equation

$$H(u)(T)+\int_0^T \gamma(u)(t)dt = H(u_0)+\int_0^T \frac{1}{|\Omega|}(\nabla\times f,u)dt, \tag{5.2}$$

where $\gamma(u)$ is the helicity dissipation rate given by

$$\gamma(u) := \frac{\nu}{|\Omega|}(\nabla\times u, \nabla\times\nabla\times u). \tag{5.3}$$

The interaction of helicity and energy is thought to play a key role in organizing flows. However, much less is known about helicity than energy and its mathematical study is more difficult than that of energy because more derivatives are involved and neither H nor γ has one sign. Helical modes with both signs exist, see Ditlevsen and Giuliani [DG01a], [DG01b], and are fundamental to the analysis of helicity cascades, studied in [LMNR06], Andre and Lesieur [AL77], Brissaud, Frisch, Leorat and Lesieur [BFL73], Chen, Chen and Eyink and Holm [CCE03], [CCEH03] and Ditlevsen and Giuliani [DG01a], [DG01b]. The best current mathematical (i.e., directly from the NSE) result on helicity appears to be Foias, Hoang and Nicolenko [FHN04] in which it is proven that if the body force is potential (and can thus be incorporated into the pressure) then $H(u)(T)\to 0$ as $T\to\infty$ for $\nu>0$. Nevertheless, a similarity theory of coupled helicity and energy cascades has recently been developed for both the NSE and the family of ADM turbulence models [LMNR06]. In this theory the time averaged helicity dissipation rate plays a key role analogous to that of the time averaged energy dissipation rate.

This section considers bounds on the time averaged helicity dissipation rate for both the NSE and the ADM turbulence model (1.4). We consider the NSE case first and derive a partial result. The expected result predicted by dimensional analysis in the NSE case is recovered if it is known that the helicity of a solution of the NSE is eventually bounded-a property that seems physically obvious but mathematically intractable. Because of the enhanced kinetic energy bound available for the ADM turbulence model, we are able to prove a bound on the ADM helicity, $<\gamma_{ADM-N}(w)>$.

5.1. Helicity dissipation for the NSE. Because of the incomplete nature of the final result, we proceed formally. Dividing the helicity balance equation by T gives

$$\frac{1}{T}H(u)(T)+\frac{1}{T}\int_0^T \gamma(u)(t)dt = \frac{1}{T}H(u_0)+\frac{1}{T}\int_0^T \frac{1}{|\Omega|}(\nabla\times f,u)dt$$

If the initial velocity is smooth, $\frac{1}{T}H(u_0)\to 0$ as $T\to\infty$. Further, if $\nabla\times f$ is square integrable, then

$$\limsup_{T\to\infty}|\frac{1}{T}\int_0^T \frac{1}{|\Omega|}(\nabla\times f,u)dt|<\infty.$$

Thus, the following limit superiors satisfy

$$\limsup_{T\to\infty}|\frac{1}{T}H(u)(T)+\frac{1}{T}\int_0^T \gamma(u)(t)dt| \le \limsup_{T\to\infty}|\frac{1}{T}\int_0^T \frac{1}{|\Omega|}(\nabla\times f,u)dt|<\infty.$$

[3] Derived formally by multiplication by the vorticity, integration over the flow domain and integration by parts.

By the Cauchy-Schwarz inequality

$$\limsup_{T\to\infty} |\frac{1}{T}\int_0^T \frac{1}{|\Omega|}(\nabla\times f, u)dt| < \frac{FU}{L}(<\infty). \tag{5.4}$$

In the NSE case, Foias[F97] and Doering and Foias[DF02] prove, as a step to bounds on energy dissipation rates, the following intermediate result on the numerator of the RHS:

$$FU \leq \frac{U^3}{L} + \nu^{\frac{1}{2}}\frac{U}{L} < \varepsilon >^{\frac{1}{2}} . \tag{5.5}$$

(For example, formally set $\delta = 0$ in the estimate (4.9).) Using this and the bound of energy dissipation of Foias[F97] and Doering and Foias[DF02], $<\varepsilon>\leq 2\frac{U^3}{L} + \mathrm{Re}^{-1}\frac{U^3}{L}$, in (5.5) gives

$$\limsup_{T\to\infty} |\frac{1}{T}H(u)(T) + \frac{1}{T}\int_0^T \gamma(u)(t)dt| \leq \frac{U^3}{L^2} + \sqrt{2}\,\mathrm{Re}^{-\frac{1}{2}}\frac{U^3}{L^2} + \mathrm{Re}^{-1}\frac{U^3}{L^2}. \tag{5.6}$$

If $H(T)$ is eventually bounded (as is expected physically but unknown mathematically) this gives the upper bound

$$| < \gamma > | \leq \frac{U^3}{L^2} + \sqrt{2}\,\mathrm{Re}^{-\frac{1}{2}}\frac{U^3}{L^2} + \mathrm{Re}^{-1}\frac{U^3}{L^2}. \tag{5.7}$$

Thus, it is clear that obtaining a bound on the helicity dissipation, i.e., completing the step between (5.6) and (5.7) depends on proving boundedness $H(T)$.

