A Multiscale V-P Discretization for Flow Problems

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Abstract

This paper gives a comprehensive numerical analysis of a multiscale method for equilibrium Navier Stokes equations. The method includes pressure regularization and eddy viscosity stabilizations both acting only on the finest scales. This method allows for equal order velocity-pressure spaces as well as the linear constant pair and the usual (P_k, P_{k-1}) pair. We show the method is optimal in a natural energy norm for all of these pairs of spaces, and provide guidance in choosing the regularization parameters.

keywords : Navier Stokes, multiscale, subgrid eddy viscosity, pressure regularization, equal order interpolations

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1 Introduction

Many practical simulations of fluid flow are underresolved: significant velocity and pressure scales can be lost on a computationally feasible mesh. Thus, one central question in Computational Fluid Dynamics is how to account for the effects of the unresolved velocity and pressure scales upon the resolved ones in a discretization. This is the motivation behind turbulence modeling, Large Eddy Simulation, and recent important algorithmic advances such as the Variational Multiscale Method of Hughes [1],[2] and the Dynamic Multilevel Methods of Temam [3].

One recent proposal to accomplish this reduction was to solve the non-discretized Navier-Stokes equations using the *constraint* that the velocity and pressure could be resolved on a given mesh as a model reduction. Hence, a new problem can be formulated using two Lagrange multipliers [4], and using penalizations, new eddy viscosity and new pressure regularization algorithms arise. This leads to a method with (naturally arising from this approach to model reduction) multiscale regularizations for both the incompressibility constraint and the nonlinear convective term, where the regularizations are used *only* on the finest resolved scales. Similar multiscale regularizations to the fine scale eddy viscosity regularization used here have been previously studied in [5],[6],and [7], and hence some of the analysis tools used in this report will be quite similar.

An analysis of this new method for the linear Stokes problem was performed in [4], with interesting results. The method is shown to be stable under a *different and less restrictive* inf-sup condition than typically used. More specifically, the infimum is taken over all elements in the *coarse* pressure mesh, but the supremum is taken over all velocity elements in the *fine* velocity mesh. Using this different inf-sup condition, it was shown that this new method is also optimal for linear-linear and linear-constant velocity-pressure element choices.

The pressure regularizations of the method would be even more beneficial in the nonlinear Navier Stokes equations. The ubiquitious need for "one more mesh" when solving flow problems often forces the use of low order elements in order to get results in a reasonable turn-around time or within storage limitations. The velocity regularization of the method is attractive for flow problems for the additional reason that it does not create non-physical energy dissipation in the large scale velocity scales.

The mathematical development of this method has only been performed for the Stokes problem [4]; the goal of this report is to give a precise analysis for the nonlinear Navier-Stokes equations. In particular, we show that the method applied to the NSE is stable and optimally accurate for linear-linear and linear-constant choices of velocity-pressure elements.

Consider the equilibrium Navier Stokes equations on a polygonal domain $\Omega \subset R^d (d = 2 \text{ or } 3)$.

$$-\nu \Delta u + u \cdot \nabla u + \nabla p = f \quad \text{in } \Omega \tag{1}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{2}$$

$$u = 0 \text{ on } \partial\Omega \tag{3}$$

$$\int_\Omega p \, dx = 0 \tag{4}$$

Let $X = H_0^1(\Omega)^d$, and $Q = L_0^2(\Omega)$. Then (1)-(4) can be written in its usual variational form as: Find $u \in X, v \in Q$ satisfying

$$(u \cdot \nabla u, v) + \nu(\nabla u, \nabla v) - (p, \nabla \cdot v) = (f, v) \quad \forall v \in X \tag{5}$$

$$(\nabla \cdot u, q) = 0 \quad \forall q \in Q \tag{6}$$

Pick finite dimensional subspaces $\overline{X} \subset X, \overline{Q} \subset Q$, and solve (5)-(6) subject to the constraint that $u \in \overline{X}, p \in \overline{Q}$. This is exactly the assumption that the pressure and velocity can be resolved. Decompose X and Q by

$$X = \overline{X} \bigoplus X' \qquad Q = \overline{Q} \bigoplus Q'$$

Decompose the space of velocity gradients $L := \{\nabla v \mid v \in X\}$ into

$$\overline{L} := \{\nabla \overline{v} \mid \overline{v} \in \overline{X}\} \quad L' := L^{\perp} \quad L = \overline{L} \bigoplus L'$$

