Numerical Analysis of a Higher Order Time Relaxation Model of Fluids

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Abstract. We study the numerical errors in finite element discretizations of a time relaxation model of fluid motion:

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} + \chi\mathbf{u}^* = \mathbf{f} \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0$$

In this model, introduced by Stolz, Adams and Kleiser, $\mathbf{u}^*$ is a generalized fluctuation and χ the time relaxation parameter. The goal of inclusion of the $\chi\mathbf{u}^*$ is to drive unresolved fluctuations to zero exponentially. We study convergence of discretization of the model to the model's solution as $h,\ \Delta t \to 0$. Next we complement this with an experimental study of the effect the time relaxation term (and a nonlinear extension of it) has on the large scales of a flow near a transitional point. We close by showing that the time relaxation term does not alter shock speeds in the inviscid, compressible case, giving analytical confirmation of a result of Stolz, Adams and Kleiser.

Key words. generalized Stokes problem, defective boundary conditions

AMS Mathematics subject classifications. 65N30

We dedicate this paper to Max Gunzburger on the occasion of his 60th birthday.

1 Introduction

A fluid's velocity at higher Reynolds numbers contains many spatial scales not economically resolvable on computationally feasible meshes. For this reason, many turbulence models, large eddy simulation models, numerical regularization and computational stabilizations have been explored in computational fluid dynamics. One of the simplest such regularization and most recent has been proposed by Adams, Stoltz and Kleiser [1, 2]. Briefly, if $\mathbf{u}$ represents the fluid velocity, h the characteristic mesh width, and $\delta = O(h)$ a chosen length scale, let $\mathbf{u}^*$ denote some representation of the part of $\mathbf{u}$ varying over length scales $< O(\delta)$, i.e. the fluctuating part of $\mathbf{u}$. (This will be made specific in Section 2.) The fluid regularization model of Adams, Stoltz and Kleiser, considered herein, arises

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by adding a simple, linear, lower order time regularization term, $\chi\mathbf{u}^*$, (where $\chi > 0$ has units of $1/time$) to the Navier-Stokes equations, giving:

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} + \chi\mathbf{u}^* = \mathbf{f}\,,\ \in\Omega\,, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0\,,\ \in\Omega\,. \tag{1.2}$$

The term $\chi\mathbf{u}^*$ is intended to drive unresolved velocity scales to zero exponentially fast. Adams, Kleiser and Stoltz have performed extensive computational tests of this time relaxation model on compressible flows with shocks and on turbulent flows, for example, [1, 2] as has Guenanff [7] on aerodynamic noise. The originating study of (1.1),(1.2) was the work of Rosenau [11] and Schochet and Tadmor [12] in which the time relaxation model was developed from a regularized Chapman-Enskog expansion of conservation laws. Most recently, in [10] it was shown that at high Reynolds number, solutions to (1.1),(1.2), possess an energy cascade which terminates at the mesh scale δ with the proper choice of relaxation coefficient χ.

Our goal in this report is to connect the work studying (1.1)-(1.2) as a continuum model with the computational experiments using (1.1)-(1.2) by a numerical analysis of discretizations of (1.1)-(1.2). We thus consider stability and convergence of finite element discretizations of (1.1)-(1.2) as $h \to 0$. Our goal is to elucidate the interconnections between δ, h, χ, ν, and the algorithms used to compute the fluctuation $\mathbf{u}^*$ as a discrete function.

In Section 2 we give a precise definition of the discrete averaging operator and the de-convolution procedure that are used to define the generalized fluctuation $\mathbf{u}^*$. We also give preliminaries about the finite element discretizations studied. Section 3 gives the convergence analysis of this method. This analysis is for $\nu > 0$. The Euler equations, $\nu = 0$ in (1.1),(1.2), include shocks – a phenomenon excluded when $\nu > 0$. In Section 5 we complement the case $\nu > 0$ by considering a conservation law in one space dimension. We show that adding the time relaxation term $\chi\mathbf{u}^*$ does not alter shock speeds – thus confirming theoretically a result of Stoltz and Adams [1]. In Section 4 we give some numerical tests. Our primary goal in these tests is to study the effect the time relaxation term has on O(1) scales. We study a flow very close to its transition from one regime to another: from equilibrium to time dependent via eddy shedding behind the forward-backward step. We investigate experimentally which of several natural formulations of this time relaxation term least retards this transition.

2 Analysis of the Time Relaxation Model

In order to discuss the effects of the regularization we introduce the following notation. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space W_2^k, and $\|\cdot\|_k$ denotes the norm in H^k. For functions $v(\mathbf{x},t)$ defined on the entire time interval $(0,T)$, we define

$$\|v\|_{\infty,k} := \sup_{0<t<T}\|v(\cdot,t)\|_k\,, \qquad \text{and} \qquad \|v\|_{m,k} := \left(\int_0^T \|v(\cdot,t)\|_k^m\,dt\right)^{1/m}.$$

The following function spaces are used in the analysis:

$$\text{Velocity Space} \ :\ X := H_0^1(\Omega)\,,$$

$$\text{Pressure Space} \;:\; P := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \,:\, \int_\Omega q\, d\Omega = 0 \right\},$$

$$\text{Divergence} - \text{free Space} \;:\; Z := \left\{ v \in X : \int_\Omega q \nabla \cdot \mathbf{v}\, d\Omega \,=\, 0,\ \forall\, q \in P \right\}.$$

We denote the dual space of X as X', with norm $\| \cdot \|_{-1}$.

A variational solution of the N-S equations may be stated as: *Find* $\mathbf{w} \in L^2(0,T;X) \cap L^\infty(0,T;L^2(\Omega))$, $r \in L^2(0,T;P)$ *with* $\mathbf{w}_t \in L^2(0,T;X')$ *satisfying*

$$(\mathbf{w}_t, \mathbf{v}) + (\mathbf{w} \cdot \nabla \mathbf{w}, \mathbf{v}) - (r, \nabla \cdot \mathbf{v}) + \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}),\ \forall \mathbf{v} \in X, \tag{2.1}$$

$$(q, \nabla \cdot \mathbf{w}) = 0,\ \forall q \in P. \tag{2.2}$$

We consider in comparison to (2.1),(2.2) the problem: *Find* $\mathbf{u} \in L^2(0,T;X) \cap L^\infty(0,T;L^2(\Omega))$, $p \in L^2(0,T;P)$ *with* $\mathbf{u}_t \in L^2(0,T;X')$ *satisfying*

$$(\mathbf{u}_t, \mathbf{v}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \chi(\mathbf{u} - G_N \bar{\mathbf{u}}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}),\ \forall \mathbf{v} \in X, \tag{2.3}$$

$$(q, \nabla \cdot \mathbf{u}) = 0,\ \forall q \in P. \tag{2.4}$$

In (2.3) $\bar{\mathbf{u}}$ denotes a spatially averaged function of $\mathbf{u}$ defined as: $\bar{\mathbf{u}} := G(\mathbf{u})$ satisfying

$$-\delta^2 \Delta \bar{\mathbf{u}} + \bar{\mathbf{u}} = \mathbf{u},\ \text{in } \Omega, \tag{2.5}$$

$$\bar{\mathbf{u}} = \mathbf{0},\ \text{on } \partial\Omega, \tag{2.6}$$

where δ represents the filter length scale. The operator G_N in (2.3) represents the Nth van Cittert approximate deconvolution operator defined by

$$G_N \phi := \sum_{n=0}^{N} (I - G)^n \phi, \quad N = 0, 1, 2, \ldots. \tag{2.7}$$

Lemma 2.1 *[3, 5] For* $\phi \in L^2(\Omega)$ *we have that*

$$\phi - G_N \bar{\phi} = \delta^{2N+2} (-\Delta G)^{N+1} \phi. \tag{2.8}$$

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As the operator $(I - G_N G)$ is Symmetric Positive Definite (SPD), [10], the operator $B : L^2(\Omega) \to L^2(\Omega)$ satisfying

$$B^2 \phi := \delta^{-(2N+2)} (I - G_N G) \phi = \delta^{-(2N+2)} (\phi - G_N \bar{\phi}) \tag{2.9}$$

is bounded and well defined, (i.e. $B = \delta^{-(N+1)} \sqrt{I - G_N G}$).

$$\text{Let} \qquad \phi^* := \delta^{(N+1)}\, B\phi \quad (\approx \phi - \bar{\phi}). \tag{2.10}$$

Then, from (2.8) we have

$$\begin{aligned} \|\phi^*\| &= (\phi - G_N \bar{\phi}, \phi)^{1/2} = \left(\delta^{2N+2} (B\phi, B\phi) \right)^{1/2} \\ &= \delta^{N+1} \|B\phi\|. \end{aligned}$$

Letting $\mathbf{e}(\mathbf{x},t) := \mathbf{w}(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t)$, subtracting (2.3) from (2.1) we have that

$$(\mathbf{e}_t,\mathbf{v}) + (\mathbf{e}\cdot\nabla\mathbf{w},\mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{e},\mathbf{v}) + \nu(\nabla\mathbf{e},\nabla\mathbf{v}) + \chi(\mathbf{e}-G_N\bar{\mathbf{e}},\mathbf{v}) \;=\; \chi(\mathbf{w}-G_N\bar{\mathbf{w}},\mathbf{v})\,,\ \forall\mathbf{v}\in Z\,. \tag{2.11}$$

With the choice $\mathbf{v} = \mathbf{e}$ we obtain (using $(\mathbf{u}\cdot\nabla\mathbf{e},\mathbf{e}) = 0$)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{e}\|^2 \,+\, (\mathbf{e}\cdot\nabla\mathbf{w},\mathbf{e}) \,+\, \nu\|\nabla\mathbf{e}\|^2 \,+\, \chi\,(\mathbf{e}-G_N\bar{\mathbf{e}},\mathbf{e}) \;&=\; \chi\,(\mathbf{w}-G_N\bar{\mathbf{w}},\mathbf{e}),\\ \frac{1}{2}\frac{d}{dt}\|\mathbf{e}\|^2 \,-\, |(\mathbf{e}\cdot\nabla\mathbf{w},\mathbf{e})| \,+\, \nu\|\nabla\mathbf{e}\|^2 \,+\, \chi\,\|\mathbf{e}^*\|^2 \;&\le\; \chi\,\delta^{2N+2}\|B\mathbf{w}\|\,\|B\mathbf{e}\|\,. \end{aligned} \tag{2.12}$$

With the estimate (using Young's inequality),

$$\begin{aligned}|(\mathbf{e}\cdot\nabla\mathbf{w},\mathbf{e})| \;&\le\; C\sqrt{\|\mathbf{e}\|\,\|\nabla\mathbf{e}\|}\,\|\nabla\mathbf{w}\|\,\|\nabla\mathbf{e}\| \;=\; C\|\mathbf{e}\|^{1/2}\,\|\nabla\mathbf{w}\|\,\|\nabla\mathbf{e}\|^{3/2}\\ &\le\; \frac{1}{2}\nu\|\nabla\mathbf{e}\|^2 \,+\, C_1\nu^{-3}\|\nabla\mathbf{w}\|^4\,\|\mathbf{e}\|^2\,,\end{aligned}$$

equation (2.12) becomes

$$\frac{d}{dt}\|\mathbf{e}\|^2 \,-\, C_1\,\nu^{-3}\|\nabla\mathbf{w}\|^4\,\|\mathbf{e}\|^2 \,+\, \nu\|\nabla\mathbf{e}\|^2 \,+\, \chi\,\|\mathbf{e}^*\|^2 \;\le\; C_2\,\chi\,\delta^{2N+2}\|B\mathbf{w}\|^2. \tag{2.13}$$

Proceeding as in Gronwall's Lemma, multiplying through by the integrating factor $\exp(-\,C_1\,\nu^{-3}\int_0^\tau \|\nabla\mathbf{w}\|^4\,ds)$ and using $\|\mathbf{e}\|(0) = 0$, we obtain

$$\begin{aligned}\|\mathbf{e}\|^2 \;\;+\;\; &\int_0^t e^{(C_1\,\nu^{-3}\int_\tau^t \|\nabla\mathbf{w}\|^4\,ds)}\left(\nu\|\nabla\mathbf{e}\|^2 \,+\, \chi\,\|\mathbf{e}^*\|^2\right)\,d\tau\\ \le\;\; &\int_0^t e^{(C_1\,\nu^{-3}\int_\tau^t \|\nabla\mathbf{w}\|^4\,ds)}\left(C_2\,\chi\,\delta^{2N+2}\|B\mathbf{w}\|^2\right)\,d\tau\,,\end{aligned} \tag{2.14}$$

i.e.,

$$\|\mathbf{e}\|^2 \,+\, \nu\int_0^t \|\nabla\mathbf{e}\|^2\,d\tau \;+\; \int_0^t \chi\,\|\mathbf{e}^*\|^2\,d\tau \;\le\; C_2 e^{C_1\,\nu^{-3}\,\|\mathbf{w}\|_{4,1}^4}\,\chi\,\delta^{2N+2}\int_0^t \|B\mathbf{w}\|^2\,d\tau\,, \tag{2.15}$$

from which the following lemma follows.

