New Discretizations of Turbulent Flow Problems

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Abstract

A suitable discretization for the Zeroth Order Model in Large Eddy Simulation of turbulent flows is sought. This is a low order model, but its importance lies in the insight that it provides for the analysis of the higher order models actually used in practice by the pioneers Stolz and Adams [1, 2] and others. The higher order models have proven to be of high accuracy. However, stable discretizations of them have proven to be tricky and other stabilizations, such as time relaxation and eddy viscosity, are often added. We propose a discretization based on a mixed variational formulation that gives the correct energy balance. We show it to be unconditionally stable and prove convergence.

1 Introduction

In this report, we consider a new discretization of the Approximate Deconvolution Models in Large Eddy Simulation (LES), focusing on the pivotal zeroth order model. The usual finite element approach was already considered in [15] and its stability was proven to be dependent on the exact way the filtering operation is performed. The discretization we propose here grows out of the natural formulation for the continuous model, i.e. it comes directly from a formulation that gives the correct energy balance for the large scales. It is inspired in the technique used in [12] to prove existence and uniqueness of strong solutions in the continuous case. In contrast to the approach of [15], it is less sensitive to the details of the filter, but its implementation introduces more degrees of freedom. After all, Ferziger [5], "... there is a close connection between the numerical methods and the modeling approach used in simulation; this connection has not been sufficiently appreciated ...." We prove that the new discretization is stable and give optimal convergence rates, including an analysis of time averaged errors.

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We are interested in designing a numerical method for approximating flow averages of flows at higher Reynolds number subject to the no-slip boundary condition, described by the incompressible Navier-Stokes equations (NSE)

$$\begin{aligned}
\mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } (0,T] \times \Omega, \\
\nabla \cdot \mathbf{u} &= 0 && \text{in } [0,T] \times \Omega, \\
\mathbf{u} &= \mathbf{0} && \text{in } [0,T] \times \partial\Omega, \\
\mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega \\
\textstyle\int_\Omega p \, d\mathbf{x} &= 0,
\end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the fluid velocity, p the pressure, $\mathbf{f}$ the external force, $\nu > 0$ the kinematic viscosity, and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ a bounded, simply connected domain with polygonal boundary $\partial\Omega$.

Let overbar denote a spacial averaging operator which preserves the no-slip condition and let $\delta > 0$ denote the averaging radius. Following [13], we define the filtering operation as the solution of a shifted Poisson problem. The differential filter $\overline{\boldsymbol{\phi}}$ of a quantity $\boldsymbol{\phi}$ is the solution to the boundary value problem

$$\begin{aligned}
-\delta^2 \Delta \overline{\boldsymbol{\phi}} + \overline{\boldsymbol{\phi}} &= \boldsymbol{\phi} && \text{in } \Omega, \\
\overline{\boldsymbol{\phi}} &= 0 && \text{on } \partial\Omega.
\end{aligned} \tag{1.2}$$

Applying this filtering operation to the Navier-Stokes equations (1.1) results

$$\begin{aligned}
\overline{\mathbf{u}}_t + \overline{\nabla \cdot (\mathbf{u}\mathbf{u})} - \nu \overline{\Delta \mathbf{u}} + \overline{\nabla p} &= \mathbf{f} && \text{in } (0,T] \times \Omega, \\
\overline{\nabla \cdot \mathbf{u}} &= 0 && \text{in } [0,T] \times \Omega, \\
\overline{\mathbf{u}} &= \mathbf{0} && \text{in } [0,T] \times \partial\Omega, \\
\overline{\mathbf{u}}(0,\mathbf{x}) &= \overline{\mathbf{u}}_0(\mathbf{x}) && \text{in } \Omega \\
\textstyle\int_\Omega p \, d\mathbf{x} &= 0,
\end{aligned} \tag{1.3}$$

The system of equations (1.3) is not closed, suggesting that $\mathbf{u}$ must be modeled in terms of $\overline{\mathbf{u}}$. We choose here the simplest closure model, $\mathbf{u} \simeq \overline{\mathbf{u}}(+O(\delta^2))$, known as the Zeroth Order Model because it is exact on constant flows. Since we are interested in non periodic flow, an $O(\delta^2)$ commutation error is introduced in the incompressibility constraint. Letting $\mathbf{w}$ be an approximation to $\overline{\mathbf{u}}$, and imposing $\nabla \cdot \mathbf{w} = 0$, system (1.3) becomes

$$\begin{aligned}
\mathbf{w}_t - \nu \Delta \mathbf{w} + \overline{\nabla \cdot (\mathbf{w}\mathbf{w})} + \overline{\nabla p} &= \overline{\mathbf{f}} && \text{in } (0,T] \times \Omega, \\
\nabla \cdot \mathbf{w} &= 0 && \text{in } [0,T] \times \Omega, \\
\mathbf{w} &= \mathbf{0} && \text{in } [0,T] \times \partial\Omega. \\
\mathbf{w}(0,\mathbf{x}) &= \overline{\mathbf{u}}_0(\mathbf{x}) && \text{in } \Omega
\end{aligned} \tag{1.4}$$

This model has been extensively studied from an analytical point of view in the case of periodic boundary conditions. In [11] it is shown that weak solutions exist and that it is stable. In [12] it was proven that strong solutions exist and are unique, the modeling error was bounded and convergence as $\delta \to 0$ to a solution of the NSE is proven. The Zeroth Order Model is the simplest model in the family of Approximate Deconvolution Models (ADM) introduced by Stolz and Adams [1, 2]. Despite being a low order model, it is the key in understanding mathematically how the higher order models in the family behave. The methods used in [11, 12] were extended for the whole family in [4] to prove an energy inequality, existence, uniqueness and regularity of strong solutions and also to give a rigorous bound on the modeling error.

Another remarkable property of this family of models, including (1.4), is that their time averaged consistency error converges to zero uniformly in the Reynolds number as $O(\delta^{1/3})$ [14].

Here, we were inspired by the idea in [12], in which the variational formulation is in $H^2(\Omega)$ (see Section 3). This would be computationally expensive, requiring the use of C^1 elements. Instead, we study a mixed formulation that requires less regularity of the true solution $\mathbf{w}$. The error analysis is performed and optimal convergence rates are derived. We also include a section on time averaged errors, since this method is designed for simulation of turbulent flows. In such cases, the usual procedure is to compute time averages of the physical quantities of interest. [16, 3].

The report is organized as follows. In Section 2, we introduce notation and give some preliminaries. The derivation of the discretization and its stability properties are explained in Section 3. Optimal convergence rates are derived in Section 4, with the help of a modified Stokes projection. In Section 5, time averaged errors are analyzed and finally, some conclusions and remarks are presented in Section 6.

2 Notation and Preliminaries

We now introduce the notation for the functional settings. The inner product and norm in $(L^2(\Omega))^d$, $d = 2, 3$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. The norm in $(H^k(\Omega))^d$ is denoted by $\|\cdot\|_k$ and the norms in Lebesgue spaces $(L^p(\Omega))^d$, $1 \leq p < \infty$, $p \neq 2$ by $\|\cdot\|_{L^p}$. The velocity and pressure spaces are $\mathbf{X} = (H^1_0(\Omega))^d$ and $Q = L^2_0(\Omega)$, respectively. For $\mathbf{f}$ an element in the dual space of $\mathbf{X}$, its norm is defined by

$$||\mathbf{f}\,||_{-1} = \sup_{\mathbf{v}\in\mathbf{X}} \frac{|(\mathbf{f},\mathbf{v})\,|}{||\mathbf{v}\,||_1}.$$

The space of divergence-free functions is defined as

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{X} : (q, \nabla\cdot\mathbf{v}) = 0, \forall\, q \in Q\}\,.$$

The following trilinear form,

$$b(\mathbf{w},\mathbf{u},\mathbf{v}) = \frac{1}{2}\left(((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v}) - ((\mathbf{w}\cdot\nabla)\mathbf{v},\mathbf{u})\right),$$

is the skew-symmetric form of the convective term. We often use the following properties:

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}) \qquad \text{and} \qquad b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \quad \forall\, \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{X}.$$

Furthermore (see [6] for a proof), if $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, there exists a constant $M = M(\Omega) < \infty$ such that

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \leq M \,||\, \nabla \mathbf{u} \,||\, ||\, \nabla \mathbf{v} \,||\, ||\, \nabla \mathbf{w} \,||, \qquad \forall\, \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{X}. \tag{2.1}$$

Particularly, when $d = 3$, this can be improved to

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \leq M \sqrt{||\, \mathbf{u} \,||\, ||\, \nabla \mathbf{u} \,||}\, ||\, \nabla \mathbf{v} \,||\, ||\, \nabla \mathbf{w} \,||, \qquad \forall\, \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{X}. \tag{2.2}$$

The discrete analogue of the model begins with constructing conforming finite element spaces $\mathbf{X}^h \subset \mathbf{X}$, $Q^h \subset Q$ where h denotes mesh width for $(\mathbf{X}^h, Q^h)$. These spaces satisfy the usual approximation theoretic conditions and the inf-sup condition or Babuska-Brezzi condition i.e. there is a constant β independent of mesh size h such that

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in \mathbf{X}^h} \frac{(q^h,\, \nabla \cdot \mathbf{v}^h)}{||\, \nabla \mathbf{v}^h \,||\, ||\, q^h \,||} \geq \beta > 0. \tag{2.3}$$

For examples of such compatible spaces see e.g., Gunzburger [7], Girault and Raviart [6]. The space of discretely divergence free functions is defined by

$$\mathbf{V}^h = \left\{ \mathbf{v} \in \mathbf{X}^h : (\nabla \cdot \mathbf{v}^h, q^h) = 0 \; \forall\, q^h \in Q^h \right\}$$

which is a nontrivial closed subspace of $\mathbf{X}^h$ under the inf-sup condition (2.3). It is known that even if typically $\mathbf{V}^h \not\subseteq \mathbf{V}$, under (2.3), the functions in $\mathbf{V}$ are well approximated by ones in $\mathbf{V}^h$ [7, 6].