Associated with these spaces are the following orthogonal projectors:

$$P : L \to \overline{L} \quad P_Q : Q \to \overline{Q} \quad P' : L \to L' \quad P'_Q : Q \to Q'$$

Then for $q \in Q$,

$$q = P_Q(q) + P'_Q(q) := \overline{q} + q',$$

and similarly for $\nabla v \in L$,

$$\nabla v = P(\nabla v) + P'(\nabla v) := \overline{\nabla v} + (\nabla v)'$$

Formulating this method with Lagrange multipliers and then eliminating the multipliers via penalty methods leads to the following reformulation of (5)-(6): Find $u \in X, v \in Q$ satisfying

$$(u \cdot \nabla u, v) + \nu(\nabla u, \nabla v) - (p, \nabla \cdot v) + \varepsilon_1((\nabla u)', (\nabla v)') = (f, v) \quad \forall v \in X \quad (7)$$

$$(q, \nabla \cdot u) + \varepsilon_2^{-1}(p', q') = 0 \quad \forall q \in Q \quad (8)$$

To obtain the discrete problem, choose velocity-pressure spaces to be the (P_k, P_k) or the (P_k, P_{k-1}) pair $(k \geq 1)$, where P_k denotes the space of polynomials of degree less than or equal to k. Choose meshes $T_H(\Omega), T_h(\Omega),\ H \geq h$ and for $j = k$ or $j = k-1$, define

$$X_H = \{v \epsilon C_0(\Omega) : v|_\Delta \epsilon P_k(\Delta) \ \ \forall \Delta \epsilon T_H(\Omega)\} \cap H_0^1(\Omega)$$

$$X_h = \{v \epsilon C_0(\Omega) : v|_\Delta \epsilon P_k(\Delta) \ \ \forall \Delta \epsilon T_h(\Omega)\} \cap H_0^1(\Omega)$$

$$Q_H^j = \{q : q|_\Delta \epsilon P_j(\Delta) \ \ \forall \Delta \epsilon T_H(\Omega)\} \cap L_0^2(\Omega)$$

$$Q_h^j = \{q : q|_\Delta \epsilon P_j(\Delta) \ \ \forall \Delta \epsilon T_h(\Omega)\} \cap L_0^2(\Omega)$$

For notational convenience, we will denote $P_{Q_H}(q_h)$ by q_H, and $(I - P_{Q_H})(q_h)$ by q_h' This leads finally to the discretization: Find $(u_h, p_h) \in (X_h, Q_h)$ satisfying

$$(u_h \cdot \nabla u_h, v_h) + \nu(\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) + \epsilon_1((\nabla u_h)', (\nabla v_h)') = (f, v_h) \qquad \forall v_h \epsilon X_h \quad (9)$$

$$(\nabla \cdot u_h, q_h) + \varepsilon_2^{-1}(p_h', q_h') = 0 \qquad \forall q_h \epsilon Q_h \quad (10)$$

The method's regularizations can be considered similar to recent work in [8], where a pressure stabilization dependes on the difference between a finite element pressure solution and its average over a small "patch".

We now present the main result of this report: If velocity and pressure spaces satisfy a less restrictive discrete inf-sup condition (described above, and presented in detail in Section 2), and for usual choices of $\epsilon_1, \epsilon_2, \nu$, a solution to (9)-(10) exists. Furthermore, if we assume a global uniqueness condition $M\nu^{-2}\|f\|_* < 1$, where $\|\cdot\|_*$ is the norm of the dual space of X and M is a constant (defined in Section 2) depending only on Ω, there holds a quasi-optimal error bound for the difference between the solution to (9)-(10) and the solution to (5)-(6). This bound is given in the naturally arising energy norm for the method, $\|(\cdot,\cdot)\|_\epsilon$, which will be carefully defined in Section 2.