Lemma 2.2 *With $\mathbf{w}\in L^4(0,T;W_4^1)$ satisfying (2.1),(2.2) and $\mathbf{u}$ given by (2.3),(2.4) we have that there exists constants C_1, $C_2 > 0$, such that*

$$\begin{aligned}\|\mathbf{w} \,-\, \mathbf{u}\|^2 \;\;+\;\; \nu\int_0^t \|\nabla(\mathbf{w} \,-\, \mathbf{u})\|^2\,d\tau \,+\, \chi\int_0^t \|(\mathbf{w} \,-\, \mathbf{u})^*\|^2\,d\tau&\\ \le\;\; C_2 e^{C_1\,\nu^{-3}\,\|\mathbf{w}\|_{4,1}^4}\,\chi\,\delta^{2N+2}\int_0^t \|B\mathbf{w}\|^2\,d\tau\,.&\end{aligned} \tag{2.16}$$

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3 Numerical Approximation of the Navier-Stokes equations using Time Relaxation

In this section we address the error between the *stabilized* approximation computed using equations (2.3),(2.4) and the solution to the Navier-Stokes equations. In view of estimate (2.16), and with the aid of the triangle inequality, the desired error estimate reduces to finding the error between the numerical approximation of (2.3),(2.4) and its true solution.

We begin by describing the finite element approximation framework and listing the approximating properties used in the analysis.

Let $\Omega \subset \mathbb{R}^{\acute{d}}$ $(\acute{d} = 2,3)$ be a polygonal domain and let T_h be a triangulation of Ω made of triangles (in $\mathbb{R}^2$) or tetrahedrals (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$\Omega = \cup K; \ \ K \in T_h.$$

We assume that there exist constants c_1, c_2 such that

$$c_1 h \leq h_K \leq c_2 \rho_K$$

where h_K is the diameter of triangle (tetrahedral) K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{K\in T_h} h_K$. Let $P_k(A)$ denote the space of polynomials on A of degree no greater than k. Then we define the finite element spaces as follows.

$$\begin{aligned}
X_h &:= \left\{ \mathbf{v} \in X \cap C(\bar{\Omega})^2 : \mathbf{v}|_K \in P_k(K), \ \forall K \in T_h \right\} , \\
P_h &:= \left\{ q \in P \cap C(\bar{\Omega}) : q|_K \in P_s(K), \ \forall K \in T_h \right\} , \\
Z_h &:= \left\{ \mathbf{v} \in X_h : (q, \nabla \cdot \mathbf{v}) = 0, \ \forall q \in P_h \right\} .
\end{aligned}$$

We assume that the spaces X_h, P_h satisfy the discrete inf-sup condition, namely there exists $\gamma \in \mathbb{R}$, $\gamma > 0$,

$$\gamma \leq \inf_{q_h \in P_h} \sup_{\mathbf{v}_h \in X_h} \frac{\int_\Omega q_h \, \nabla \cdot \mathbf{v}_h \, dA}{\|q_h\|_P \, \|\mathbf{v}_h\|_X} \, . \tag{3.1}$$

Let Δt be the step size for t so that $t_n = n\Delta t$, $n = 0, 1, 2, \ldots, N_T$, with $T := N_T \Delta t$, and $d_t f^n := \frac{f(t_n) - f(t_{n-1})}{\Delta t}$. We define the following additional norms:

$$|||v|||_{\infty,k} := \max_{0\leq n \leq N_T} \|v^n\|_k \, , \qquad |||v_{1/2}|||_{\infty,k} := \max_{1\leq n \leq N_T} \|v^{n-1/2}\|_k \, ,$$

$$|||v|||_{m,k} := \left(\sum_{n=0}^{N_T} \|v^n\|_k^m \Delta t \right)^{1/m} , \qquad |||v_{1/2}|||_{m,k} := \left(\sum_{n=1}^{N_T} \|v^{n-1/2}\|_k^m \Delta t \right)^{1/m} .$$

In addition, we make use of the following approximation properties,[4]:

$$\begin{aligned}
\inf_{\mathbf{v}\in X_h} \|\mathbf{u} - \mathbf{v}\| &\leq Ch^{k+1} \|\mathbf{u}\|_{k+1}, \ \ \mathbf{u} \in H^{k+1}(\Omega)^{\acute{d}}, \\
\inf_{\mathbf{v}\in X_h} \|\mathbf{u} - \mathbf{v}\|_1 &\leq Ch^{k} \|\mathbf{u}\|_{k+1}, \ \ \mathbf{u} \in H^{k+1}(\Omega)^{\acute{d}}, \\
\inf_{r\in P_h} \|p - r\| &\leq Ch^{s+1} \|p\|_{s+1}, \ \ p \in H^{s+1}(\Omega).
\end{aligned} \tag{3.2}$$

We define the *skew-symmetric trilinear form* $b^*(\cdot,\cdot,\cdot) : X \times X \times X \to \mathbb{R}$ as

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}) . \tag{3.3}$$

Note that for $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\in X$, with $\int_\Omega q\,\nabla\cdot\mathbf{u}\,dA = 0$, $\forall q \in P$,

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) = b(\mathbf{u},\mathbf{v},\mathbf{w}) := (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) .$$

For ease of notation in discussion the Crank-Nicolson temporal discretization we let

$$\breve{\mathbf{u}}^n = \frac{u^n + u^{n-1}}{2} .$$

The time relaxed, discrete approximation to (2.3),(2.4) on the time interval $(0,T]$, is given by:
For $n = 1,2,\ldots,N_T$, find $\mathbf{u}_h^n \in X_h$, $p_h^n \in P_h$, such that

$$\begin{aligned}(\mathbf{u}_h^n,\mathbf{v}) + \Delta t\, b^*(\breve{\mathbf{u}}_h^n,\breve{\mathbf{u}}_h^n,\mathbf{v}) - \Delta t\,(\breve{p}_h^n,\nabla\cdot\mathbf{v}) &+ \Delta t\,\nu(\nabla\breve{\mathbf{u}}_h^n,\nabla\mathbf{v}) + \Delta t\,\chi\,(\breve{\mathbf{u}}_h^n - G_N\bar{\breve{\mathbf{u}}}_h^n,\mathbf{v}) \\ &= (\mathbf{u}_h^{n-1},\mathbf{v}) + \Delta t\,(\breve{\mathbf{f}}^n,\mathbf{v}),\ \forall \mathbf{v}\in X_h , \qquad (3.4)\\ (q,\nabla\cdot\mathbf{u}_h^n) &= 0\ ,\ \forall q \in P_h . \qquad (3.5)\end{aligned}$$

As the spaces X_h and P_h satisfy the discrete inf-sup condition (3.1), we can equivalent consider the problem:
For $n = 1,2,\ldots,N_T$ find $\mathbf{u}_h^n \in Z_h$, $p_h \in P_h$, such that

$$\begin{aligned}(\mathbf{u}_h^n,\mathbf{v}) &+ \Delta t\, b^*(\breve{\mathbf{u}}_h^n,\breve{\mathbf{u}}_h^n,\mathbf{v}) + \Delta t\,\nu(\nabla\breve{\mathbf{u}}_h^n,\nabla\mathbf{v}) + \Delta t\,\chi\,(\breve{\mathbf{u}}_h^n - G_N\bar{\breve{\mathbf{u}}}_h^n,\mathbf{v}) \\ &= (\mathbf{u}_h^{n-1},\mathbf{v}) + \Delta t\,(\breve{\mathbf{f}}^n,\mathbf{v}),\ \forall \mathbf{v}\in Z_h . \end{aligned} \tag{3.6}$$

The discrete Gronwall's lemma plays an important role in the following analysis.

Lemma 3.1 (Discrete Gronwall's Lemma) *[9] Let Δt, H, and a_n, b_n, c_n, γ_n (for integers $n \geq 0$) be nonnegative numbers such that*

$$a_l + \Delta t\sum_{n=0}^{l} b_n \leq \Delta t\sum_{n=0}^{l}\gamma_n\, a_n + \Delta t\sum_{n=0}^{l} c_n + H \quad \text{for } l \geq 0 .$$

Suppose that $\Delta t\,\gamma_n < 1$, for all n, and set $\sigma_n = (1-\Delta t\,\gamma_n)^{-1}$. Then,

$$a_l + \Delta t\sum_{n=0}^{l} b_n \leq \exp\left(\Delta t\sum_{n=0}^{l}\sigma_n\,\gamma_n\right)\left\{\Delta t\sum_{n=0}^{l} c_n + H\right\} \quad \text{for } l \geq 0 . \tag{3.7}$$

For the approximation scheme given by (3.6) we have that the iteration is computable and satisfies the following a priori estimate.

Lemma 3.2 *For the approximation scheme (3.6) we have that a solution $\mathbf{u}_h^l$, $l = 1,\ldots N_T$, exists at each iteration and, for $\Delta t < 1$, satisfies the following a priori bounds:*

$$\|\mathbf{u}_h^l\|^2 + 2\Delta t\,\chi\sum_{n=1}^{l}\|\breve{\mathbf{u}}_h^{n\,*}\|^2 + 2\Delta t\,\nu\sum_{n=1}^{l}\|\nabla\breve{\mathbf{u}}_h^n\|^2 \leq C\left(|||f|||_{2,0}^2 + \|\mathbf{u}_h^0\|^2\right) . \tag{3.8}$$

Proof: The existence of a solution $\mathbf{u}_h^n$ to (3.6) follows from the Leray-Schauder Principle [16]. Specifically, with $A : Z_h \to Z_h$, defined by $\mathbf{y} = A(\mathbf{w})$

$$(\mathbf{y},\mathbf{v}) \; := \; -\Delta t\, b^*((\mathbf{w}+\mathbf{u}_h^{n-1})/2, (\mathbf{w}+\mathbf{u}_h^{n-1})/2, \mathbf{v}) \;-\; \Delta t \nu (\nabla(\mathbf{w}+\mathbf{u}_h^{n-1})/2, \nabla \mathbf{v}) \\ + \Delta t\, \chi((\mathbf{w}+\mathbf{u}_h^{n-1})/2 - G_N(\bar{\mathbf{w}}+\bar{\mathbf{u}}_h^{n-1})/2, \mathbf{v}) \;+\; (\mathbf{u}_h^{n-1}, \mathbf{v}) \;+\; \Delta t(\breve{\mathbf{f}}^n, \mathbf{v}) \;,$$

the operator A is compact and any solution of $\mathbf{u} \; = \; s\, A(\mathbf{u})\,, \quad$ for $\; 0 \le s < 1$, satisfied the bound $\|\mathbf{u}\| \le \gamma$, where γ is independent of s.