In the analysis, we often use the following inequalities:

Young's Inequality:

$$ab \leq \frac{\epsilon}{q} a^q + \frac{\epsilon^{-q/p}}{p} b^p, \quad 1 < q,\, p < \infty, \quad \frac{1}{q} + \frac{1}{p} = 1.$$

Poincaré's Inequality:

$$\|\mathbf{v}\| \leq C_{PF} \|\nabla \mathbf{v}\| \quad \forall\, \mathbf{v} \in \mathbf{X},$$

where C_{PF} is a constant depending on Ω.

Time averages are denoted by $< \cdot >$; for example, the time average of $\boldsymbol{\psi}$ is

$$< \boldsymbol{\psi} > = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \boldsymbol{\psi}(t)\, dt.$$

Time averages satisfy a Cauchy-Schwarz type of inequality [10]:

$$< (\boldsymbol{\psi}, \boldsymbol{\chi}) > \leq < ||\, \boldsymbol{\psi} \,||^2 >^{1/2} < ||\, \boldsymbol{\chi} \,||^2 >^{1/2}.$$

3 Derivation of the New Discretization

This section develops a mixed variational formulation for (1.4) and its finite element discretization. We recall that the operations of differentiation and filtering do not commute and use a strategy that gives the correct balance of energy for the model. The stability of the discrete solution is also investigated.

By choosing a differential filter as an averaging operator, following the discussion in [12], we define $A\mathbf{v} = -\delta^2\Delta\mathbf{v} + \mathbf{v}$, for all $\mathbf{v} \in \mathbf{X} \cap (H^2(\Omega))^d$, so that $A\overline{\boldsymbol{\phi}} = \boldsymbol{\phi}$. Note that since the Laplace operator Δ is self-adjoint, so is A.

Let $\mathbf{w}$ be a smooth strong solution of (1.4). The development of the model starts with multiplying (1.4) by $A\mathbf{v}$ and integrating over the domain. One has

$$(\mathbf{w}_t, A\mathbf{v}) - \nu(\Delta\mathbf{w}, A\mathbf{v}) + (\overline{\nabla\cdot(\mathbf{w}\,\mathbf{w})}, A\mathbf{v}) + (\overline{\nabla p}, A\mathbf{v}) = (\overline{\mathbf{f}}, A\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X} \cap (H^2(\Omega))^d.$$

By using the self-adjointness of the operator A together with property (1.2), followed by integration by parts, we derive the following variational formulation: Find $\mathbf{w} : [0,T] \to \mathbf{X} \cap (H^2(\Omega))^d$, $p : (0,T] \to Q$ satisfying $\mathbf{w}(0,\mathbf{x}) = \overline{\mathbf{u}}_0(\mathbf{x})$ and

$$\begin{aligned} (\mathbf{w}_t, \mathbf{v}) + \delta^2(\nabla\mathbf{w}_t, \nabla\mathbf{v}) \quad + \quad & \nu[(\nabla\mathbf{w}, \nabla\mathbf{v}) + \delta^2(\Delta\mathbf{w}, \Delta\mathbf{v})] \\ & +(\nabla\cdot(\mathbf{w}\,\mathbf{w}), \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ & (\nabla\cdot\mathbf{w}, q) = 0, \end{aligned} \tag{3.1}$$

for all $(\mathbf{v}, q) \in (\mathbf{X} \cap (H^2(\Omega))^d, Q)$.

No similar formulation follows by the usual approach of multiplying by $\mathbf{v}$ and integrating by parts. The formulation (3.1) contains the term $(\Delta\mathbf{w}, \Delta\mathbf{v})$ which is a fourth order term. This suggests using C^1 elements. Instead, we consider a mixed formulation of (3.1): Find $\mathbf{w} : [0,T] \to \mathbf{X}, \boldsymbol{\phi} : [0,T] \to \mathbf{X}$ and $p : (0,T] \to Q$ satisfying $\mathbf{w}(0,\mathbf{x}) = \overline{\mathbf{u}}_0(\mathbf{x})$ and:

$$(\mathbf{w}_t, \mathbf{v}) + \delta^2(\nabla\mathbf{w}_t, \nabla\mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v})$$

$$+\nu(\nabla\mathbf{w}, \nabla\mathbf{v}) + \nu\delta^2(\nabla\boldsymbol{\phi}, \nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \tag{3.2}$$

$$(\nabla\mathbf{w}, \nabla\boldsymbol{\xi}) = (\boldsymbol{\phi}, \boldsymbol{\xi}), \tag{3.3}$$

$$(\nabla\cdot\mathbf{w}, q) = 0, \tag{3.4}$$

for all $(\mathbf{v}, \boldsymbol{\xi}, q) \in (\mathbf{X}, \mathbf{X}, Q)$.

In $\mathbf{V}$, this formulation becomes: Find $(\mathbf{w}, \boldsymbol{\phi}) : [0,T] \to (\mathbf{V}, \mathbf{X})$ such that

$$(\mathbf{w}_t, \mathbf{v}) + \delta^2(\nabla\mathbf{w}_t, \nabla\mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v})$$

$$+\nu(\nabla\mathbf{w}, \nabla\mathbf{v}) + \nu\delta^2(\nabla\boldsymbol{\phi}, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \tag{3.5}$$

$$(\nabla\mathbf{w}, \nabla\boldsymbol{\xi}) = (\boldsymbol{\phi}, \boldsymbol{\xi}). \tag{3.6}$$

for all $(\mathbf{v}, \boldsymbol{\xi}) \in (\mathbf{V}, \mathbf{X})$.

The kinetic energy and the energy dissipation rate at time t associated with this model are defined as

$$\kappa(\mathbf{w}) = \frac{1}{2}\left(||\,\mathbf{w}(t)\,||^2 + \delta^2\,||\,\nabla \mathbf{w}(t)\,||^2\right) \qquad \text{and} \qquad \varepsilon(\mathbf{w}, \boldsymbol{\phi}) = \frac{\nu}{|\,\Omega\,|}\left(||\,\nabla \mathbf{w}(t)\,||^2 + \delta^2\,||\,\boldsymbol{\phi}(t)\,||^2\right),$$

where $|\,\Omega\,|$ is the measure of Ω.

We first establish uniformly boundness of the kinetic energy of $\mathbf{w}$ at time T.

Lemma 3.1. *Let $\mathbf{f} \in L^\infty(0, \infty, H^{-1}(\Omega))$. Then kinetic the energy $\kappa(\mathbf{w})$ at time T is uniformly bounded as*

$$\begin{aligned}\kappa(\mathbf{w}) \;\; &\le \;\; (||\,\mathbf{w}(0)\,||^2 + \delta^2\,||\,\nabla \mathbf{w}(0)\,||^2)\, e^{-\nu\, C_{PF}^{-2} T} \\ &\quad + \nu^{-2} C_{PF}^2 ||\,\mathbf{f}\,||^2_{L^\infty(0,\infty;H^{-1}(\Omega))}.\end{aligned}$$

In particular,

$$\lim_{T\to\infty} \frac{1}{T}\kappa(\mathbf{w}) = 0.$$

Proof. Set $\mathbf{v} = \mathbf{w}$ in (3.5) and $\boldsymbol{\xi} = \boldsymbol{\phi}$ in (3.6). Then, since $b(\mathbf{w}, \mathbf{w}, \mathbf{w}) = 0$, we get

$$\frac{1}{2}\frac{d}{dt}\left(||\,\mathbf{w}\,||^2 + \delta^2||\,\nabla \mathbf{w}\,||^2\right) + \nu(||\,\nabla \mathbf{w}\,||^2 + 2\,\delta^2||\,\boldsymbol{\phi}\,||^2) \le \frac{1}{2\,\nu}||\,\mathbf{f}\,||^2_{-1}. \tag{3.7}$$

Using Poincaré's inequality in (3.6) yields $||\,\boldsymbol{\phi}\,|| \ge C_{PF}^{-1}||\,\nabla \mathbf{w}\,||$, then (3.7) becomes

$$\frac{d}{dt}\left(||\,\mathbf{w}\,||^2 + \delta^2||\,\nabla \mathbf{w}\,||^2\right) + \nu\, C_{PF}^{-2}(||\,\mathbf{w}\,||^2 + \delta^2||\,\nabla \mathbf{w}\,||^2) \le \nu^{-1}||\,\mathbf{f}\,||^2_{-1}. \tag{3.8}$$