Theorem 1.1. *Given finite element spaces (X_h, Q_H) satisfying the different discrete inf-sup condition (12), $0 < \epsilon_2, \epsilon_1, \nu \leq 1$, then there exists a solution (u_h, p_h) to (9),(10). If (u,p) satisfies (5)-(6) and we assume a global uniqueness condition $M(\Omega)\nu^{-2}\|f\|_* < 1$, then*

$$\|(u-u_h, p-p_h)\|_\epsilon^2 \leq \inf_{v_h \in X_h} \inf_{q_h \in Q_h} C(\beta_h, \nu, \alpha)\{\epsilon_1 \|(\nabla u)'\|^2 + \epsilon_2^{-1}\|p'\|^2 + \|(u-v_h, p-q_h)\|_\epsilon^2\} \quad (11)$$

Proof. (See Section 4 for proof of this theorem) □

In Section 3, we use this theorem to show that the method is optimally accurate for both the (P_k, P_k) and (P_k, P_{k-1}) pairs ($k \geq 1$), when T_h is generated by refinements (refinements which depend on the choice of elements) of T_H. The proof is then given in Section 4.

2 Notation and Preliminaries

Theorem 1.1 assumes the following discrete inf-sup condition with the infimum taken over the coarse pressure scales and the supremum over the fine velocity scales. It is less restrictive than the usual inf-sup condition, and when coupled with properly chosen meshes (examples given in Section 3) allows the method to be stable with velocity-pressure element choices of linear-linear and linear-constant.

$$\inf_{q_H \in Q_H} \sup_{v_h \in X_h} \frac{(q_H, \nabla \cdot v_h)}{\|q_h\| \|\nabla v_h\|} \geq \beta_h > 0 \tag{12}$$

Next we define the naturally occurring energy norm $\|(\cdot,\cdot)\| : (X, Q) \to R$, by

$$\|(v,q)\|_\varepsilon^2 := \nu \|\nabla v\|^2 + \varepsilon_1 \|(\nabla v)'\|^2 + \varepsilon_2^{-1} \|q'\|^2 + \|q\|^2$$

The following well known lemma (whose proof we include for completeness) gives the existence of a constant M which is used in the assumed global uniqueness condition and when bounding the trilinear forms that occur throughout the proof.

Lemma 2.1. *There is a constant $M = M(\Omega)$ such that $\forall\ u, v, w \in X$,*

$$|(u \cdot \nabla v, w)| \leq M \|\nabla u\| \|\nabla v\| \|\nabla w\| \tag{13}$$

Proof. By Holder,

$$|(u \cdot \nabla v, w)| \leq M\|u\|_{L^p}\|\nabla v\|\|w\|_{L^r},\ \frac{1}{p}+\frac{1}{r}=\frac{1}{2}$$

Then by the Sobolev embedding theorem, $\|u\|_{L^4} \leq C\|\nabla u\|$ in 2d and 3d. Thus the result follows by picking $p = r = 4$. □

Another assumption of Theorem 1.1 was a global uniqueness condition on the data, i.e. that $\alpha \stackrel{\text{def}}{:=} M\nu^{-2}\|f\|_* < 1$. This combination of terms appears often in the proof the Theorem, and hence it will be convenient to replace it with a single letter. Thus the assumed global uniqueness condition can now be restated as $\alpha < 1$.

Lastly, recall that we shall denote the fine scales of the velocity gradient by

$$(\nabla v)' = (I - P_{L_H})(\nabla v)$$

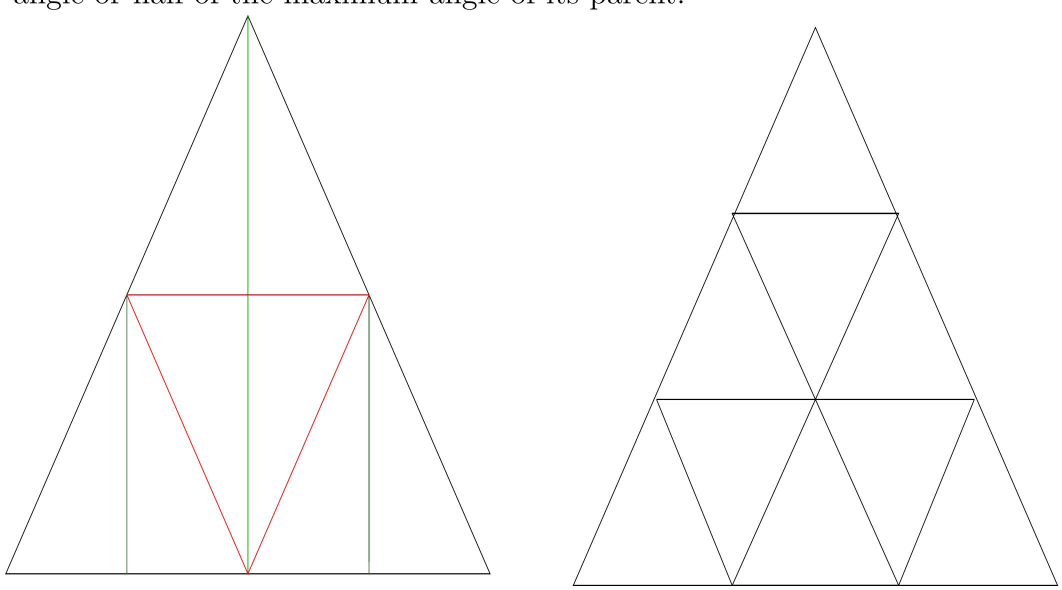
and the notation for the splitting of the pressure into its parts on and off of the large scales by