To obtain the a priori estimates, in (3.6) setting $\mathbf{v} = \breve{\mathbf{u}}_h^n$ we have

$$\|\mathbf{u}_h^n\|^2 \;-\; \|\mathbf{u}_h^{n-1}\|^2 \;+\; 2\Delta t\, \nu \|\nabla \breve{\mathbf{u}}_h^n\|^2 \;+\; 2\Delta t\, \chi\, \|\breve{\mathbf{u}}_h^{n\,*}\|^2 \;\le\; \Delta t\, \|\breve{\mathbf{u}}_h^n\|^2 \;+\; \Delta t\, \|\breve{\mathbf{f}}^n\|^2 \,. \tag{3.9}$$

Summing (3.9) from $n = 1$ to l, implies

$$\begin{aligned} \|\mathbf{u}_h^l\|^2 \;+\; 2\Delta t\, \chi \sum_{n=1}^{l} \|\breve{\mathbf{u}}_h^{n\,*}\|^2 \quad + \quad 2\Delta t\, \nu \sum_{n=1}^{l} \|\nabla \breve{\mathbf{u}}_h^n\|^2 & \\ \le \quad \|\mathbf{u}_h^0\|^2 \;+\; \Delta t \sum_{n=1}^{l} \|\breve{\mathbf{u}}_h^n\|^2 \;+\; \Delta t \sum_{n=1}^{l} \|\breve{\mathbf{f}}^n\|^2 \;, & \\ \le \quad \|\mathbf{u}_h^0\|^2 \;+\; \Delta t \sum_{n=0}^{l} \|\mathbf{u}_h^n\|^2 \;+\; \Delta t \sum_{n=0}^{l} \|\mathbf{f}^n\|^2 \;. & \end{aligned} \tag{3.10}$$

Applying (3.7) we obtain (3.8), with C explicitly given by $C = \exp(T/(1-\Delta t))$.

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For the approximation error between $\mathbf{u}_h^n$ satisfying (3.6) and $\mathbf{u}^n$ satisfying (2.3) we have the following.

Theorem 3.1 *For* $\mathbf{u} \in L^\infty(0,T;W_4^{k+1}) \cap W_2^3(0,T;L^2) \cap W_4^2(0,T;W_2^1)$, $p \in L^4(0,T;W_4^{s+1}) \cap W_2^2(0,T;L^2)$, $f \in L^2(0,T;W_2^2)$ *satisfying (2.3),(2.4), and* $\mathbf{u}_h$ *given by (3.4),(3.5) we have that*

$$|||\mathbf{u} \,-\, \mathbf{u}_h|||_{\infty,0} \;\le\; \mathbf{F}(\Delta t, h, \delta, \chi) \;+\; C h^{k+1} |||\mathbf{u}|||_{\infty,k+1}\,, \tag{3.11}$$

$$\left(\nu \Delta t \sum_{n=1}^{l} \|\nabla(\mathbf{u}^{n+1/2} \,-\, (\mathbf{u}_h^n + \mathbf{u}_h^{n-1})/2)\|^2 \right)^{1/2} \;\le\; \mathbf{F}(\Delta t, h, \delta, \chi) \;+\; C\nu^{1/2}(\Delta t)^2 \|\nabla \mathbf{u}_{tt}\|_{2,0} \\ + C\nu^{1/2} h^k |||\mathbf{u}|||_{2,k+1} \;, \; \textit{for } 1 \le l \le N_T \tag{3.12}$$

where

$$\begin{aligned} \mathbf{F}(\Delta t, h, \delta, \chi) \;\; := \;\; & C\nu^{-1/2} \left(h^k |||\mathbf{u}|||_{4,k+1}^2 \;+\; h^{k+1/2} |||\nabla \mathbf{u}|||_{4,0}^2 \;+\; h^{s+1} |||p_{1/2}|||_{2,s+1} \right) \\ & + C\nu^{-1/2}\, h^k \left(|||\mathbf{f}|||_{2,0} \;+\; \|\mathbf{u}_h^0\| \right) \;+\; C\nu^{1/2}\, h^k |||u|||_{2,k+1} + C\chi^{1/2}\, h^{k+1} |||u|||_{2,k+1} \\ & + C(\Delta t)^2 \Big(\|\mathbf{u}_{ttt}\|_{2,0} \;+\; \nu^{-1/2} \|p_{tt}\|_{2,0} \;+\; \|\mathbf{f}_{tt}\|_{2,0} \\ & \quad + \nu^{1/2} \|\nabla \mathbf{u}_{tt}\|_{2,0} \;+\; \nu^{-1/2} \|\nabla \mathbf{u}_{tt}\|_{4,0}^2 \\ & \quad + \nu^{-1/2} |||\nabla \mathbf{u}|||_{4,0}^2 \;+\; \nu^{-1/2} |||\nabla \mathbf{u}_{1/2}|||_{4,0}^2 \\ & \quad + \; \chi^{1/2}\, \delta^{2N+2} \|\mathbf{u}_{tt}\|_{2,0} + \chi^{1/2}\, \|\mathbf{u}_{tt}\|_{2,0} \Big) \;. \end{aligned}$$

Proof:

Let $\mathcal{A} : X \times X \to \mathbb{R}$ be defined by

$$\mathcal{A}(\mathbf{u},\mathbf{v}) \ := \ \nu(\nabla\mathbf{u}\,,\nabla\mathbf{v}) \, + \, \chi(\mathbf{u}-G_N\bar{\mathbf{u}}\,,\,\mathbf{v}) \ , \tag{3.13}$$

and note that

$$\mathcal{A}(\mathbf{u},\mathbf{u}) \ = \ \nu\|\nabla\mathbf{u}\|^2 + \chi\|\mathbf{u}^*\|^2 \ . \tag{3.14}$$

Then, (3.6) may be written as

$$(\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\mathbf{v}) \, + \, \Delta t\,\mathcal{A}(\breve{\mathbf{u}}_h^n,\mathbf{v}) \, + \, \Delta t\, b^*(\breve{\mathbf{u}}_h^n,\breve{\mathbf{u}}_h^n,\mathbf{v}) \ = \ \Delta t\,(\breve{\mathbf{f}},\mathbf{v}) \ , \quad \forall \mathbf{v}\in Z_h \, . \tag{3.15}$$

Also, at time $t=(n-1/2)\Delta t$, $\mathbf{u}$ given by (2.3)-(2.4) satisfies

$$(\mathbf{u}^n-\mathbf{u}^{n-1},\mathbf{v}) + \Delta t\,\mathcal{A}(\breve{\mathbf{u}}^n,\mathbf{v}) + \Delta t\, b^*(\breve{\mathbf{u}}^n,\breve{\mathbf{u}}^n,\mathbf{v}) - \Delta t\,(\breve{p}^n,\nabla\cdot\mathbf{v}) \ = \ \Delta t\,(\breve{\mathbf{f}},\mathbf{v}) + \Delta t\, Intp(\mathbf{u}^n,p^n;\mathbf{v}) \ , \tag{3.16}$$

for all $\mathbf{v}\in Z_h$, where $Intp(\mathbf{u}^n,p^n;\mathbf{v})$, representing the interpolating error, denotes

$$\begin{aligned} Intp(\mathbf{u}^n,p^n;\mathbf{v}) \ &= \ \Big(d_t\mathbf{u}^n - \mathbf{u}_t^{n-1/2},\mathbf{v}\Big) \, + \, \mathcal{A}(\breve{\mathbf{u}}^n \, - \, \mathbf{u}^{n-1/2},\mathbf{v}) \, + \, b^*(\breve{\mathbf{u}}^n,\breve{\mathbf{u}}^n,\mathbf{v}) \\ & \quad - b^*(\mathbf{u}^{n-1/2},\mathbf{u}^{n-1/2},\mathbf{v}) \, - \, (\breve{p}^n - p^{n-1/2},\nabla\cdot\mathbf{v}) \, + \, (\mathbf{f}^{n-1/2}-\breve{\mathbf{f}},\mathbf{v}) \, . \end{aligned} \tag{3.17}$$

Subtracting (3.15) from (3.16), we have for $\mathbf{e}^n \ = \ \mathbf{u}^n \ - \ \mathbf{u}_h^n$,

$$(\mathbf{e}^n-\mathbf{e}^{n-1},\mathbf{v}) + \Delta t\mathcal{A}(\breve{\mathbf{e}}^n,\mathbf{v}) + \Delta t\,(b^*(\breve{\mathbf{e}}^n,\breve{\mathbf{u}}^n,\mathbf{v}) \, + \, b^*(\breve{\mathbf{u}}_h^n,\breve{\mathbf{e}}^n,\mathbf{v})) \ = \ \Delta t(\breve{p}^n,\nabla\cdot\mathbf{v}) + \Delta t\, Intp(\mathbf{u}^n,p^n;\mathbf{v}) \ , \tag{3.18}$$

for all $\mathbf{v}\in Z_h$.

Let $\mathbf{e}^n \ = \ \mathbf{u}^n \ - \ \mathbf{u}_h^n \ = \ (\mathbf{u}^n - \mathbf{U}^n) \, + \, (\mathbf{U}^n - \mathbf{u}_h^n) := \Lambda^n \, + \, E^n \,$, where $\mathbf{U}^n \in Z_h$.

With the choice $\mathbf{v} = \breve{E}^n$, and using $(q,\nabla\cdot\breve{E}^n) = 0, \ \ \forall q\in P_h$, equation (3.18) becomes

$$\begin{aligned} (E^n - E^{n-1},\breve{E}^n) \ \ &+ \ \ \Delta t\mathcal{A}(\breve{E}^n,\breve{E}^n) \, + \, \Delta t\,\Big(b^*(\breve{E}^n,\breve{\mathbf{u}}^n,\breve{E}^n) \, + \, b^*(\breve{\mathbf{u}}_h^n,\breve{E}^n,\breve{E}^n)\Big) \\ &= \ \ -(\Lambda^n-\Lambda^{n-1},\breve{E}^n) \, - \, \Delta t\mathcal{A}(\breve{\Lambda}^n,\breve{E}^n) \, - \, \Delta t\,\Big(b^*(\breve{\Lambda}^n,\breve{\mathbf{u}}^n,\breve{E}^n) \, + \, b^*(\breve{\mathbf{u}}_h^n,\breve{\Lambda}^n,\breve{E}^n)\Big) \\ & \quad\ \ +\Delta t(\breve{p}^n - q,\nabla\cdot\breve{E}^n) \, + \, \Delta t\, Intp(\mathbf{u}^n,p^n;\breve{E}^n) \ , \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{1}{2}\left(\|E^n\|^2 \, - \, \|E^{n-1}\|^2\right) \ \ &+ \ \ \Delta t\,\Big(\nu\|\nabla\breve{E}^n\|^2 \, + \, \chi\|\breve{E}^{n\,*}\|^2\Big) \\ &= \ \ -\Delta t\, b^*(\breve{E}^n,\breve{\mathbf{u}}^n,\breve{E}^n) \, - \, (\Lambda^n-\Lambda^{n-1},\breve{E}^n) \, - \, \Delta t\mathcal{A}(\breve{\Lambda}^n,\breve{E}^n) \\ & \quad\ \ - \Delta t\,\Big(b^*(\breve{\Lambda}^n,\breve{\mathbf{u}}^n,\breve{E}^n) \, + \, b^*(\breve{\mathbf{u}}_h^n,\breve{\Lambda}^n,\breve{E}^n)\Big) \\ & \quad\ \ +\Delta t(\breve{p}^n - q,\nabla\cdot\breve{E}^n) \, + \, \Delta t\, Intp(\mathbf{u}^n,p^n;\breve{E}^n) \ . \end{aligned} \tag{3.19}$$

Next we estimate the terms on the RHS of (3.19).