Setting $y = ||\,\mathbf{w}\,||^2 + \delta^2||\,\nabla \mathbf{w}\,||^2$ and using again an integrating factor, equation (3.8) gives

$$y(T) \le y(0)\, e^{-\nu\, C_{PF}^{-2} T} + \nu^{-2} C_{PF}^2\,||\,\mathbf{f}\,||^2_{L^\infty(0,T;H^{-1}(\Omega))}.$$

This proves the uniform boundedness. Now, diving by T and taking the limit as $T \to \infty$ gives the second claim. □

Remark 3.1. *One can also show that the total energy is bounded. We only present the proof for the discrete case (Lemma 3.3), since the idea is the same in both cases.*

Our goal is to understand the behavior of numerical methods based on (3.2)-(3.4). Therefore, we consider a continuous-in time finite element discretization of the problem (3.2)-(3.4). Let $\mathbf{X}^h \subset \mathbf{X}$ and $Q^h \subset Q$ satisfy (2.3). The finite element approximation to $(\mathbf{w}, \boldsymbol{\phi}, p)$ are maps $\mathbf{w}^h : [0, T] \to$

$\mathbf{X}^h, \boldsymbol{\phi}^h : [0,T] \to \mathbf{X}^h$ and $p^h : (0,T] \to Q^h$ such that

$$
\begin{aligned}
&(\mathbf{w}_t^h, \mathbf{v}^h) + \delta^2(\nabla \mathbf{w}_t^h, \nabla \mathbf{v}^h) + b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) \\
&+\nu(\nabla \mathbf{w}^h, \nabla \mathbf{v}^h) + \nu\delta^2(\nabla \boldsymbol{\phi}^h, \nabla \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h), \qquad (3.9) \\
&(\nabla \mathbf{w}^h, \nabla \boldsymbol{\xi}^h) = (\boldsymbol{\phi}^h, \boldsymbol{\xi}^h), \qquad (3.10) \\
&(q^h, \nabla \cdot \mathbf{w}^h) = 0, \qquad (3.11)
\end{aligned}
$$

for all $(\mathbf{v}^h, \boldsymbol{\xi}^h, q^h) \in (\mathbf{X}^h, \mathbf{X}^h, Q^h)$.

In $\mathbf{V}^h$, the semi-discrete approximation of (3.9)-(3.11) is: Find $(\mathbf{w}^h, \boldsymbol{\phi}^h) \in (\mathbf{V}^h, \mathbf{X}^h)$ such that

$$
\begin{aligned}
&(\mathbf{w}_t^h, \mathbf{v}^h) + \delta^2(\nabla \mathbf{w}_t^h, \nabla \mathbf{v}^h) + b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) \\
&+\nu(\nabla \mathbf{w}^h, \nabla \mathbf{v}^h) + \nu\delta^2(\nabla \boldsymbol{\phi}^h, \nabla \mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h), \qquad (3.12) \\
&(\nabla \mathbf{w}^h, \nabla \boldsymbol{\xi}^h) = (\boldsymbol{\phi}^h, \boldsymbol{\xi}^h), \qquad (3.13)
\end{aligned}
$$

for all $(\mathbf{v}^h, \boldsymbol{\xi}^h) \in (\mathbf{V}^h, \mathbf{X}^h)$.

A discrete version of Lemma 3.1 shows that the kinetic energy of the discrete solution is also uniformly bounded.

Lemma 3.2. *Let $\mathbf{f} \in L^\infty(0,\infty, H^{-1}(\Omega))$. Then the kinetic energy $\kappa(\mathbf{w}^h)$ is uniformly bounded as*

$$
\begin{aligned}
\kappa(\mathbf{w}^h) \quad \leq \quad & (||\, \mathbf{w}^h(0)\,||^2 + \delta^2\, ||\, \nabla \mathbf{w}^h(0)\,||^2)\, e^{-\nu\, C_{PF}^{-2} T} \\
& +\nu^{-2} C_{PF}^2 ||\, \mathbf{f}\,||^2_{L^\infty(0,\infty;H^{-1}(\Omega))}.
\end{aligned}
$$

As a consequence,

$$
\lim_{T\to\infty} \frac{1}{T}\kappa(\mathbf{w}^h) = 0.
$$

Proof. The claim exactly follows as in the continuous case of Lemma 3.1. □

In addition, the next result shows that the total energy of the approximate solution $\mathbf{w}^h$ is bounded.

Lemma 3.3. *(Stability of $\mathbf{w}^h$) Let $\mathbf{f} \in L^2(0,T,H^{-1}(\Omega))$. Then any solution $(\mathbf{w}^h, \boldsymbol{\phi}^h)$ of (3.12)-(3.13) satisfies the following stability bound:*

$$
\begin{aligned}
\frac{1}{2}\|\mathbf{w}^h(t)\|^2 + \frac{\delta^2}{2}\|\nabla \mathbf{w}^h(t)\|^2 + \int_0^t [\frac{\nu}{2}\|\nabla \mathbf{w}^h\|^2 + \nu\delta^2\|\boldsymbol{\phi}^h\|^2]dt' & \\
\leq \frac{1}{2}\|\mathbf{w}^h(0)\|^2 + \frac{\delta^2}{2}\|\nabla \mathbf{w}^h(0)\|^2 + \frac{1}{2\nu}\int_0^t \|\mathbf{f}\|_{-1}^2 dt'. &
\end{aligned}
$$

Proof. Set $\mathbf{v}^h = \mathbf{w}^h$ in (3.12), $\boldsymbol{\xi}^h = \boldsymbol{\phi}^h$ in (3.13) and use $b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{w}^h) = 0$, we get:

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}^h\|^2 + \delta^2\|\nabla\mathbf{w}^h\|^2\right) + \nu\|\nabla\mathbf{w}^h\|^2 + \nu\delta^2(\nabla\boldsymbol{\phi}^h, \nabla\mathbf{w}^h) = (\mathbf{f}, \mathbf{w}^h) \tag{3.14}$$

$$(\nabla\mathbf{w}^h, \nabla\boldsymbol{\phi}^h) = (\boldsymbol{\phi}^h, \boldsymbol{\phi}^h). \tag{3.15}$$

Multiply (3.15) by $\nu\delta^2$ and add to (3.14) and use Cauchy Schwarz inequality, one has

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}^h\|^2 + \delta^2\|\nabla\mathbf{w}^h\|^2\right) + \frac{\nu}{2}\|\nabla\mathbf{w}^h\|^2 + \nu\delta^2\|\boldsymbol{\phi}^h\|^2 \leq \frac{1}{2\nu}\|\mathbf{f}\|_{-1}^2.$$

Integrating the last equation over $(0, t)$ with $t \leq T$ gives the required result. □

4 Convergence Analysis

It is useful to define the following modified Stokes projection aiming at simplifying the error analysis. In the analysis, it is useful to have a clear description of the Stokes projection.

Definition 4.1. *(Modified Stokes Projection) The Stokes projection operator $P_S : (\mathbf{X}, \mathbf{X}, Q) \to (\mathbf{X}^h, \mathbf{X}^h, Q^h)$ is defined as follows: Let $P_S(\mathbf{w}, \boldsymbol{\phi}, p) = (\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}, \tilde{p})$ where $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}, \tilde{p})$ satisfies*

$$\begin{aligned}
\nu(\nabla(\mathbf{w} - \tilde{\mathbf{w}}), \nabla\mathbf{v}^h) + \nu\delta^2(\nabla(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}), \nabla\mathbf{v}^h) - (p - \tilde{p}, \nabla\cdot\mathbf{v}^h) &= 0,\\
(\nabla(\mathbf{w} - \tilde{\mathbf{w}}), \nabla\boldsymbol{\xi}^h) &= (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}, \boldsymbol{\xi}^h),\\
(q^h, \nabla\cdot(\mathbf{w} - \tilde{\mathbf{w}})) &= 0,
\end{aligned} \tag{4.1}$$

for all $(\mathbf{v}^h, \boldsymbol{\xi}^h, q^h) \in (\mathbf{X}^h, \mathbf{X}^h, Q^h)$.

In $(\mathbf{V}^h, \mathbf{X}^h)$, this formulation reads: Given $(\mathbf{w}, \boldsymbol{\phi})$, find $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}) \in (\mathbf{V}^h, \mathbf{X}^h)$ satisfying

$$\nu(\nabla(\mathbf{w} - \tilde{\mathbf{w}}), \nabla\mathbf{v}^h) + \nu\delta^2(\nabla(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}), \nabla\mathbf{v}^h) - (p - q^h, \nabla\cdot\mathbf{v}^h) = 0, \tag{4.2}$$

$$(\nabla(\mathbf{w} - \tilde{\mathbf{w}}), \nabla\boldsymbol{\xi}^h) = (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}, \boldsymbol{\xi}^h), \tag{4.3}$$

for all $(\mathbf{v}^h, \boldsymbol{\xi}^h) \in (\mathbf{V}^h, \mathbf{X}^h)$ and any $q^h \in Q^h$.

Under the discrete inf-sup condition (2.3), $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}, \tilde{p})$ is a quasi optimal approximation of $(\mathbf{w}, \boldsymbol{\phi}, p)$. Since the stability and error estimation of the projection operator will be used to approximate the error between $\mathbf{w}$ and $\mathbf{w}^h$, we now give two related results.