5.2. Helicity dissipation in ADM turbulence models. From the NSE case, it is clear that the key to completing the argument of Section 5.1 for the ADM turbulence model will be proving $H_{ADM-N}(w)(T)$ is bounded. We begin by recalling the ADM turbulence model's helicity balance (the essential first ingredient in the analysis) discovered by Rebholz [R06]. Define the conserved, Rebholz [R06], ADM helicity

$$H_{ADM-N}(w)(t) := \frac{1}{|\Omega|}[(w, \nabla\times w)_N + \delta^2(\nabla\times w, \nabla\times\nabla\times w)_N]. \tag{5.8}$$

For the ADM, it is known, Rebholz [R06], that the ADM helicity satisfies the balance equation

$$H_{ADM-N}(w)(T) + \int_0^T \gamma_{ADM-N}(w)(t)dt = H(w(0)) + \int_0^T \frac{1}{|\Omega|}(\nabla\times f, w)dt, \tag{5.9}$$

where $\gamma_{ADM-N}(w)$ is the model's helicity dissipation rate given by

$$\gamma_{ADM-N}(w) := \frac{\nu}{|\Omega|}[(\nabla\times w, \nabla\times\nabla\times w)_N + \delta^2(\nabla\times\nabla\times w, \nabla\times\nabla\times\nabla\times w)_N]. \tag{5.10}$$

Note that (in the zeroth order model-to simplify notation)

$$H_{ADM}(w) = H(w) + \delta^2 H(\nabla\times w) \text{ and}$$
$$\gamma_{ADM}(w) = \gamma(w) + \delta^2\gamma(\nabla\times w) .$$

5.3. Bounding model helicity. Arguing as in [LN05], the zeroth order model, (1.4) with $N = 0$, is equivalent to $\nabla \cdot Aw = 0$ and

$$Aw_t + w \cdot \nabla w + \nabla Aq - \nu \triangle Aw = f. \tag{5.11}$$

LEMMA 5.1. *Suppose $\delta > 0$ then*

$$\sup_{0 \le T < \infty} [||w(T)|| + ||\nabla w(T)|| + ||\triangle w(T)||] \le C(data, \delta) < \infty. \tag{5.12}$$

Proof. First we note that it has been proven that the ADM turbulence model (1.4) has a unique strong solution that is as smooth as the problem data so formal manipulations of the model are mathematically justified. The bound on w and ∇w follow from the energy inequality for the model in Proposition 3.1. Taking the inner product of (5.11) with Aw gives

$$\frac{1}{2}\frac{d}{dt}||Aw||^2 + \nu||\nabla Aw||^2 = (Af, w) - (w \cdot \nabla w.Aw).$$

Basic inequalities and the bounds on w and ∇w give

$$|(Af, w)| \le C(data),$$
$$|(w \cdot \nabla w, Aw)| \le C||\nabla w||^2||\nabla Aw||^2 \le C(data, \delta) + \frac{\nu}{2}||\nabla Aw||^2.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}||Aw||^2 + \nu||\nabla Aw||^2 \le C(data, \delta) + \frac{\nu}{2}||\nabla Aw||^2,$$

and the result follows by a Poincaré type inequality and an integrating factor. ∎

COROLLARY 5.2. *If $\delta > 0$ then $|H_{ADM-N}(w)(T)|$ is uniformly bounded.*

Proof. There follows

$$|H_{ADM-N}(w)(T)| \le C(||w||||\nabla w|| + \delta^2||\nabla w||||\triangle w||) \le C(data, \delta).$$

∎

With these bounds, the final step lacking in the argument from Section 5.1 in the NSE case can be carried through successfully.

PROPOSITION 5.3. *Let $\delta > 0$, then*

$$| < \gamma_{ADM}(w) > | \le \frac{U^3}{L^2} + \sqrt{2}\,\mathrm{Re}^{-\frac{1}{2}}(1 + \frac{\delta^2}{L^2})^{\frac{1}{2}}\frac{U^3}{L^2} + \mathrm{Re}^{-1}(1 + \frac{\delta^2}{L^2})\frac{U^3}{L^2}$$

Proof. Time averaging the ADM turbulence model helicity balance relation, both helicity terms drop out by Corollary 5.2. Thus we have

$$\lim\sup_{T\to\infty} |\frac{1}{T}\int_0^T \gamma_{ADM}(w)(t)dt| \le \lim\sup_{T\to\infty} |\frac{1}{T}\int_0^T \frac{1}{|\Omega|}(\nabla \times f, u)dt| \le \frac{FU}{L}.$$

Inserting the bounds on FU and $< \varepsilon_{ADM}(w) >$ from Section 4 gives

$$| < \gamma_{ADM}(w) > | \le \frac{FU}{L} \le \frac{U^3}{L^2} + \frac{U}{L}(\frac{\nu}{L^2} + \frac{\nu\delta^2}{L^4})^{\frac{1}{2}} < \varepsilon_{ADM-0}(w) >^{\frac{1}{2}} \le$$
$$\le \frac{U^3}{L^2} + \mathrm{Re}^{-\frac{1}{2}}(\frac{U}{L})^{\frac{3}{2}}(1 + \delta^2)^{\frac{1}{2}}[2\frac{U^3}{L} + \mathrm{Re}^{-1}\frac{U^3}{L}(1 + \frac{\delta^2}{L^2})]^{\frac{1}{2}} \le$$
$$\le \frac{U^3}{L^2} + \sqrt{2}\,\mathrm{Re}^{-\frac{1}{2}}(1 + \delta^2)^{\frac{1}{2}}\frac{U^3}{L^2} + \mathrm{Re}^{-1}(1 + \delta^2)\frac{U^3}{L^2},$$

as claimed. ∎

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