Using $b^*(\mathbf{u},\mathbf{v},\mathbf{w}) \le C(\Omega)\sqrt{\|\mathbf{u}\|\,\|\nabla\mathbf{u}\|}\,\|\nabla\mathbf{v}\|\,\|\nabla\mathbf{w}\|$, for $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \ \in X$, and Young's inequality,

$$b^*(\breve{E}^n,\breve{\mathbf{u}}^n,\breve{E}^n) \ \le \ C\|\breve{E}^n\|^{1/2}\,\|\nabla\breve{E}^n\|^{3/2}\,\|\nabla\breve{\mathbf{u}}^n\| \tag{3.20}$$

$$\le \ \frac{\nu}{10}\|\nabla\breve{E}^n\|^2 \, + \, C\,\nu^{-3}\|\breve{E}^n\|^2\|\nabla\breve{\mathbf{u}}^n\|^4 \, . \tag{3.21}$$

$$(\Lambda^n - \Lambda^{n-1}, \breve{E}^n) \;\le\; \frac{1}{2}\|\Lambda^n - \Lambda^{n-1}\|^2 + \frac{1}{2}\|\breve{E}^n\|^2 . \tag{3.22}$$

$$\begin{aligned}
\mathcal{A}(\breve{\Lambda}^n, \breve{E}^n) &= \nu(\nabla\breve{\Lambda}^n, \nabla\breve{E}^n) + \chi\,(\breve{\Lambda}^n - G_N\bar{\breve{\Lambda}}^n, \breve{E}^n) \\
&\le \frac{\nu}{10}\|\nabla\breve{E}^n\|^2 + C\,\nu\|\nabla\breve{\Lambda}^n\|^2 + \chi\delta^{2N+2}\|B\breve{\Lambda}^n\|\|B\breve{E}^n\| \\
&\le \frac{\nu}{10}\|\nabla\breve{E}^n\|^2 + C\,\nu\|\nabla\breve{\Lambda}^n\|^2 + \chi\frac{1}{2}(\breve{\Lambda}^n - G_N\bar{\breve{\Lambda}}^n, \breve{\Lambda}^n) + \chi\,\frac{1}{2}\|\breve{E}^{n\,*}\|^2 \\
&\le \frac{\nu}{10}\|\nabla\breve{E}^n\|^2 + C\,\nu\|\nabla\breve{\Lambda}^n\|^2 + \chi\,\frac{1}{4}\|\breve{\Lambda}^n - G_N\bar{\breve{\Lambda}}^n\|^2 + \chi\,\frac{1}{4}\|\breve{\Lambda}^n\|^2 + \chi\,\frac{1}{2}\|\breve{E}^{n\,*}\|^2.
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
b^*(\breve{\Lambda}^n, \breve{\mathbf{u}}^n, \breve{E}^n) &\le C\sqrt{\|\breve{\Lambda}^n\|\,\|\nabla\breve{\Lambda}^n\|}\,\|\nabla\breve{\mathbf{u}}^n\|\,\|\nabla\breve{E}^n\| \\
&\le \frac{\nu}{10}\|\nabla\breve{E}^n\|^2 + \nu^{-1}\,C\,\|\breve{\Lambda}^n\|\,\|\nabla\breve{\Lambda}^n\|\,\|\nabla\breve{\mathbf{u}}^n\|^2 .
\end{aligned} \tag{3.24}$$

$$b^*(\breve{\mathbf{u}}_h^n, \breve{\Lambda}^n, \breve{E}^n) \;\le\; \frac{\nu}{10}\|\nabla\breve{E}^n\|^2 + \nu^{-1}\,C\,\|\breve{\mathbf{u}}_h^n\|\,\|\nabla\breve{\mathbf{u}}_h^n\|\,\|\nabla\breve{\Lambda}^n\|^2 . \tag{3.25}$$

$$\begin{aligned}
(\breve{p}^n - q, \nabla\cdot\breve{E}^n) &\le \|\breve{p}^n - q\|\,\|\nabla\cdot\breve{E}^n\| \\
&\le \frac{\nu}{10}\|\nabla\breve{E}^n\|^2 + \nu^{-1}\,C\,\|\breve{p} - q\|^2 .
\end{aligned} \tag{3.26}$$

Substituting (3.21)-(3.26) into (3.19), and summing from $n = 1$ to l (assuming that $\|E^0\| = 0$), we have

$$\begin{aligned}
\|E^l\|^2 \;+\; & \Delta t\sum_{n=1}^{l}\nu\|\nabla\breve{E}^n\|^2 + \Delta t\chi\sum_{n=1}^{l}\|\breve{E}^{n\,*}\|^2 \\
\le\; & \Delta t\sum_{n=1}^{l} C\,(\nu^{-3}\|\nabla\breve{\mathbf{u}}^n\|^4 + 1)\|\breve{E}^n\|^2 \\
& + \Delta t\sum_{n=1}^{l}\|\Lambda^n - \Lambda^{n-1}\|^2 + 2\Delta t\sum_{n=1}^{l} C\,\nu\|\nabla\breve{\Lambda}^n\|^2 \\
& + \Delta t\,\chi\,\frac{1}{2}\left(\sum_{n=1}^{l}\|\breve{\Lambda}^n - G_N\bar{\breve{\Lambda}}^n\|^2 + \sum_{n=1}^{l}\|\breve{\Lambda}^n\|^2\right) \\
& + 2\Delta t\sum_{n=1}^{l} C\,\nu^{-1}\left(\|\breve{\Lambda}^n\|\,\|\nabla\breve{\Lambda}^n\|\,\|\nabla\breve{\mathbf{u}}^n\|^2 + \|\breve{\mathbf{u}}_h^n\|\,\|\nabla\breve{\mathbf{u}}_h^n\|\,\|\nabla\breve{\Lambda}^n\|^2\right) \\
& + 2\Delta t\sum_{n=1}^{l} C\,\nu^{-1}\|\breve{p}^n - q\|^2 \\
& + 2\Delta t\sum_{n=1}^{l}|Intp(\mathbf{u}^n, p^n; \breve{E}^n)| .
\end{aligned} \tag{3.27}$$

The next step in the proof is to bound the terms on the RHS of (3.27). We have that

$$\begin{aligned} 2\Delta t \sum_{n=1}^{l} C\,\nu \|\nabla \breve{\Lambda}^n\|^2 &\leq 2\Delta t\, C\,\nu \sum_{n=0}^{l} \|\nabla \Lambda^n\|^2 \;\leq\; 2C\,\nu\,\Delta t \sum_{n=0}^{l} h^{2k} |\mathbf{u}^n|^2_{k+1} \\ &\leq 2C\,\nu\, h^{2k} \||\mathbf{u}|\|^2_{2,k+1}\,. \end{aligned} \tag{3.28}$$

Also,

$$\begin{aligned} \Delta t \sum_{n=1}^{l} \|\Lambda^n - \Lambda^{n-1}\|^2 &\leq 4\,\Delta t \sum_{n=0}^{l} \|\Lambda^n\|^2 \;\leq\; 4\,\Delta t \sum_{n=0}^{l} C\,h^{2k+2} |u^n|^2_{k+1} \\ &\leq Ch^{2k+2} \||u|\|^2_{2,k+1}\,. \end{aligned} \tag{3.29}$$

Using (2.8), and that $G_N G$ is a bounded operator from $L^2(\Omega) \to L^2(\Omega)$,

$$\begin{aligned} \Delta t\,\chi\,\frac{1}{2}\left(\sum_{n=1}^{l} \|\breve{\Lambda}^n - G_N \bar{\breve{\Lambda}}^n\|^2 + \sum_{n=1}^{l} \|\breve{\Lambda}^n\|^2\right) &\leq C\Delta t\,\chi \sum_{n=1}^{l} \|\breve{\Lambda}^n\|^2 \,+\, \Delta t\,\chi \sum_{n=1}^{l} \|G_N G \breve{\Lambda}^n\|^2 \\ &\leq C\Delta t\,\chi \sum_{n=0}^{l} \|\Lambda^n\|^2 \,+\, \Delta t\,\chi \sum_{n=1}^{l} C_N \|\breve{\Lambda}^n\|^2 \\ &\leq C\Delta t\,\chi \sum_{n=0}^{l} C\,h^{2k+2} |u^n|^2_{k+1} \,+\, \Delta t\,\chi\, C_N \sum_{n=0}^{l} \|\Lambda^n\|^2 \\ &\leq C\,\chi\, h^{2k+2} \Delta t \sum_{n=0}^{l} |u^n|^2_{k+1} \\ &\leq C\,\chi\, h^{2k+2} \||u|\|^2_{2,k+1}\,. \end{aligned} \tag{3.30}$$

For the term

$$\begin{aligned} 2\Delta t \sum_{n=1}^{l} C\,\nu^{-1} \|\breve{\Lambda}^n\|\,\|\nabla \breve{\Lambda}^n\|\,\|\nabla \breve{\mathbf{u}}^n\|^2 &\leq C\,\nu^{-1}\,\Delta t \sum_{n=1}^{l} \big(\|\Lambda^n\|\,\|\nabla \Lambda^n\| \,+\, \|\Lambda^{n-1}\|\,\|\nabla \Lambda^{n-1}\| \\ &\qquad + \|\Lambda^{n-1}\|\,\|\nabla \Lambda^n\| + \|\Lambda^n\|\,\|\nabla \Lambda^{n-1}\|\big)\, \|\nabla \breve{\mathbf{u}}^n\|^2 \\ &\leq C\,\nu^{-1}\,h^{2k+1} \left(\Delta t \sum_{n=1}^{l} |\mathbf{u}^n|^2_{k+1}\,\|\nabla \breve{\mathbf{u}}^n\|^2 + \Delta t \sum_{n=1}^{l} |\mathbf{u}^n|_{k+1} |\mathbf{u}^{n-1}|_{k+1}\,\|\nabla \breve{\mathbf{u}}^n\|^2 \right. \\ &\qquad \left. +\; \Delta t \sum_{n=1}^{l} |\mathbf{u}^{n-1}|^2_{k+1}\,\|\nabla \breve{\mathbf{u}}^n\|^2 \right) \\ &\leq C\,\nu^{-1}\,h^{2k+1} \left(\Delta t \sum_{n=0}^{l} |\mathbf{u}^n|^4_{k+1} \right. \\ &\qquad \left. +\; \Delta t \sum_{n=0}^{l} \|\nabla \mathbf{u}^n\|^4 \right) \\ &= C\,\nu^{-1}\,h^{2k+1} \left(\||\mathbf{u}|\|^4_{4,k+1} \,+\, \||\nabla \mathbf{u}|\|^4_{4,0}\right)\,. \end{aligned} \tag{3.31}$$

Using the a priori estimate for $\|\mathbf{u}_h^n\|$, (3.8),

$$
\begin{aligned}
2\Delta t \sum_{n=1}^{l} C\,\nu^{-1}\left(\|\breve{\mathbf{u}}_h^n\|\,\|\nabla\breve{\mathbf{u}}_h^n\|\,\|\nabla\breve{\Lambda}^n\|^2\right) &\leq C\,\nu^{-1}\,\Delta t\sum_{n=1}^{l}\|\nabla\breve{\mathbf{u}}_h^n\|\,\|\nabla\breve{\Lambda}^n\|^2 \\
&\leq C\,\nu^{-1}\,\Delta t\sum_{n=1}^{l}\left(\|\nabla\Lambda^n\|^2 + \|\nabla\Lambda^{n-1}\|^2\right)\|\nabla\breve{\mathbf{u}}_h^n\| \\
&\leq C\,\nu^{-1}\,h^{2k}\,\Delta t\sum_{n=1}^{l}\left(|\mathbf{u}^n|_{k+1}^2 + |\mathbf{u}^{n-1}|_{k+1}^2\right)\|\nabla\breve{\mathbf{u}}_h^n\| \\
&\leq C\,\nu^{-1}\,h^{2k}\left(\Delta t\sum_{n=0}^{l}\|\mathbf{u}^n\|_{k+1}^4 + \Delta t\sum_{n=1}^{l}\|\nabla\breve{\mathbf{u}}_h^n\|^2\right) \\
&\leq C\,\nu^{-1}\,h^{2k}\left(|||\mathbf{u}|||_{4,k+1}^4 + \nu^{-1}\left(|||\mathbf{f}|||_{2,0}^2 + \|\mathbf{u}_h^0\|^2\right)\right). \quad (3.32)
\end{aligned}
$$

From (6.1),

$$
\begin{aligned}
2\Delta t\sum_{n=1}^{l} C\,\nu^{-1}\|\breve{p}^n - q\|^2 &\leq C\,\nu^{-1}\,\Delta t\sum_{n=1}^{l}\|p^{n-1/2} - q\|^2 + \|\breve{p}^n - p^{n-1/2}\|^2 \\
&\leq C\,\nu^{-1}\left(h^{2s+2}\,\Delta t\sum_{n=1}^{l}\|p^{n-1/2}\|_{s+1}^2 + \Delta t\sum_{n=1}^{l}\frac{1}{48}(\Delta t)^3\int_{t_{n-1}}^{t_n}\|p_{tt}\|^2\,dt\right) \\
&\leq C\,\nu^{-1}\left(h^{2s+2}\,|||p_{1/2}|||_{2,s+1}^2 + (\Delta t)^4\,\|p_{tt}\|_{2,0}^2\right) \qquad (3.33)
\end{aligned}
$$