Proposition 4.1. *(Stability of the Stokes projection) Let $(\mathbf{w}, \boldsymbol{\phi})$ be given. Then, $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ satisfies*

$$\nu\|\nabla\tilde{\mathbf{w}}\|^2 + 2\nu\delta^2\|\tilde{\boldsymbol{\phi}}\|^2 \leq C\{(\nu + \nu\delta^2 h^{-2})\|\nabla\mathbf{w}\|^2 + \nu\delta^4\|\nabla\boldsymbol{\phi}\|^2 + \nu\delta^2\|\boldsymbol{\phi}\|^2 + \nu^{-1}\|p - q^h\|^2\},$$

where C is independent of ν, δ and h.

Proof. We first set $\mathbf{v}^h = \tilde{\mathbf{w}}$ in (4.2) and $\boldsymbol{\xi}^h = \tilde{\boldsymbol{\phi}}$ in (4.3). Then, we obtain

$$\nu\|\nabla\tilde{\mathbf{w}}\|^2 = \nu(\nabla\mathbf{w},\nabla\tilde{\mathbf{w}}) + \nu\delta^2(\nabla(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}),\nabla\tilde{\mathbf{w}})) - (p-q^h,\nabla\cdot\tilde{\mathbf{w}}), \tag{4.4}$$

$$(\nabla\tilde{\mathbf{w}},\nabla\tilde{\boldsymbol{\phi}}) = (\nabla\mathbf{w},\nabla\tilde{\boldsymbol{\phi}}) + (\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi},\tilde{\boldsymbol{\phi}}). \tag{4.5}$$

Multiplying (4.5) by $\nu\delta^2$, substituting in (4.4) and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}\nu\|\nabla\tilde{\mathbf{w}}\|^2 + \nu\delta^2\|\tilde{\boldsymbol{\phi}}\|^2 &\le \nu\|\nabla\mathbf{w}\|\|\nabla\tilde{\mathbf{w}}\| + \nu\delta^2\|\nabla\boldsymbol{\phi}\|\|\nabla\tilde{\mathbf{w}}\| \\ &\quad +\nu\delta^2\|\nabla\mathbf{w}\|\|\nabla\tilde{\boldsymbol{\phi}}\| + \nu\delta^2\|\boldsymbol{\phi}\|\|\tilde{\boldsymbol{\phi}}\| + \|p-q^h\|\|\nabla\cdot\tilde{\mathbf{w}}\|.\end{aligned}$$

We apply the following inverse inequality

$$\|\nabla\tilde{\boldsymbol{\phi}}\| \le C\,h^{-1}\|\tilde{\boldsymbol{\phi}}\|,$$

and Young's inequality to obtain the final result. □

Proposition 4.2. *(Error in the Stokes projection) Suppose the discrete inf-sup condition (2.3) holds. Then,* $(\tilde{\mathbf{w}},\tilde{\boldsymbol{\phi}},\tilde{p})$ *exists uniquely in* $(\mathbf{X}^h,\mathbf{X}^h,Q^h)$ *and satisfies*

$$\begin{aligned}\nu\|\nabla(\mathbf{w}-\tilde{\mathbf{w}})\|^2 + 2\nu\delta^2\|\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}\|^2 \le C \inf_{\substack{\tilde{\mathbf{v}^h}\in\mathbf{X}^h\\ \boldsymbol{\phi}^*\in\mathbf{X}^h}} \Big\{&(\nu+\nu\delta^2h^{-2})\|\nabla(\mathbf{w}-\tilde{\mathbf{v}^h})\|^2 + \nu\delta^4\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}^*)\|^2 \\ &+ \inf_{q^h\in Q^h}\nu^{-1}\|p-q^h\|^2\Big\},\end{aligned}$$

where C is a constant independent of ν,δ and h.

Proof. Set $\mathbf{e} = \mathbf{w}-\tilde{\mathbf{w}}$ and decompose the error in two parts as $\mathbf{e} = \boldsymbol{\eta}-\boldsymbol{\psi}^h = (\mathbf{w}-I(\mathbf{w}))-(\tilde{\mathbf{w}}-I(\mathbf{w}))$, where $I(\mathbf{w})$ is the best approximation of $\mathbf{w}\in\mathbf{V}^h$. Then equation (4.2) becomes

$$\nu(\nabla\boldsymbol{\psi}^h,\nabla\mathbf{v}^h) + \nu\delta^2(\nabla\tilde{\boldsymbol{\phi}},\nabla\mathbf{v}^h) = \nu(\nabla\boldsymbol{\eta},\nabla\mathbf{v}^h) + \nu\delta^2(\nabla\boldsymbol{\phi},\nabla\mathbf{v}^h) - (p-q^h,\nabla\cdot\mathbf{v}^h) \tag{4.6}$$

$$(\nabla(\boldsymbol{\eta}-\boldsymbol{\psi}^h),\nabla\boldsymbol{\xi}^h) = (\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}},\boldsymbol{\xi}^h). \tag{4.7}$$

We pick $\mathbf{v}^h = \boldsymbol{\psi}^h$ and subtract $\nu\delta^2(\nabla I(\boldsymbol{\phi}),\nabla\boldsymbol{\psi}^h)$ in both sides of (4.6), where $I(\boldsymbol{\phi})$ is the L^2 orthogonal projection of $\boldsymbol{\phi}$ in $\mathbf{X}^h$. Also, in (4.7), we choose $\boldsymbol{\xi}^h = \tilde{\boldsymbol{\phi}}-I(\boldsymbol{\phi})$ and use the orthogonality. This gives

$$\begin{aligned}\nu\|\nabla\boldsymbol{\psi}^h\|^2 + \nu\delta^2(\nabla(\tilde{\boldsymbol{\phi}}-I(\boldsymbol{\phi})),\nabla\boldsymbol{\psi}^h) &= \nu(\nabla\boldsymbol{\eta},\nabla\boldsymbol{\psi}^h) + \nu\delta^2(\nabla(\boldsymbol{\phi}-I(\boldsymbol{\phi})),\nabla\boldsymbol{\psi}^h) \\ &\quad -(p-q^h,\nabla\cdot\boldsymbol{\psi}^h), \end{aligned}\tag{4.8}$$

$$(\nabla\boldsymbol{\eta},\nabla(\tilde{\boldsymbol{\phi}}-I(\boldsymbol{\phi}))) + \|\tilde{\boldsymbol{\phi}}-I(\boldsymbol{\phi})\|^2 = (\nabla\boldsymbol{\psi}^h,\nabla(\tilde{\boldsymbol{\phi}}-I(\boldsymbol{\phi}))). \tag{4.9}$$

Multiply (4.9) by $\nu\delta^2$ and substitute the resulting expression in the left hand side of (4.8). With the application of the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\nu\|\nabla\boldsymbol{\psi}^h\|^2 + \nu\delta^2\|\tilde{\boldsymbol{\phi}}-I(\boldsymbol{\phi})\|^2 &\le \nu\|\nabla\boldsymbol{\eta}\|\|\nabla\boldsymbol{\psi}^h\| + \nu\delta^2\|\nabla\boldsymbol{\eta}\|\|\nabla(\tilde{\boldsymbol{\phi}}-I(\boldsymbol{\phi}))\| \\ &\quad +\nu\delta^2\|\nabla(\boldsymbol{\phi}-I(\boldsymbol{\phi}))\|\|\nabla\boldsymbol{\psi}^h\| + \|p-q^h\|\|\nabla\cdot\boldsymbol{\psi}^h\|. \end{aligned}\tag{4.10}$$

By using the following inverse inequality,

$$\|\nabla(\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi}))\| \leq Ch^{-1}\|\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\|,$$

and using Young's inequality for the terms on the right hand side of (4.10), we get

$$\begin{aligned} \nu\|\nabla\boldsymbol{\psi}^h\|^2 + 2\nu\delta^2\|\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\|^2 \quad &\leq \quad C\Big\{(\nu + \nu\delta^2 h^{-2})\|\nabla\boldsymbol{\eta}\|^2 \\ &\quad +\nu\delta^4\|\nabla(\boldsymbol{\phi} - I(\boldsymbol{\phi}))\|^2 + \nu^{-1}\|p - q^h\|^2\Big\}. \end{aligned} \tag{4.11}$$

The result follows from applying the triangle inequality and taking the infimum over $\mathbf{V}^h$. Note that, under the inf-sup condition and the condition $\nabla \cdot \mathbf{w} = 0$, the infimum over $\mathbf{V}^h$ can be replaced by infimum over $\mathbf{X}^h$. $\square$

Remark 4.1. *The statements of Proposition 4.1 and Proposition 4.2 suggest that to get an optimal bound, one has to choose $\delta = \mathcal{O}(h)$.*

Remark 4.2. *The error in the Stokes projection $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ is bounded by approximation theoretic terms, according to Proposition 4.2.*

The semi-discrete convergence analysis of the new discretization uses properties of the Stokes projection defined above. It follows the usual finite element technique and calls for the use of Gronwall's inequality. A term similar to the nonlinear one that arises in the analysis of the Navier-Stokes equations appears here, making it necessary to make *a priori* assumptions on $\mathbf{w}$.