We now bound the terms in $Intp(\mathbf{u}^n, p^n; \breve{E}^n)$. Using (6.1),(6.2),(6.3),

$$
\begin{aligned}
\left(d_t\mathbf{u}^n - \mathbf{u}_t^{n-1/2}, \breve{E}^n\right) &\leq \frac{1}{2}\|\breve{E}^n\|^2 + \frac{1}{2}\|d_t\mathbf{u}^n - \mathbf{u}_t^{n-1/2}\|^2 \\
&\leq \frac{1}{2}\|E^n\|^2 + \frac{1}{2}\|E^{n-1}\|^2 + \frac{1}{2}\frac{(\Delta t)^3}{1280}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{ttt}\|^2\,dt\,, \qquad (3.34)
\end{aligned}
$$

$$
\begin{aligned}
(\breve{p}^n - p^{n-1/2}, \nabla\cdot\breve{E}^n) &\leq \epsilon_1\nu\|\nabla\breve{E}^n\|^2 + C\,\nu^{-1}\|\breve{p}^n - p^{n-1/2}\|^2 \\
&\leq \epsilon_1\nu\|\nabla\breve{E}^n\|^2 + C\,\nu^{-1}\frac{(\Delta t)^3}{48}\int_{t_{n-1}}^{t_n}\|p_{tt}\|^2\,dt\,, \qquad (3.35)
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{f}^{n-1/2} - \breve{\mathbf{f}}^n, \breve{E}^n) &\leq \frac{1}{2}\|\breve{E}^n\|^2 + \frac{1}{2}\|\mathbf{f}^{n-1/2} - \breve{\mathbf{f}}^n\|^2 \\
&\leq \frac{1}{2}\|E^n\|^2 + \frac{1}{2}\|E^{n-1}\|^2 + \frac{(\Delta t)^3}{48}\int_{t_{n-1}}^{t_n}\|\mathbf{f}_{tt}\|^2\,dt\,, \qquad (3.36)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2}, \breve{E}^n) &= \nu(\nabla(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2}), \nabla\breve{E}^n) \\
&\quad + \chi\left((\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2}) - G_N\overline{(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2})}, \breve{E}^n\right) \\
&\leq \epsilon_2\nu\,\|\nabla\breve{E}^n\|^2 + C\,\nu\|\nabla(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2})\|^2 + \chi\frac{1}{4}\|\breve{E}^{n*}\|^2
\end{aligned}
$$

$$
\begin{aligned}
& + \chi\left((\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2}) - G_N\overline{(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2})}, \breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2}\right) \\
\leq\ & \epsilon_2\nu\,\|\nabla\breve{E}^n\|^2 + \frac{\chi}{4}\|\breve{E}^{n*}\|^2 + C\,\nu\frac{(\Delta t)^3}{48}\int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_{tt}\|^2\,dt \\
& + \chi\frac{1}{2}\delta^{4N+4}\|B^2(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2})\|^2 + \chi\frac{1}{2}\|\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2}\|^2 \\
\leq\ & \epsilon_2\nu\,\|\nabla\breve{E}^n\|^2 + \frac{\chi}{4}\|\breve{E}^{n*}\|^2 + C\,\nu\frac{(\Delta t)^3}{48}\int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_{tt}\|^2\,dt \\
& + C\,\chi\delta^{4N+4}\,(\Delta t)^3\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{tt}\|^2\,dt + C\,\chi(\Delta t)^3\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{tt}\|^2\,dt\,, \quad (3.37)
\end{aligned}
$$

where in the estimate for the last term, in the last step, we use that B is a bounded operator from $L^2 \to L^2$ and (6.1).

$$
\begin{aligned}
b^*(\breve{\mathbf{u}}^n, \breve{\mathbf{u}}^n, \breve{E}^n) - b^*(\mathbf{u}^{n-1/2}, \mathbf{u}^{n-1/2}, \breve{E}^n) =\ & b^*(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2}, \breve{\mathbf{u}}^n, \breve{E}^n) + b^*(\mathbf{u}^{n-1/2}, \breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2}, \breve{E}^n) \\
\leq\ & C\,\|\nabla(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2})\|\,\|\nabla\breve{E}^n\|\,\left(\|\nabla\breve{\mathbf{u}}^n\| + \|\nabla\mathbf{u}^{n-1/2}\|\right) \\
\leq\ & C\,\nu^{-1}\left(\|\nabla\breve{\mathbf{u}}^n\|^2 + \|\nabla\mathbf{u}^{n-1/2}\|^2\right)\frac{(\Delta t)^3}{48}\int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_{tt}\|^2\,dt \\
& + \epsilon_3\nu\|\nabla\breve{E}^n\|^2 \\
\leq\ & C\,\nu^{-1}\frac{(\Delta t)^3}{48}\left(\int_{t_{n-1}}^{t_n} 2(\|\nabla\breve{\mathbf{u}}^n\|^4 + \|\nabla\mathbf{u}^{n-1/2}\|^4)\,dt \right. \\
& \left. + \int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_{tt}\|^4\,dt\right) + \epsilon_3\nu\|\nabla\breve{E}^n\|^2 \\
\leq\ & C\,\nu^{-1}\,(\Delta t)^4(\|\nabla\breve{\mathbf{u}}^n\|^4 + \|\nabla\mathbf{u}^{n-1/2}\|^4) \\
& + C\,\nu^{-1}\,(\Delta t)^3\int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_{tt}\|^4\,dt + \epsilon_3\nu\|\nabla\breve{E}^n\|^2\,. \quad (3.38)
\end{aligned}
$$

Combining (3.34)-(3.38) we have that

$$
\begin{aligned}
2\Delta t\sum_{n=1}^{l}|Intp(\mathbf{u}^n, p^n; \breve{E}^n)| \leq\ & \Delta t\,C\sum_{n=0}^{l}\|E^n\|^2 + \Delta t\,\chi\frac{1}{2}\sum_{n=1}^{l}\|\breve{E}^{n*}\|^2 \\
& + (\epsilon_1 + \epsilon_2 + \epsilon_3)\Delta t\,\nu\sum_{n=0}^{l}\|\nabla\breve{E}^n\|^2 \\
& + C(\Delta t)^4\left(\|\mathbf{u}_{ttt}\|_{2,0}^2 + \nu^{-1}\|p_{tt}\|_{2,0}^2 + \|\mathbf{f}_{tt}\|_{2,0}^2\right. \\
& \quad + \nu\|\nabla\mathbf{u}_{tt}\|_{2,0}^2 + \nu^{-1}\|\nabla\mathbf{u}_{tt}\|_{4,0}^4 \\
& \quad + \nu^{-1}|||\nabla\mathbf{u}|||_{4,0}^4 + \nu^{-1}|||\nabla\mathbf{u}_{1/2}|||_{4,0}^4 \\
& \quad \left. + \chi\,\delta^{4N+4}\|\mathbf{u}_{tt}\|_{2,0}^2 + \chi\|u_{tt}\|_{2,0}^2\right)\,. \quad (3.39)
\end{aligned}
$$

Thus, with (3.28)-(3.33) and (3.39), from (3.27) we obtain

$$
\|E^l\|^2 + \Delta t\sum_{n=1}^{l}\nu\|\nabla\breve{E}^n\|^2 + \Delta t\,\chi\,\frac{1}{2}\sum_{n=1}^{l}\|\breve{E}^{n*}\|^2
$$

$$
\begin{aligned}
&\leq \;\Delta t \sum_{n=0}^{l} C(\,\nu^{-3}\|\nabla\breve{\mathbf{u}}^n\|^4+1)\|E^n\|^2 \\
&\quad + C\nu^{-1}\left(h^{2k}|||\mathbf{u}|||_{4,k+1}^4 \,+\, h^{2k+1}|||\nabla\mathbf{u}|||_{4,0}^4 \,+\, h^{2s+2}|||p_{1/2}|||_{2,s+1}^2\right) \\
&\quad + C\nu^{-1}\,h^{2k}(|||\mathbf{f}|||_{2,0}^2 \,+\, \|\mathbf{u}_h^0\|^2) \,+\, C\,\chi\,h^{2k+2}|||u|||_{2,k+1}^2 \,+\, C\,\nu\,h^{2k}|||\mathbf{u}|||_{2,k+1}^2 \\
&\quad + C(\Delta t)^4\left(\|\mathbf{u}_{ttt}\|_{2,0}^2 \,+\, \nu^{-1}\|p_{tt}\|_{2,0}^2 \,+\, \|\mathbf{f}_{tt}\|_{2,0}^2\right. \\
&\qquad\quad + \nu\|\nabla\mathbf{u}_{tt}\|_{2,0}^2 \,+\, \nu^{-1}\|\nabla\mathbf{u}_{tt}\|_{4,0}^4 \\
&\qquad\quad + \nu^{-1}|||\nabla\mathbf{u}|||_{4,0}^4 \,+\, \nu^{-1}|||\nabla\mathbf{u}_{1/2}|||_{4,0}^4 \\
&\qquad\quad \left. + \;\chi\,\delta^{4N+4}\|\mathbf{u}_{tt}\|_{2,0}^2 + \chi\|u_{tt}\|_{2,0}^2\right)\,. \qquad (3.40)
\end{aligned}
$$

Hence, with Δt sufficiently small, i.e. $\Delta t < C(\nu^{-3}|||\nabla\mathbf{u}|||_{\infty,0}^4+1)^{-1}$, from Gronwall's Lemma (see (3.7), we have

$$
\begin{aligned}
&\|E^l\|^2 \;+\; \Delta t\sum_{n=1}^{l}\nu\|\nabla\breve{E}^n\|^2 \;+\; \Delta t\,\chi\frac{1}{2}\sum_{n=1}^{l}\|\breve{E}^{n\,*}\|^2 \\
&\leq \;C\nu^{-1}\left(h^{2k}|||\mathbf{u}|||_{4,k+1}^4 \,+\, h^{2k+1}|||\nabla\mathbf{u}|||_{4,0}^4 \,+\, h^{2s+2}|||p_{1/2}|||_{2,s+1}^2\right) \\
&\quad + C\nu^{-1}\,h^{2k}(|||\mathbf{f}|||_{2,0}^2 \,+\, \|\mathbf{u}_h^0\|^2) \,+\, C\,\chi\,h^{2k+2}|||u|||_{2,k+1}^2 \,+\, C\,\nu\,h^{2k}|||\mathbf{u}|||_{2,k+1}^2 \\
&\quad + C(\Delta t)^4\left(\|\mathbf{u}_{ttt}\|_{2,0}^2 \,+\, \nu^{-1}\|p_{tt}\|_{2,0}^2 \,+\, \|\mathbf{f}_{tt}\|_{2,0}^2\right. \\
&\qquad\quad + \nu\|\nabla\mathbf{u}_{tt}\|_{2,0}^2 \,+\, \nu^{-1}\|\nabla\mathbf{u}_{tt}\|_{4,0}^4 \\
&\qquad\quad + \nu^{-1}|||\nabla\mathbf{u}|||_{4,0}^4 \,+\, \nu^{-1}|||\nabla\mathbf{u}_{1/2}|||_{4,0}^4 \\
&\qquad\quad \left. + \;\chi\,\delta^{4N+4}\|\mathbf{u}_{tt}\|_{2,0}^2 + \chi\|u_{tt}\|_{2,0}^2\right)\,. \qquad (3.41)
\end{aligned}
$$

Estimate (3.11) then follows from the triangle inequality and (3.41).