Theorem 4.1. *Let $(\mathbf{w}, p)$ be the solution of (3.2)-(3.4). Under the assumption that $\nabla\mathbf{w} \in L^4(0,T;L^2)$, the error $\mathbf{e} = \mathbf{w} - \mathbf{w}^h$ satisfies*

$$\begin{aligned} &\|\mathbf{e}\|^2_{L^\infty(0,T;L^2)} + \delta^2\|\nabla\mathbf{e}\|^2_{L^\infty(0,T;L^2)} + \nu\|\nabla\mathbf{e}\|^2_{L^2(0,T;L^2)} + \nu\delta^2\|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|^2_{L^2(0,T;L^2)} \\ &\leq CC^*(\|\mathbf{w}(0) - \mathbf{w}^h(0)\|^2 + \|\nabla(\mathbf{w}(0) - \mathbf{w}^h(0))\|^2) + C\,\mathcal{F}(\mathbf{w} - \tilde{\mathbf{w}}, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}, p - q^h) \end{aligned}$$

where $C^*(T) = \exp(\frac{C}{\nu^3}\int_0^T \|\nabla\mathbf{w}\|^4 dt')$ and

$$\begin{aligned} \mathcal{F}(\mathbf{w} - \tilde{\mathbf{w}}, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}, p - q^h) \quad &= \quad \|\mathbf{w} - \tilde{\mathbf{w}}\|^2 + \delta^2\|\nabla(\mathbf{w} - \tilde{\mathbf{w}})\|^2_{L^2(0;T;L^2)} \\ &\quad +\nu\delta^2\|\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}\|^2_{L^2(0;T;L^2)} + C^*(T)\Big[\|\mathbf{w}(0) - \tilde{\mathbf{w}}(0)\|^2 \\ &\quad +\|\nabla(\mathbf{w}(0) - \tilde{\mathbf{w}}(0))\|^2 + \nu^{-1}\|(\mathbf{w} - \tilde{\mathbf{w}})_t\|^2_{L^2(0;T;H^{-1})} \\ &\quad +\nu^{-1}\delta^4\|\nabla(\mathbf{w} - \tilde{\mathbf{w}})_t\|^2_{L^2(0;T;L^2)} \\ &\quad +(\|\nabla\mathbf{w}^h\|_{L^2(0;T;L^2)} + \|\nabla\mathbf{w}\|^2_{L^4(0;T;L^2)})\|\nabla(\mathbf{w} - \tilde{\mathbf{w}})\|^2_{L^4(0;T;L^2)}\Big]. \end{aligned}$$

Proof. We first set $\mathbf{v} = \mathbf{v}^h$ in (3.2) and $\xi = \xi^h$ in (3.3). Then, subtracting (3.2) from (3.12) and (3.3) from (3.13) and letting $\mathbf{e} = \mathbf{w} - \mathbf{w}^h$ give the following error equations:

$$
\begin{aligned}
&(\mathbf{e}_t, \mathbf{v}^h) + \delta^2(\nabla \mathbf{e}_t, \nabla \mathbf{v}^h) + \nu(\nabla \mathbf{e}, \nabla \mathbf{v}^h) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}^h) - b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) \\
&+\nu\delta^2(\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}^h), \nabla \mathbf{v}^h) - (p - q^h, \nabla \cdot \mathbf{v}^h) = 0 \quad \forall \mathbf{v}^h \in \mathbf{V}^h \qquad (4.12) \\
&(\nabla \mathbf{e}, \nabla \boldsymbol{\xi}^h) = (\boldsymbol{\phi} - \boldsymbol{\phi}^h, \boldsymbol{\xi}^h) \quad \forall \boldsymbol{\xi}^h \in \mathbf{X}^h. \qquad (4.13)
\end{aligned}
$$

Decompose the error in two parts: $\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\psi}^h$ where $\boldsymbol{\eta} = \mathbf{w} - \tilde{\mathbf{w}}$, $\boldsymbol{\psi}^h = \mathbf{w}^h - \tilde{\mathbf{w}}$, and add and subtract $\tilde{\boldsymbol{\phi}}$ in (4.12), where $\tilde{\mathbf{w}} \in \mathbf{V}^h, \tilde{\boldsymbol{\phi}} \in \mathbf{X}^h$ are chosen as the Stokes projection, defined via (4.2)-(4.3). Putting all these together and setting $\mathbf{v}^h = \boldsymbol{\psi}^h$ in (4.12), and $\boldsymbol{\xi} = \boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}$ in (4.13) yield

$$
\begin{aligned}
&(\boldsymbol{\psi}_t^h, \boldsymbol{\psi}^h) + \delta^2(\nabla \boldsymbol{\psi}_t^h, \nabla \boldsymbol{\psi}^h) + \nu(\nabla \boldsymbol{\psi}^h, \nabla \boldsymbol{\psi}^h) + \nu\delta^2(\nabla(\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}), \nabla \boldsymbol{\psi}^h) \\
&= (\boldsymbol{\eta}_t, \boldsymbol{\psi}^h) + \delta^2(\nabla \boldsymbol{\eta}_t, \nabla \boldsymbol{\psi}^h) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}^h) - b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) \qquad (4.14) \\
&(\nabla \boldsymbol{\psi}^h, \nabla(\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}})) = (\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}, \boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}). \qquad (4.15)
\end{aligned}
$$

Note that, since $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ is taken to be the Stokes projection of $(\mathbf{w}, \boldsymbol{\phi})$ in $(\mathbf{V}^h, \mathbf{X}^h)$, some of the terms in the error equation (4.14) vanish.

We then multiply both sides of (4.15) by $\nu\delta^2$, substitute in (4.14), and apply Cauchy-Schwarz inequality. This gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\psi}^h\|^2 + \frac{\delta^2}{2}\frac{d}{dt}\|\nabla \boldsymbol{\psi}^h\|^2 + \nu\|\nabla \boldsymbol{\psi}^h\|^2 + \nu\delta^2\|\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}\|^2 \\
\le&\|\boldsymbol{\eta}_t\|_{-1}\|\nabla \boldsymbol{\psi}^h\| + \delta^2\|\nabla \boldsymbol{\eta}_t\|\|\nabla \boldsymbol{\psi}^h\| + |b(\mathbf{w}, \mathbf{w}, \boldsymbol{\psi}^h) - b(\mathbf{w}^h, \mathbf{w}^h, \boldsymbol{\psi}^h)|. \qquad (4.16)
\end{aligned}
$$

The nonlinear term on the right hand side of (4.16) is decomposed into three parts. This reduces to

$$
b(\mathbf{w}, \mathbf{w}, \boldsymbol{\psi}^h) - b(\mathbf{w}^h, \mathbf{w}^h, \boldsymbol{\psi}^h) = b(\boldsymbol{\eta}, \mathbf{w}, \boldsymbol{\psi}^h) - b(\boldsymbol{\psi}^h, \mathbf{w}, \boldsymbol{\psi}^h) + b(\mathbf{w}^h, \boldsymbol{\eta}, \boldsymbol{\psi}^h).
$$

By applying the improved estimate (2.2), Poincaré's and Young's inequalities, the nonlinear terms are bounded as:

$$
\begin{aligned}
b(\boldsymbol{\eta}, \mathbf{w}, \boldsymbol{\psi}^h) &\le C\|\boldsymbol{\eta}\|^{1/2}\|\nabla \boldsymbol{\eta}\|^{1/2}\|\nabla \mathbf{w}\|\|\nabla \boldsymbol{\psi}^h\| \\
&\le \frac{\epsilon}{4}\|\nabla \boldsymbol{\psi}^h\|^2 + \frac{C}{\epsilon}\|\nabla \boldsymbol{\eta}\|^2\|\nabla \mathbf{w}\|^2 \\
b(\boldsymbol{\psi}^h, \mathbf{w}, \boldsymbol{\psi}^h) &\le \|\nabla \boldsymbol{\psi}^h\|^{3/2}\|\boldsymbol{\psi}^h\|^{1/2}\|\nabla \mathbf{w}\| \\
&\le \frac{\epsilon}{2}\|\nabla \boldsymbol{\psi}^h\|^2 + \frac{C}{\epsilon^3}\|\nabla \mathbf{w}\|^4\|\boldsymbol{\psi}^h\|^2 \\
b(\mathbf{w}^h, \boldsymbol{\eta}, \boldsymbol{\psi}^h) &\le C\|\mathbf{w}^h\|^{1/2}\|\nabla \mathbf{w}^h\|^{1/2}\|\nabla \boldsymbol{\eta}\|\|\nabla \boldsymbol{\psi}^h\| \\
&\le \frac{\epsilon}{4}\|\nabla \boldsymbol{\psi}^h\|^2 + \frac{C}{\epsilon}\|\mathbf{w}^h\|\|\nabla \mathbf{w}^h\|\|\nabla \boldsymbol{\eta}\|^2
\end{aligned}
$$

On the right hand side of (4.16), we apply Young's inequality and choose $\epsilon = \nu/4$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\psi}^h\|^2 + \frac{\delta^2}{2}\frac{d}{dt}\|\nabla\boldsymbol{\psi}^h\|^2 + \frac{\nu}{2}\|\nabla\boldsymbol{\psi}^h\|^2 + \nu\delta^2\|\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}\|^2\\ \le& 2\nu^{-1}\|\boldsymbol{\eta}_t\|_{-1}^2 + \nu^{-1}\delta^4\|\nabla\boldsymbol{\eta}_t\|^2 + \frac{C}{\nu}(\|\nabla\mathbf{w}\|^2 + \|\mathbf{w}^h\|\|\nabla\mathbf{w}^h\|)\|\nabla\boldsymbol{\eta}\|^2 + \frac{C}{\nu^3}\|\nabla\mathbf{w}\|^4\|\boldsymbol{\psi}^h\|^2\end{aligned}$$

Since by assumption $\|\nabla\mathbf{w}\|^4 \in L^1(0,T)$, Gronwall inequality implies that

$$\begin{aligned}&\|\boldsymbol{\psi}^h\|^2 + \delta^2\|\nabla\boldsymbol{\psi}^h\|^2 + \int_0^t [\nu\|\nabla\boldsymbol{\psi}^h\|^2 + 2\nu\delta^2\|\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}\|^2]dt'\\ \le& C^*(t)(\|\boldsymbol{\psi}^h(0)\|^2 + \|\nabla\boldsymbol{\psi}^h(0)\|^2) + CC^*(t)\int_0^t \Big[\nu^{-1}\|\boldsymbol{\eta}_t\|_{-1}^2 + \nu^{-1}\delta^4\|\nabla\boldsymbol{\eta}_t\|^2\\ &+ \frac{1}{\nu}(\|\nabla\mathbf{w}\|^2 + \|\mathbf{w}^h\|\|\nabla\mathbf{w}^h\|)\|\nabla\boldsymbol{\eta}\|^2\Big]dt',\end{aligned}$$

where $C^*(t) = \exp(\frac{C}{\nu^3}\int_0^t \|\nabla\mathbf{w}\|^4 dt')$. In order to complete proof, one has to study the $L^1(0,T)$ regularity of terms in the last inequality. Note that we can bound by using Cauchy-Schwarz inequality

$$\int_0^t \|\nabla\mathbf{w}\|^2\|\nabla\boldsymbol{\eta}\|^2 dt' \le \|\nabla\mathbf{w}\|_{L^4(0,t;L^2)}^2\|\nabla\boldsymbol{\eta}\|_{L^4(0,t;L^2)}^2 < \infty.$$

Similarly, using Hölders inequality and Lemma 3.3 imply that

$$\begin{aligned}\int_0^T \|\mathbf{w}^h\|\|\nabla\mathbf{w}^h\|\|\nabla\boldsymbol{\eta}\|^2 dt' &\le \|\mathbf{w}^h\|_{L^\infty(0,t;L^2)}\|\nabla\mathbf{w}^h\|_{L^2(0,t;L^2)}\|\nabla\boldsymbol{\eta}\|_{L^4(0,t;L^2)}^2\\ &\le C(\frac{1}{\nu^{1/2}}\|\mathbf{w}^h(0)\|^2 + \frac{\delta^2}{\nu^{1/2}}\|\nabla\mathbf{w}^h(0)\|^2\\ &\quad + \frac{1}{\nu^{3/2}}\|\mathbf{f}\|_{L^2(0,t;H^{-1})}^2)\|\nabla\boldsymbol{\eta}\|_{L^4(0,t;L^2)}^2 < \infty.\end{aligned}$$

The stated error estimate now follows by applying the triangle inequality. □

5 Time Averaged Errors

In this section, we analyze time averaged errors. In practical flow computations, pointwise flow quantities may not make sense, whereas statistics of flow quantities do. The analysis we employ here follows the same idea as the one in [10], where statistics of weak solutions of the Navier-Stokes are investigated. Accordingly, we put ourselves in the case where the solution to the time dependent problem converges to a stationary solution, provided the steady-state body force is small enough. In this context, statistics are optimally computable. In the general case, it is not known if a closed estimate is feasible (see [10]).

We will need properties of the steady-state solution, denoted with superscript $*$, so we first consider the equilibrium variational formulation of problem (1.4) when $\mathbf{f}(x,t) \to \mathbf{f}^*(x)$ as $t \to \infty$:

Find $(\mathbf{w}^*, \boldsymbol{\phi}^*, p^*) \in (\mathbf{X}, \mathbf{X}, Q)$ such that

$$\nu(\nabla \mathbf{w}^*, \nabla \mathbf{v}) + \nu\delta^2(\nabla \boldsymbol{\phi}^*, \nabla \mathbf{v}) + b(\mathbf{w}^*, \mathbf{w}^*, \mathbf{v}) - (p^*, \nabla \cdot \mathbf{v}) = (\mathbf{f}^*, \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{X} \tag{5.1}$$

$$(\nabla \mathbf{w}^*, \nabla \boldsymbol{\xi}) = (\boldsymbol{\phi}^*, \boldsymbol{\xi}), \quad \forall\, \boldsymbol{\xi} \in \mathbf{X} \tag{5.2}$$

$$(q, \nabla \cdot \mathbf{w}^*) = 0, \quad \forall\, q \in Q \tag{5.3}$$

In $\mathbf{V}$, the variational formulation becomes: Find $(\mathbf{w}^*, \boldsymbol{\phi}^*) \in (\mathbf{V}, \mathbf{X})$ satisfying

$$\nu(\nabla \mathbf{w}^*, \nabla \mathbf{v}) + \nu\delta^2(\nabla \boldsymbol{\phi}^*, \nabla \mathbf{v}) + b(\mathbf{w}^*, \mathbf{w}^*, \mathbf{v}) = (\mathbf{f}^*, \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{V} \tag{5.4}$$

$$(\nabla \mathbf{w}^*, \nabla \boldsymbol{\xi}) = (\boldsymbol{\phi}^*, \boldsymbol{\xi}), \quad \forall\, \boldsymbol{\xi} \in \mathbf{X} \tag{5.5}$$

Our first result in this section gives an *a priori* bound on $(\mathbf{w}^*, \boldsymbol{\phi}^*)$.

Lemma 5.1. *The solution* $(\mathbf{w}^*, \boldsymbol{\phi}^*)$ *to (5.4)-(5.5) is bounded such that*

$$||\, \nabla \mathbf{w}^* \,||^2 + 2\delta^2 \,||\, \boldsymbol{\phi}^* \,||^2 \leq \nu^{-2} ||\, \mathbf{f}^* \,||_{-1}^2.$$

Proof. Setting $\mathbf{v} = \mathbf{w}^*$ in (5.4) and $\boldsymbol{\xi} = \boldsymbol{\phi}^*$ in (5.5) gives the claimed result. □

This result, together with assumptions on the steady state body force $\mathbf{f}^*$ and on its relationship with the time dependent body force $\mathbf{f}$ give, in turn, a relationship between the solutions $\mathbf{w}$ and $\mathbf{w}^*$.

Proposition 5.1. *Let* $\mathbf{f} \in L^\infty(0, \infty, H^{-1}(\Omega))$*. If* $||\, \mathbf{f} - \mathbf{f}^* \,||_{-1}$ *is bounded for* $0 \leq t \leq T/2$ *and* $\mathbf{f}(x,t) \to \mathbf{f}^*(x)$ *in* $L^2(T/2, T; H^{-1}(\Omega))$ *as* $T \to \infty$*, then* $\mathbf{w}(x,t) \to \mathbf{w}^*(x)$ *in* $H^1(\Omega)$*, whenever* $M\,\nu^{-2} ||\, \mathbf{f}^* \,||_{-1} := \alpha < 1$*.*

Proof. We first subtract (5.4) from (3.5) and (5.5) from (3.6) and set $\mathbf{W} = \mathbf{w} - \mathbf{w}^*$ and $\boldsymbol{\Phi} = \boldsymbol{\phi} - \boldsymbol{\phi}^*$. Then, we have an equation of the form

$$\begin{aligned} &(\mathbf{W}_t, \mathbf{v}) + \delta^2(\nabla \mathbf{W}_t, \nabla \mathbf{v}) + \nu(\nabla \mathbf{W}, \nabla \mathbf{v}) \\ &+ \nu\delta^2(\nabla \boldsymbol{\Phi}, \nabla \mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}) - b(\mathbf{w}^*, \mathbf{w}^*, \mathbf{v}) = (\mathbf{f} - \mathbf{f}^*, \mathbf{v}), \end{aligned} \tag{5.6}$$

$$(\nabla \mathbf{W}, \nabla \boldsymbol{\xi}) = (\boldsymbol{\Phi}, \boldsymbol{\xi}), \tag{5.7}$$

for all $(\mathbf{v}, \boldsymbol{\xi}) \in (\mathbf{V}, \mathbf{X})$.