To obtain (3.12), we use (3.41) and

$$
\begin{aligned}
\|\nabla\left(\mathbf{u}^{n+1/2}-(\mathbf{u}_h^n+\mathbf{u}_h^{n-1})/2\right)\|^2 \;&\leq\; \|\nabla(\mathbf{u}^{n+1/2}-\breve{\mathbf{u}}^n)\|^2 \,+\, \|\nabla\breve{\Lambda}^n\|^2 \,+\, \|\nabla\breve{E}^n\|^2 \\
&\leq\; \frac{(\Delta t)^3}{48}\int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_{tt}\|^2\,dt \,+\, Ch^{2k}|\mathbf{u}^n|_{k+1}^2 \,+\, Ch^{2k}|\mathbf{u}^{n-1}|_{k+1}^2 \,+\, \|\nabla\breve{E}^n\|^2\,.
\end{aligned}
$$

■

Corollary 3.1 *Under the assumptions of Lemma 2.2 and Theorem 3.1 we have that*

$$
|||\mathbf{w}\,-\,\mathbf{u}_h|||_{\infty,0} \;\leq\; C_2 e^{C_1\,\nu^{-3/2}\,\|\mathbf{w}\|_{4,1}^2}\,\chi^{1/2}\,\delta^{N+1}\,\|B\mathbf{w}\|_{\infty,0} \,+\, \mathbf{F}(\Delta t,h,\delta,\chi) \,+\, Ch^{k+1}|||\mathbf{u}|||_{\infty,k+1} \quad (3.42)
$$

with $\mathbf{F}(\Delta t,h,\delta,\chi)$ *defined as in Theorem 3.1.*

Proof: Equation (3.42) follows immediately from Lemma 2.2, Theorem 3.1, and the triangle inequality.

4 A Numerical Illustration

We study herein a simple, underresolved flow with recirculation: the flow across a step. The most distinctive feature of this flow is a recirculating vortex behind the step, see Figure 4.1 for illustration.

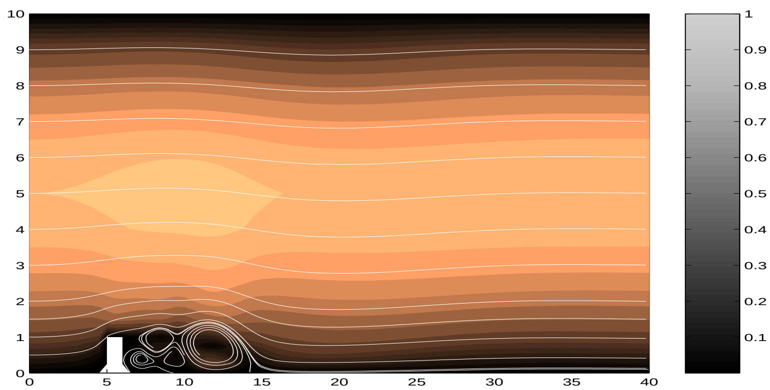

Figure 4.1: NSE at $T = 40$ and $\nu = 1/600$

We will study a flow in the transition via shedding of eddies behind the step using Navier-Stokes equations + Time Relaxation, i.e. (2.3),(2.4) with $N = 0$ (NSE + TR0), (2.3),(2.4) with $N = 1$ (NSE + TR1) and NSE + nonlinear Time Relaxation with $N = 0$ (NSE + NTR0), [10]. We will compare these models with a LES model - the Smagorinsky model. The difference between NSE + TR0 and NSE + NTR0 is in the time relaxation term which has the form:

$$\chi|u - \overline{u}|(u - \overline{u})$$

in the NSE + NTR0. In this notation, by $|\cdot|$ we mean the Euclidean norm of the corresponding vector. We used $\chi = 0.01$ in the computations presented in this section. The only difference between the Navier-Stokes equations (NSE) and the Smagorinsky model (NSE + SMA) is in the viscous term, which has the following form:

$$\nabla \cdot ((2\nu + c_s\delta^2||\mathbb{D}(u)||_F)\mathbb{D}(u)) \,.$$

Here, c_s is a positive constant (usually $c_s \sim 0.01$, see [18]), $\mathbb{D}(u)$ is the deformation tensor and $||\cdot||_F$ denotes the Frobenius norm of a tensor. We used $c_s = 0.01$ in the computations presented in this section. Although the Smagorinsky model is widely used, it has some drawbacks. These are well documented in the literature, e.g. see [20]. For instance, the Smagorinsky model constant c_s is an á priori input and this single constant is not capable of representing correctly various turbulent flows. Another drawback of this model is that it introduces too much diffusion into the flow, e.g., see [19] or Figure 4.2.

The domain of the two-dimensional flow across a step is presented in Figure 4.3. We present results for a parabolic inflow profile, which is given by $u = (u_1, u_2)^T$, with $u_1 = y(10 - y)/25$, $u_2 = 0$. No-slip boundary condition is prescribed on the top and bottom boundary as well as on the step. At the outflow we have "do nothing" boundary condition, an accepted outflow condition in CFD.

The computations were performed on various grids. For instance, for the fully resolved NSE simulation, which is our "truth" solution, we used a fine grid (level 3) whereas a much coarser grid

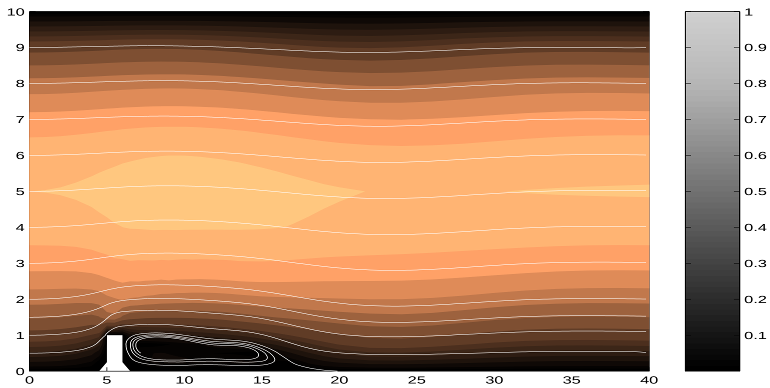

Figure 4.2: SM at $T = 40$, $\nu = 1/600$ and $\delta = 1.5$

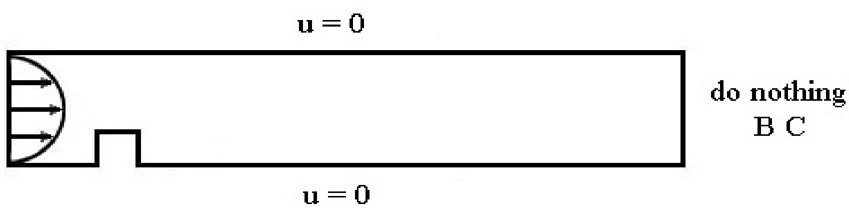

Figure 4.3: Boundary conditions

(level 1) has been used for NSE + TR0, NSE + TR1, NSE + NTR0 and NSE + SMA. The point is obviously to compare the performance of the various options in underresolved simulations by comparison against a "truth"/fully-resolved solution.

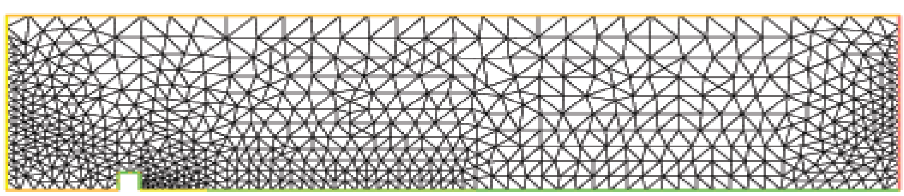

Figure 4.4: Mesh at level 1

The computations were performed with the software FreeFem++; see [17] for its description. The models were discretized in time with the Crank Nicolson (an implicit scheme of second order) and in space with the Taylor Hood finite-element method, i.e., the velocity is approximated by continuous piecewise quadratics and the pressure by continuous piecewise linears. The coarse grid which was used in the computations (level 1) is given in Figure 4.4. The fine grid computations were performed on the grid for which we chose approximately two times more degrees of freedom (level 3). The background color represents the norm of the velocity vectors.

The results pictured in Figure 4.5 give strong, although admittedly very preliminary, support for the general form of the Time Relaxation. Comparing the Figures 4.7, 4.8, 4.9, 4.10 with 4.6 we conclude that the NSE + TR0, NSE + TR1 and NSE + NTR0 tests replicate the shedding of eddies and the Smagorinsky eddy remains attached. Clearly, the Smagorinsky model is too stabilizing: eddies

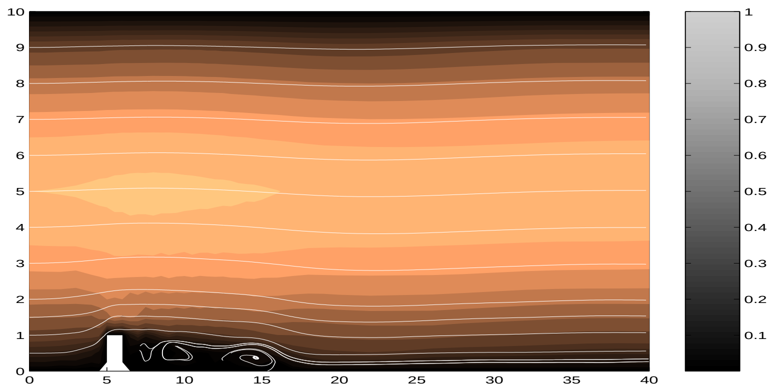

Figure 4.5: NSE + TR0 at $T = 50$, $\nu = 1/600$ and $\delta = 1.5$

which should separate and evolve remain attached and attain steady state. However, regarding the main point of study, the effects of the Time Relaxation on the truncation of scales, it is clear that this approach of regularization of NSE improved the simulation results for this transition problem. Figures 4.9 and 4.10 show that the effects of the time relaxation with N=1 are the same as for the nonlinear time relaxation with N=0. Further studies and tests of this approach are thus well merited!