Setting $\mathbf{v} = \mathbf{W}$ in (5.6) and $\boldsymbol{\xi} = \boldsymbol{\Phi}$ in (5.7), adding the two resulting equations together and adding and subtracting the term $b(\mathbf{w}, \mathbf{w}^*, \mathbf{W})$, we have

$$\frac{1}{2}\frac{d}{dt}\left(||\, \mathbf{W} \,||^2 + \delta^2 ||\, \nabla \mathbf{W} \,||^2\right) + \nu\left(||\, \nabla \mathbf{W} \,||^2 + \delta^2 ||\, \boldsymbol{\Phi} \,||^2\right) = b(\mathbf{W}, \mathbf{w}^*, \mathbf{W}) + (\mathbf{f} - \mathbf{f}^*, \mathbf{W}). \tag{5.8}$$

Using the bound on nonlinear term, $b(\mathbf{W}, \mathbf{w}^*, \mathbf{W}) \leq M ||\, \nabla \mathbf{w}^* \,||\, ||\, \nabla \mathbf{W} \,||^2$, together with the *a priori* bound $||\, \nabla \mathbf{w}^* \,|| \leq \nu^{-1} ||\, \mathbf{f}^* \,||_{-1}$ (from Lemma 5.1), followed by $(\mathbf{f} - \mathbf{f}^*, \mathbf{W}) \leq ||\, \mathbf{f} - \mathbf{f}^* \,||_{-1} \,||\, \nabla \mathbf{W} \,||$

and Young's inequality, we get, for fixed $\epsilon > 0$,

$$\frac{1}{2}\frac{d}{dt}\left(||\,\mathbf{W}\,||^2 + \delta^2||\,\nabla\mathbf{W}\,||^2\right) + \nu(1-\alpha-\epsilon)||\,\nabla\mathbf{W}\,||^2 + \nu\delta^2||\,\mathbf{\Phi}\,||^2 \leq \frac{1}{4\epsilon\nu}||\,\mathbf{f}-\mathbf{f}^*\,||_{-1}^2. \tag{5.9}$$

Letting $\boldsymbol{\xi} = \mathbf{W}$ in equation (5.7) and using Poincare's inequality, we find that $||\,\nabla\mathbf{W}\,|| \leq C_{PF}||\,\mathbf{\Phi}\,||$. Application of Poincare's inequality to (5.9) yields

$$\frac{d}{dt}\left(||\,\mathbf{W}\,||^2 + \delta^2||\,\nabla\mathbf{W}\,||^2\right) + 2\,C_{PF}^{-2}\,\nu\left((1-\alpha-\epsilon)||\,\mathbf{W}\,||^2 + \delta^2||\,\nabla\mathbf{W}\,||^2\right) \leq \frac{1}{2\epsilon\nu}||\,\mathbf{f}-\mathbf{f}^*\,||_{-1}^2.$$

Set $y = ||\,\mathbf{W}\,||^2 + \delta^2\,||\,\nabla\mathbf{W}\,||^2$. Then, for ϵ small enough, $K := 2\,\nu\,C_{PF}^{-2}(1-\alpha-\epsilon) > 0$ and this inequality becomes

$$\frac{d\,y}{d\,t} + K\,y < \frac{1}{2\epsilon\nu}||\,\mathbf{f}-\mathbf{f}^*\,||_{-1}^2. \tag{5.10}$$

Choosing an integrating factor, equation (5.10) gives

$$\begin{aligned} y(T) \;\; &\leq \;\; y(0)\,e^{-K\,T} + \frac{1}{2\epsilon\nu}\int_0^{\frac{T}{2}} e^{K\,(t-T)}||\,\mathbf{f}-\mathbf{f}^*\,||_{-1}^2\,dt &(5.11)\\ &\quad + \frac{1}{2\epsilon\nu}\int_{\frac{T}{2}}^{T} e^{K\,(t-T)}||\,\mathbf{f}-\mathbf{f}^*\,||_{-1}^2\,dt. &(5.12)\end{aligned}$$

The first integral on the right hand side of (5.12) can be computed:

$$\int_0^{\frac{T}{2}} e^{K\,(t-T)}||\,\mathbf{f}-\mathbf{f}^*\,||_{-1}^2\,dt \leq K^{-1}\,(e^{-K\,\frac{T}{2}} - e^{-K\,T})||\,\mathbf{f}-\mathbf{f}^*\,||_{L^\infty(0,T/2;H^{-1}(\Omega))}^2,$$

and the second integral can be estimated:

$$\int_{\frac{T}{2}}^{T} e^{K\,(t-T)}||\,\mathbf{f}-\mathbf{f}^*\,||_{-1}^2\,dt \leq \int_{\frac{T}{2}}^{T} ||\,\mathbf{f}-\mathbf{f}^*\,||_{-1}^2\,dt,$$

since $e^{K\,(t-T)} \leq 1$ for $\frac{T}{2} \leq t \leq T$.

Combining everything together, (5.12) becomes

$$\begin{aligned} y(T) \;\; &\leq \;\; y(0)\,e^{-K\,T} + \frac{1}{2\epsilon\nu K}\,(e^{-K\,\frac{T}{2}} - e^{-K\,T})||\,\mathbf{f}-\mathbf{f}^*\,||_{L^\infty(0,T/2;H^{-1}(\Omega))}^2 \\ &\quad + \frac{1}{2\epsilon\nu}||\,\mathbf{f}-\mathbf{f}^*\,||_{L^2(T/2,T;H^{-1}(\Omega))}^2.\end{aligned}$$

Letting $T \to \infty$ concludes the proof. □

The first result involving time averages shows that the time averaged energy dissipation rate of the non stationary solution converges, as $t \to \infty$, to the steady state energy dissipation rate.

Proposition 5.2. *Under the same assumptions of Proposition 5.1, we can show that*

$$< \varepsilon(\mathbf{w}-\mathbf{w}^*, \boldsymbol{\phi}-\boldsymbol{\phi}^*) >= 0.$$

Proof. The proof is similar to the proof of Proposition 5.1, so we start directly from equation (5.9), with $\mathbf{W} = \mathbf{w} - \mathbf{w}^*$ and $\mathbf{\Phi} = \boldsymbol{\phi} - \boldsymbol{\phi}^*$. We multiply it by 2, use the fact that $\min\{1-\alpha-\epsilon, 1\} = 1-\alpha-\epsilon$ and integrate from 0 to T to get

$$\begin{aligned} ||\,\mathbf{W}(T)\,||^2 + \delta^2||\,\nabla\mathbf{W}(T)\,||^2 \quad &+ \quad (1-\alpha-\epsilon)\,\nu\int_0^T \left(||\,\nabla\mathbf{W}\,||^2 + \delta^2||\,\mathbf{\Phi}\,||^2\right)\,dt \\ &\leq \quad ||\,\mathbf{W}(0)\,||^2 + \delta^2||\,\nabla\mathbf{W}(0)\,||^2 + \frac{1}{2\epsilon\nu}\int_0^T ||\,\mathbf{f} - \mathbf{f}^*\,||_{-1}^2\,dt. \quad (5.13) \end{aligned}$$

Dividing everything by T and taking limit supremum on both sides, we see that the first and second terms on the left hand side vanish (as a consequence of Lemma 3.1 and of the fact that $\mathbf{w}^*$ does not depend on time). Observing that $||\,\mathbf{W}(0)\,||^2 + \delta^2||\,\nabla\mathbf{W}(0)\,||^2$ is a constant and using the hypothesis on $\mathbf{f}$ and $\mathbf{f}^*$, the right hand side also vanishes and we are left with

$$(1-\alpha-\epsilon)\,\nu < ||\,\nabla\mathbf{W}\,||^2 + \delta^2||\,\mathbf{\Phi}\,||^2 > \leq 0.$$

The fact that $(1-\alpha-\epsilon) > 0$ gives the desired result. ☐

Properties of the approximate solution $\mathbf{w}^{*\,h}$ are also needed. Thus, we also consider finite element approximation of (5.1)-(5.3). Finite element approximation of the equilibrium solution is to: Find $(\mathbf{w}^{*\,h}, \boldsymbol{\phi}^{*\,h}, p^{*\,h}) \in (\mathbf{X}^h, \mathbf{X}^h, Q^h)$ satisfying

$$\nu(\nabla\mathbf{w}^{*\,h}, \nabla\mathbf{v}^h) + \nu\delta^2(\nabla\boldsymbol{\phi}^{*\,h}, \nabla\mathbf{v}) + b(\mathbf{w}^{*\,h}, \mathbf{w}^{*\,h}, \mathbf{v}^h) - (p^{*\,h}, \nabla\cdot\mathbf{v}^h) = (\mathbf{f}^*, \mathbf{v}^h), \quad (5.14)$$

$$(\nabla\mathbf{w}^{*\,h}, \nabla\boldsymbol{\xi}^h) = (\boldsymbol{\phi}^{*\,h}, \boldsymbol{\xi}^h), \quad (5.15)$$

$$(q^h, \nabla\cdot\mathbf{w}^{*\,h}) = 0, \quad (5.16)$$

for all $(\mathbf{v}^h, \boldsymbol{\xi}^h, q^h) \in (\mathbf{X}^h, \mathbf{X}^h, Q^h)$, with the usual extension to the formulation in $(\mathbf{V}^h, \mathbf{X}^h)$.

Lemma 5.2. *$(\mathbf{w}^{*\,h}$, $\boldsymbol{\phi}^{*\,h})$ satisfies:*

$$||\,\nabla\mathbf{w}^{*\,h}\,||^2 + 2\delta^2\,||\,\boldsymbol{\phi}^{*\,h}\,||^2 \leq \nu^{-2}||\,\mathbf{f}^*\,||_{-1}^2.$$

Proof. The claim exactly follows the proof of Lemma 5.1. ☐

We now derive error estimates. The following result uses the Stokes projection defined via (4.2)-(4.3). Recall that according to Proposition 4.2, the error in the Stokes projection $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ is bounded.

Proposition 5.3. *Assume that $(\mathbf{X}^h, Q^h)$ satisfy an inf-sup condition. Under the small data condition, $M\nu^{-2}||\,\mathbf{f}\,||_{-1} := \alpha < 1$, the following error estimate holds:*

$$\begin{aligned} \nu||\,\nabla(\mathbf{w}^* - \mathbf{w}^{*\,h})\,||^2 + \nu\delta^2||\,\boldsymbol{\phi}^* - \boldsymbol{\phi}^{*\,h}\,||^2 \leq C\,\Big\{&(\nu + \nu^{-3}||\,\mathbf{f}^*\,||_{-1}^2)||\,\nabla(\mathbf{w}^* - \tilde{\mathbf{w}})\,||^2 \\ &+ \nu\delta^2||\,\boldsymbol{\phi}^* - \tilde{\boldsymbol{\phi}}\,||^2\Big\} \end{aligned}$$

Proof. Subtracting (5.14) from (5.1), for $\mathbf{v}^h \in \mathbf{V}^h$, and subtracting (5.15) from (5.2), for $\boldsymbol{\xi}^h \in \mathbf{X}^h$, we find the error equations:

$$\begin{aligned}\nu(\nabla(\mathbf{w}^* - \mathbf{w}^{*\,h}), \nabla \mathbf{v}^h) \quad &+ \quad \nu\delta^2(\nabla(\boldsymbol{\phi}^* - \boldsymbol{\phi}^{*\,h}), \nabla \mathbf{v}^h) \\ &\qquad + b(\mathbf{w}^*, \mathbf{w}^*, \mathbf{v}^h) - b(\mathbf{w}^{*\,h}, \mathbf{w}^{*\,h}, \mathbf{v}^h) - (p^* - q^h, \nabla \cdot \mathbf{v}^h) = 0 \\ (\nabla(\mathbf{w}^* - \mathbf{w}^{*\,h}), \nabla \boldsymbol{\xi}^h) \quad &= \quad (\boldsymbol{\phi}^* - \boldsymbol{\phi}^{*\,h}, \boldsymbol{\xi}^h)\end{aligned}$$

Decompose the error as $\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\psi}^h$, where $\boldsymbol{\eta} = \mathbf{w}^* - \tilde{\mathbf{w}}$, $\boldsymbol{\psi}^h = \mathbf{w}^{*\,h} - \tilde{\mathbf{w}}$, and add and subtract $\tilde{\boldsymbol{\phi}}$, where $\tilde{\mathbf{w}} \in \mathbf{V}^h, \tilde{\boldsymbol{\phi}} \in \mathbf{X}^h$ are chosen as the Stokes projection. Putting all these together and setting $\mathbf{v}^h = \boldsymbol{\psi}^h$ and $\boldsymbol{\xi}^h = \boldsymbol{\phi}^{*\,h} - \tilde{\boldsymbol{\phi}}$ yields

$$\nu|| \nabla \boldsymbol{\psi}^h ||^2 \quad + \quad \nu\delta^2|| \boldsymbol{\phi}^{*\,h} - \tilde{\boldsymbol{\phi}} ||^2 = b(\boldsymbol{\eta}, \mathbf{w}^*, \boldsymbol{\psi}^h) - b(\boldsymbol{\psi}^h, \mathbf{w}^*, \boldsymbol{\psi}^h) + b(\mathbf{w}^{*\,h}, \boldsymbol{\eta}, \boldsymbol{\psi}^h) \tag{5.17}$$

where the nonlinear term was decomposed into three parts (by adding and subtracting appropriate terms).

Using the bounds on the trilinear form followed by Young's inequality and the Cauchy-Schwarz inequality, together with the a priori estimates for $\mathbf{w}^*$ and $\mathbf{w}^{*\,h}$, we write

$$\begin{aligned}b(\boldsymbol{\eta}, \mathbf{w}^*, \boldsymbol{\psi}^h) \quad &\leq \quad M|| \nabla \boldsymbol{\eta} || \, || \nabla \mathbf{w}^* || \, || \nabla \boldsymbol{\psi}^h || \\ &\leq \quad \frac{\nu}{4}|| \nabla \boldsymbol{\psi}^h ||^2 + C\nu^{-1}|| \nabla \mathbf{w}^* ||^2 \, || \nabla \boldsymbol{\eta} ||^2 \\ &\leq \quad \frac{\nu}{4}|| \nabla \boldsymbol{\psi}^h ||^2 + C\nu^{-3}|| \mathbf{f}^* ||_{-1}^2 \, || \nabla \boldsymbol{\eta} ||^2\end{aligned}$$

$$\begin{aligned}b(\boldsymbol{\psi}^h, \mathbf{w}^*, \boldsymbol{\psi}^h) \quad &\leq \quad M|| \nabla \mathbf{w}^* || \, || \nabla \boldsymbol{\psi}^h ||^2 \\ &\leq \quad M\nu^{-1}|| \mathbf{f}^* ||_{-1} \, || \nabla \boldsymbol{\psi}^h ||^2\end{aligned}$$

$$\begin{aligned}b(\mathbf{w}^{*\,h}, \boldsymbol{\eta}, \boldsymbol{\psi}^h) \quad &\leq \quad M|| \nabla \boldsymbol{\eta} || \, || \nabla \mathbf{w}^{*\,h} || \, || \nabla \boldsymbol{\psi}^h || \\ &\leq \quad \frac{\nu}{4}|| \nabla \boldsymbol{\psi}^h ||^2 + C\nu^{-3}|| \mathbf{f}^* ||_{-1}^2 \, || \nabla \boldsymbol{\eta} ||^2\end{aligned}$$

With the help of the estimates above and the fact that $1 - \alpha > 0$, Equation (5.17) becomes

$$\nu|| \nabla \boldsymbol{\psi}^h ||^2 \quad + \quad \nu\delta^2|| \boldsymbol{\phi}^{*\,h} - \tilde{\boldsymbol{\phi}} ||^2 \leq C\nu^{-3}|| \mathbf{f}^* ||_{-1}^2 \, || \nabla \boldsymbol{\eta} ||^2.$$

The triangle inequality gives the result. □

The discrete counterpart of Proposition 5.2 is given in the following statement.

Proposition 5.4. *With the same assumptions as in Proposition 5.2, we show that*

$$< \varepsilon(\mathbf{w}^h - \mathbf{w}^{*\,h}, \boldsymbol{\phi}^h - \boldsymbol{\phi}^{*\,h}) >= 0.$$

Proof. The argument is the same as in the proof of Proposition 5.2, for the discrete case. □

The next theorem is the major result in this section. It shows that under the condition that the body force driving the flow when it has reached the steady state is small enough, statistics can be accurately computed.

We must point out that weak solutions of the Zeroth Order Model are indeed strong solutions and satisfy an energy equality [12]. This has been proven for the periodic case, but it is reasonable to assume that the same holds true in the non periodic scenario.

Theorem 5.1. *Assuming the hypotheses of Proposition 5.1 hold, then*

$$< \varepsilon(\mathbf{w} - \mathbf{w}^h, \boldsymbol{\phi} - \boldsymbol{\phi}^h) > \leq C\,\nu \left(||\, \nabla(\mathbf{w}^* - \mathbf{w}^{*\,h})\, ||^2 + \delta^2 ||\, \boldsymbol{\phi}^* - \boldsymbol{\phi}^{*\,h}\, ||^2 \right). \tag{5.18}$$

Proof. Add and subtract $\mathbf{w}^*$, $\mathbf{w}^{*\,h}$, $\boldsymbol{\phi}^*$, $\boldsymbol{\phi}^{*\,h}$ appropriately to the formula of $< \varepsilon(\mathbf{w} - \mathbf{w}^h, \boldsymbol{\phi} - \boldsymbol{\phi}^h) >$. Then, the proof follows by the application of triangle inequality and from Propositions 5.2 and 5.4. □

Corollary 5.1. *Suppose that the small data condition holds and* $(\mathbf{X}^h, Q^h)$ *satisfy an inf-sup condition. Then,*

$$< \varepsilon(\mathbf{w} - \mathbf{w}^h, \boldsymbol{\phi} - \boldsymbol{\phi}^h) > \leq C \Bigg\{ (\nu + \nu^{-3} ||\, \mathbf{f}^*\, ||_{-1}^2) \inf_{\tilde{\mathbf{w}} \in \mathbf{V}^h} ||\, \nabla(\mathbf{w}^* - \tilde{\mathbf{w}^*})\, ||^2 + \nu\delta^2 \inf_{\tilde{\boldsymbol{\phi}} \in \mathbf{X}^h} ||\, \boldsymbol{\phi}^* - \tilde{\boldsymbol{\phi}}\, ||^2 + \nu^{-1} \inf_{q^h \in Q^h} ||\, p^* - q^h\, ||^2 \Bigg\}$$

Proof. Use the estimates of Proposition 5.3 on the right-hand side of (5.18). □

6 Conclusions

We proposed a discretization to the Zeroth Order Model based on a mixed variational formulation that represents the natural energy properties of the model well. Optimal convergence rates are obtained if $\delta = \mathcal{O}(h)$, which is consistent with the literature and simulations with other models [8, 9]. Additionally, time averaged error estimates are presented. For the special case of asymptotically small body force, they prove to be optimally computable.

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