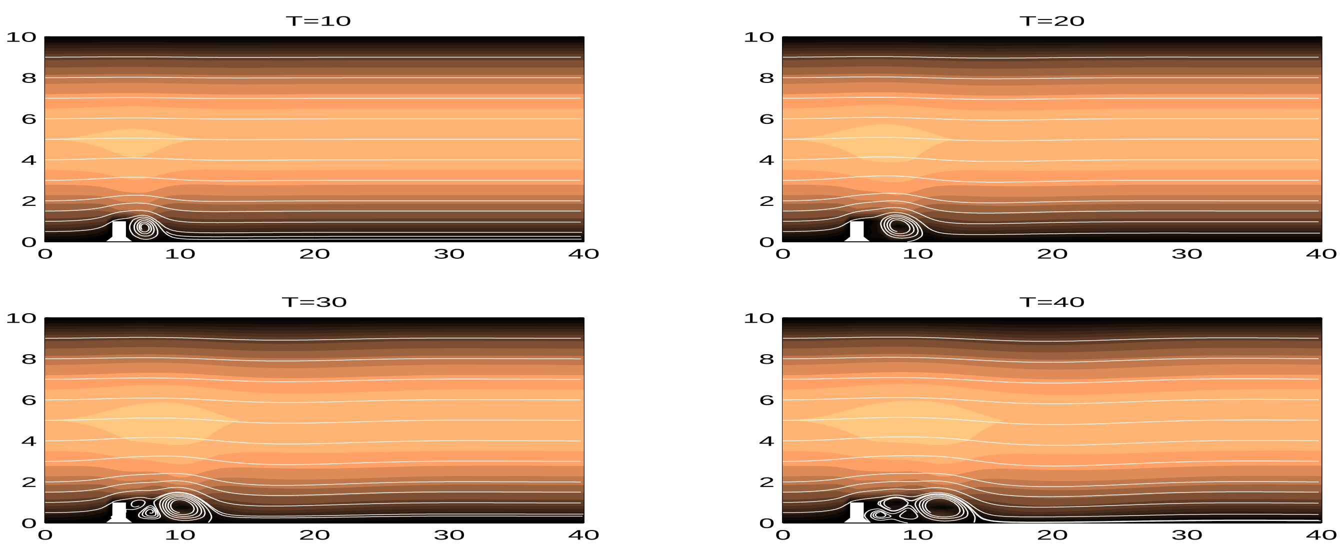

Figure 4.6: NSE at $\nu = 1/600$

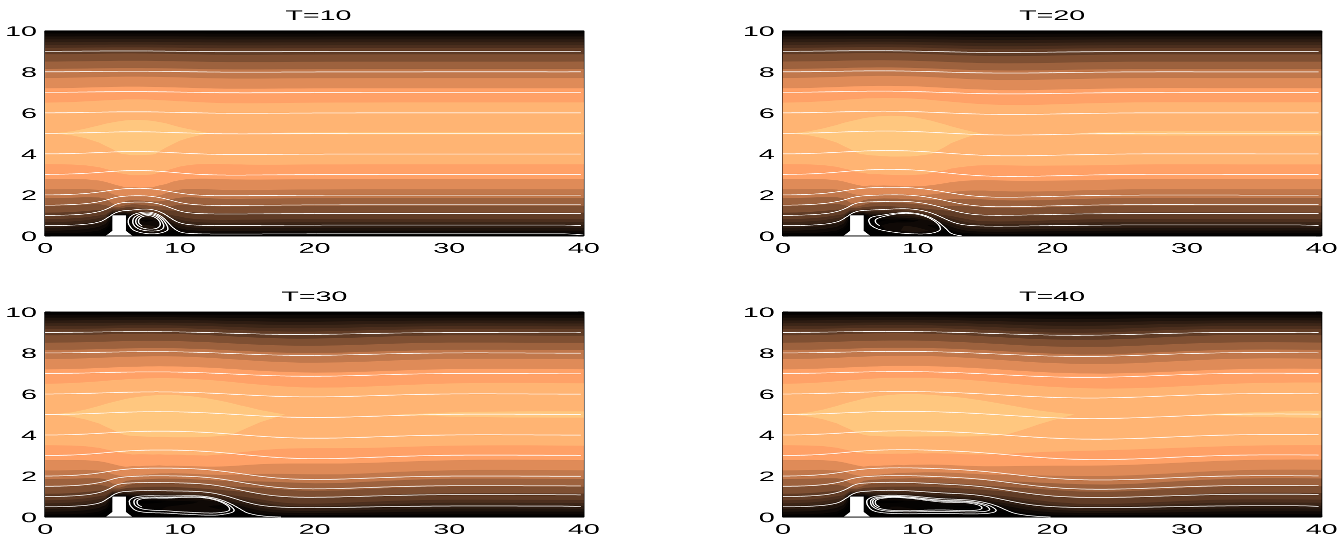

Figure 4.7: NSE + SM at $\nu = 1/600$, and $\delta = 1.5$

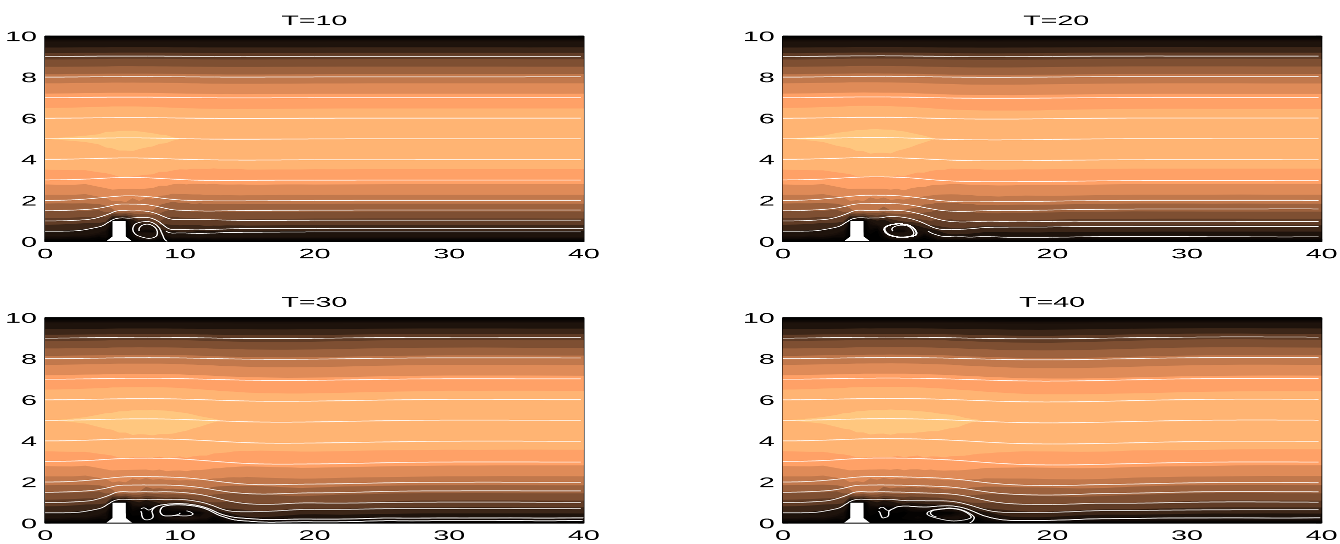

Figure 4.8: NSE + TR0 at $\nu = 1/600$, and $\delta = 1.5$

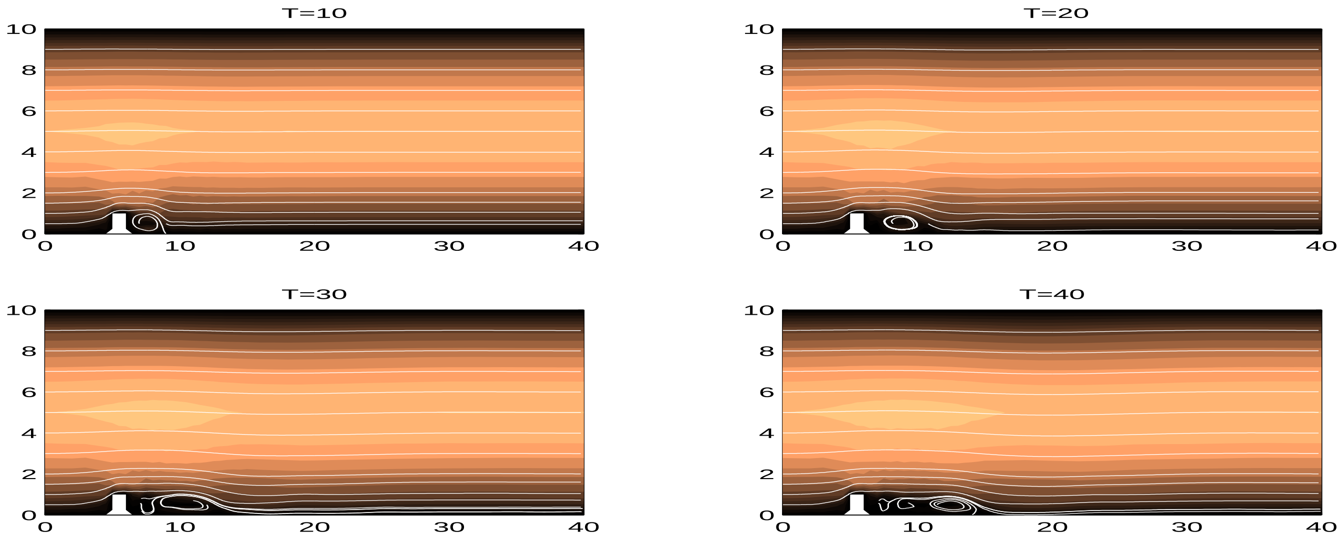

Figure 4.9: NSE + TR1 at $\nu = 1/600$, and $\delta = 1.5$

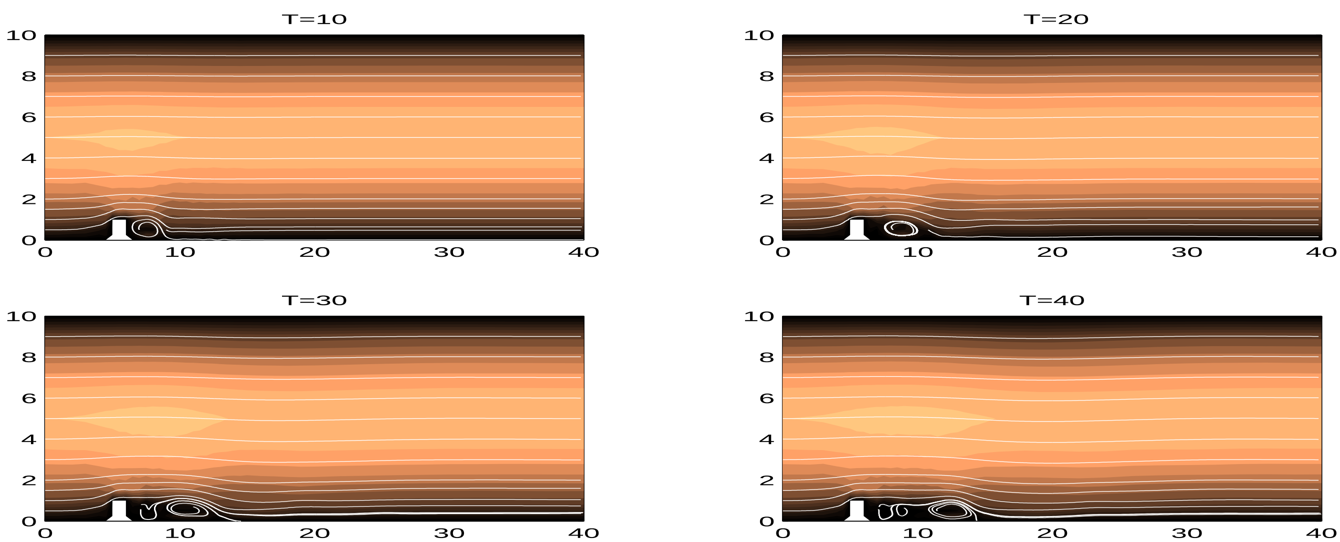

Figure 4.10: NSE + NTR0 at $\nu = 1/600$, and $\delta = 1.5$

5 Influence of Time Relaxation on Shocks

When one passes from the *incompressible*, *viscous* Navier-Stokes equations to the *compressible*, *inviscid* Euler equations, the new physical phenomenon of shocks is introduced. Often a first idea of the behavior of a model or numerical method when shocks are present is developed by considering the model or numerical method applied to 1-d conservation laws (or even Burger's equation) in the absence of boundaries. For such equations a clear understanding for the correct behavior of the shock is well known. We follow this precedent and consider the shock position, velocity, and jump conditions of solutions to

$$w_t + \frac{\partial}{\partial x} q(w) + \chi w' = 0\,, \quad -\infty < x < \infty\,, \quad t > 0\,, \tag{5.1}$$

$$w(x,0) = w_0(x)\,, \quad -\infty < x < \infty\,. \tag{5.2}$$

Here $\chi > 0$ (and we intend to compare the $\chi > 0$ case to the $\chi = 0$ case), $w' := w - \bar{w}$, and $\bar{w}$ is the differential filtered function $-\delta^2 \bar{w}_{xx} + \bar{w} = w$, given explicitly by

$$\bar{w}(x,t) = \frac{1}{2\delta} \int_{-\infty}^{\infty} e^{-|x-y|/(2\delta)}\, w(y,t)\, dy\,. \tag{5.3}$$

Stolz, Adams and Kleiser [1], [13], [14], [15] have shown in extensive tests that $\chi > 0$ does not alter the shock speed from $\chi = 0$ in (5.1),(5.2). We give in this section a theoretical justification for this result of Stolz, Adams and Kleiser.

Definition 5.1 *Let $Q_T := \mathbb{R} \times [0,T]$.*
(1) ϕ is a test function if $\phi \in C^\infty(Q_T)$ with compact support $\subseteq Q_T$.
(2) w is a weak solution of (5.1),(5.2) if for any test function ϕ,

$$\int_{Q_T} w\, \phi_t\, dx\, dt + \int_{\mathbb{R}} w_0(x)\, \phi(x,0)\, dx + \int_{Q_T} q(w) \frac{\partial}{\partial x}\phi + \chi\, w\, (\phi - \bar{\phi})\, dx\, dt = 0\,. \tag{5.4}$$

If w is a weak solution of (5.1),(5.2) with $\chi = 0$, the Rankine-Hugoniot jump condition at the shock can be calculated by an argument which is well known for conservation laws, e.g. see [21], [22]. The results is that if Γ is the shock curve $X = X(t)$ and $[\cdot]$ denotes the jump across Γ of the indicated variable then

$$[w]\frac{dX}{dt} = [q(w)]\,, \text{ or, shock speed } = \frac{dX}{dt} = \frac{[q(w)]}{[w]}\,. \tag{5.5}$$

We now (briefly – since it is standard) review this argument for $\chi > 0$ and verify that the same relation (5.5) holds for any $\chi > 0$.

Consider a test function ϕ with support strictly inside Q_T and which crosses a curve of discontinuity Γ. Partition the *support*(ϕ) into that lying to the left of Γ, denoted D_L, that lying to the right of Γ, denoted D_R, and I the portion of Γ inside *support*(ϕ),

$$support(\phi) = D_L \cup I \cup D_R\,.$$

Split the integral over Q_T (which can be restricted to the *support*(ϕ)) into integral over D_L, D_R, and I. Integrate by parts the integrals over D_L and D_R. This gives, letting $w^{L/R}$ denote the left and right limits of w on I, and $\mathbf{n} = [n_x\,, n_t]^T$ the unit normal on Γ pointing from D_L to D_R,

$$\begin{aligned} 0 = & -\int_{D_L} \left(w_t + \frac{\partial}{\partial x} q(w) + \chi(w - \bar{w}) \right) \phi\, dx\, dt \\ & + \int_I \left(w^L \phi\, n_t + q(w^L)\, \phi\, n_x \right) ds \\ & - \int_{D_R} \left(w_t + \frac{\partial}{\partial x} q(w) + \chi(w - \bar{w}) \right) \phi\, dx\, dt \\ & - \int_I \left(w^R \phi\, n_t + q(w^R)\, \phi\, n_x \right) ds\,. \end{aligned}$$

The integrals over D_L and D_R vanish, and the remaining integrals over I can be grouped to yield the jump terms

$$\int_I \left((w^L - w^R)n_t + (q(w^L) - q(w^R))n_x\right) \phi \, ds = 0 \, .$$

No jump terms arise from the time relaxation as the χw term requires no integration by parts and $\bar{w}$ is continuous across I when w has a jump discontinuity across I. Since Γ is the shock curve $X = X(t)$, the shock speed is $dX/dt = -n_t/n_x$ and (5.5) follows.

The above is a calculation. To complement it we give now a more intuitive explanation. Note that by the definition of $\bar{w}$, ($-\delta^2 \bar{w}_{xx} + \bar{w} = w$), $w' = w - \bar{w} = -\delta^2 \bar{w}_{xx}$. Thus (5.1) can be rewritten in conservation law form (with a modified flux)

$$w_t + \frac{\partial}{\partial x}\left(q(w) - \chi \, \delta^2 \bar{w}_x\right) = 0 \, .$$

The shock speed is then formally

$$\text{shock speed} = \frac{[q(w) - \chi \, \delta^2 \, \bar{w}_x]}{[w]} \, .$$

For w piecewise continuous, by (5.3), $\bar{w}_x$ will be continuous. Thus $[q(w) - \chi \, \delta^2 \, \bar{w}_x] = [q(w)]$ and the shock speeds are unchanged. Clearly, *modification of a conservation law's flux function by something continuous across the shock will not alter shock speeds.* Thus, we would expect averaging by second order differential filters, or by convolution with C^1 filter kernels, in (5.1) not to effect shock speeds (while filtering with a top-hat filter might).

6 Appendix

Lemma 6.1

$$\left\| \breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2} \right\|^2 \le \frac{1}{48}(\Delta t)^3 \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{tt}\|^2 \, dt \, . \tag{6.1}$$

Proof of Lemma 6.1:

$$\begin{aligned}
\left\| \breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2} \right\|^2 &= \left\| \frac{1}{2}(\mathbf{u}^n + \mathbf{u}^{n-1}) - \mathbf{u}^{n-1/2} \right\|^2 \\
&= \frac{1}{4} \int_\Omega \left[\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{tt}(\cdot, t) \, (t_n - t) \, dt + \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot, t) \, (t - t_{n-1}) \, dt \right]^2 d\mathbf{x} \\
&\le \frac{1}{4} \int_\Omega 2 \left[\left(\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{tt}(\cdot, t) \, (t_n - t) \, dt \right)^2 + \left(\int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot, t) \, (t - t_{n-1}) \, dt \right)^2 \right] d\mathbf{x} \\
&\le \frac{1}{2} \int_\Omega \left[\int_{t_{n-1/2}}^{t_n} (\mathbf{u}_{tt}(\cdot, t))^2 \, dt \int_{t_{n-1/2}}^{t_n} (t_n - t)^2 \, dt \right. \\
&\quad \left. + \int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot, t))^2 \, dt \int_{t_{n-1}}^{t_{n-1/2}} (t - t_{n-1})^2 \, dt \right] d\mathbf{x}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_\Omega \left[\frac{1}{3}\left(\frac{\Delta t}{2}\right)^3 \int_{t_{n-1/2}}^{t_n} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt \; + \; \frac{1}{3}\left(\frac{\Delta t}{2}\right)^3 \int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt \right] d\mathbf{x} \\
&= \frac{1}{48}(\Delta t)^3 \int_\Omega \int_{t_{n-1}}^{t_n} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt \, d\mathbf{x} \\
&= \frac{1}{48}(\Delta t)^3 \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{tt}\|^2 \, dt \,.
\end{aligned}
$$

■

Lemma 6.2

$$
\left\| d_t \mathbf{u}^n - \mathbf{u}_t^{n-1/2} \right\|^2 \; \le \; \frac{1}{1280}(\Delta t)^3 \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{ttt}\|^2 \, dt \,. \tag{6.2}
$$

Proof of Lemma 6.2:

$$
\begin{aligned}
\left\| d_t \mathbf{u}^n - \mathbf{u}_t^{n-1/2} \right\|^2 &= \left\| \frac{1}{\Delta t}(\mathbf{u}^n - \mathbf{u}^{n-1}) - \mathbf{u}_t^{n-1/2} \right\|^2 \\
&= \left(\frac{1}{4\,\Delta t}\right)^2 \int_\Omega \left[\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{ttt}(\cdot,t)\,(t_n - t)^2 \, dt \; + \; \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{ttt}(\cdot,t)\,(t - t_{n-1})^2 \, dt \right]^2 d\mathbf{x} \\
&\le \left(\frac{1}{4\,\Delta t}\right)^2 \int_\Omega 2 \left[\left(\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{ttt}(\cdot,t)\,(t_n - t)^2 \, dt \right)^2 \; + \; \left(\int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{ttt}(\cdot,t)\,(t - t_{n-1})^2 \, dt \right)^2 \right] d\mathbf{x} \\
&\le 2\left(\frac{1}{4\,\Delta t}\right)^2 \int_\Omega \left[\int_{t_{n-1/2}}^{t_n} (\mathbf{u}_{ttt}(\cdot,t))^2 \, dt \; \int_{t_{n-1/2}}^{t_n} (t_n - t)^4 \, dt \right. \\
&\qquad \left. + \int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{ttt}(\cdot,t))^2 \, dt \; \int_{t_{n-1}}^{t_{n-1/2}} (t - t_{n-1})^4 \, dt \right] d\mathbf{x} \\
&= 2\left(\frac{1}{4\,\Delta t}\right)^2 \int_\Omega \left[\frac{1}{5}\left(\frac{\Delta t}{2}\right)^5 \int_{t_{n-1/2}}^{t_n} (\mathbf{u}_{ttt}(\cdot,t))^2 \, dt \; + \; \frac{1}{5}\left(\frac{\Delta t}{2}\right)^5 \int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{ttt}(\cdot,t))^2 \, dt \right] d\mathbf{x} \\
&= \frac{1}{1280}(\Delta t)^3 \int_\Omega \int_{t_{n-1}}^{t_n} (\mathbf{u}_{ttt}(\cdot,t))^2 \, dt \, d\mathbf{x} \\
&= \frac{1}{1280}(\Delta t)^3 \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{ttt}\|^2 \, dt \,.
\end{aligned}
$$

■

For the vector $\mathbf{u}$, $\mathbf{u}^{(i)}$, $i = 1, \ldots \acute{d}$, denotes the ith component of the vector.

Lemma 6.3

$$
\left\| \nabla(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2}) \right\|^2 \; \le \; \frac{(\Delta t)^3}{48} \int_{t_{n-1}}^{t_n} \|\nabla \mathbf{u}_{tt}\|^2 \, dt \,. \tag{6.3}
$$

Proof of Lemma 6.3:

$$
\begin{aligned}
\left\|\nabla(\breve{\mathbf{u}}^n - \mathbf{u}^{n-1/2})\right\|^2 &= \frac{1}{4}\int_\Omega \nabla\left\{\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{tt}(\cdot,t)\,(t_n-t)\,dt \;+\; \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t-t_{n-1})\,dt\right\} \\
&\quad : \nabla\left\{\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{tt}(\cdot,t)\,(t_n-t)\,dt \;+\; \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t-t_{n-1})\,dt\right\}\,d\mathbf{x} \\
&\quad \text{interchanging differentiation and integration} \\
&= \frac{1}{4}\int_\Omega \left\{\int_{t_{n-1/2}}^{t_n} \nabla\mathbf{u}_{tt}(\cdot,t)\,(t_n-t)\,dt \;+\; \int_{t_{n-1}}^{t_{n-1/2}} \nabla\mathbf{u}_{tt}(\cdot,t)\,(t-t_{n-1})\,dt\right\} \\
&\quad : \left\{\int_{t_{n-1/2}}^{t_n} \nabla\mathbf{u}_{tt}(\cdot,t)\,(t_n-t)\,dt \;+\; \int_{t_{n-1}}^{t_{n-1/2}} \nabla\mathbf{u}_{tt}(\cdot,t)\,(t-t_{n-1})\,dt\right\}\,d\mathbf{x} \\
&= \sum_{i,j=1}^{\acute{d}} \frac{1}{4}\int_\Omega \left(\int_{t_{n-1/2}}^{t_n} \mathbf{u}^i_{ttx_j}(\cdot,t)\,(t_n-t)\,dt \;+\; \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}^i_{ttx_j}(\cdot,t)\,(t-t_{n-1})\,dt\right)^2 d\mathbf{x} \\
&\le \sum_{i,j=1}^{\acute{d}} \frac{1}{4}\int_\Omega 2\left[\left(\int_{t_{n-1/2}}^{t_n} \mathbf{u}^i_{ttx_j}(\cdot,t)\,(t_n-t)\,dt\right)^2 + \left(\int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}^i_{ttx_j}(\cdot,t)\,(t-t_{n-1})\,dt\right)^2\right] d\mathbf{x} \\
&\le \sum_{i,j=1}^{\acute{d}} \frac{1}{4}\int_\Omega 2\left[\int_{t_{n-1/2}}^{t_n} \left(\mathbf{u}^i_{ttx_j}(\cdot,t)\right)^2 dt \int_{t_{n-1/2}}^{t_n} (t_n-t)^2\,dt\right. \\
&\quad \left. + \int_{t_{n-1}}^{t_{n-1/2}} \left(\mathbf{u}^i_{ttx_j}(\cdot,t)\right)^2 dt \int_{t_{n-1}}^{t_{n-1/2}} (t-t_{n-1})^2\,dt\right] d\mathbf{x} \\
&= \sum_{i,j=1}^{\acute{d}} \frac{1}{4}\int_\Omega 2\,\frac{1}{3}\left(\frac{\Delta t}{2}\right)^3 \int_{t_{n-1}}^{t_n} \left(\mathbf{u}^i_{ttx_j}(\cdot,t)\right)^2 dt\; d\mathbf{x} \\
&= \frac{(\Delta t)^3}{48}\int_{t_{n-1}}^{t_n} \|\nabla\mathbf{u}_{tt}\|^2\,dt\,.
\end{aligned}
$$

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