$$q = P_{Q_H}(q) + (I - P_{Q_H})(q) = q_H + q'$$

3 Application of the Theorem

To provide analytic guidance in choosing spaces and regularization parameters, it is necessary to consider specific examples of velocity-pressure spaces. This first corollary shows that linear-linear elements can be used, and with this choice the method is optimal in the ϵ norm.

Figure 1: Two possible refinement of T_H which allow linear-linear elements on (X_h, Q_h) to satisfy the less restrictive discrete inf-sup condition. For the first refinement example, made from the red and green refinements of [9], the minimum angle for a triangle in the fine mesh will be either the minimum angle or half of the maximum angle of its parent.



Corollary 3.1. *(Linear-Linear Elements) Suppose velocity-pressure elements are chosen to be the linear-linear pair* (P_1, P_1)*, and* $0 < \epsilon_1, \epsilon_2, \nu \leq 1$*. If* T_h *is any refinement of* T_H *which generates an interior basis function for each element which vanishes on the element's boundary, (examples of such refinements are shown in figure 1) then* (X_h, Q_H) *satisfies (12), and*

$$\|(u - u_h, p - p_h)\|_\epsilon^2 \leq C(\beta_h, \alpha, |u|_2^2, |p|_2^2, \nu)\{\varepsilon_1 h^2 + \varepsilon_2^{-1} h^4 + h^2 + h^4\} \quad (14)$$

Proof. The fact that (X_h, Q_H) satisfies the discrete inf-sup follows, e.g., by the same proof as in [10] since the interior basis function can replace the cubic bubble function.

For the error bound, note that $\|p'\| \leq CH^2 |p|_2$, and $\|(\nabla u)'\| \leq CH |u|_2$. Then by Theorem 1.1,

$$\|(u - u_h, p - p_h)\|_\epsilon^2 \leq C(\beta_h, \alpha, \nu)\{\varepsilon_1 H^2 + \varepsilon_2^{-1} H^4 + \nu h^2 |u|_2^2 + \varepsilon_1 h^2 |u|_2^2 + \varepsilon_2^{-1} h^4 |p|_2^2 + h^4 |p|_2^2\} \quad (15)$$

and taking $H = 2h$ gives the result. □

The error for the linear-linear pair is $O(h)$ provided $\epsilon_2 \geq h^2$. Since the velocity space was chosen to be P_1, the error in the velocity gradient arising from the discretization is $O(h)$. Hence the $O(h)$ error in the method is optimal.

We next consider the (P_k, P_{k-1}) pair. It is known [11] that for $k \geq 2$, (X_H, Q_H) satisfies the discrete inf-sup condition on a mesh T_H, and so if $T_h \subset T_H$ and $X_h \subset X_H$, then (X_h, Q_H) satisfies (12). If $k = 1$ and T_h

is generated by one uniform mesh refinement of T_H, it is known [12] that (X_h, Q_H) satisfies (12).

Corollary 3.2. *(Higher Order Elements) Consider velocity-pressure elements to be (P_k, P_{k-1}) for $k \geq 1$, with T_h being one uniform refinement of T_H. The (X_h, Q_H) satisfies (12), and*

$$\|(u - u_h, p - p_h)\|_\epsilon^2 \leq C(\beta_h, \alpha, \nu, |u|_{k+1}, |p|_k)\{\epsilon_1 h^{2k} + h^{2k} + \epsilon_2^{-1} h^{2k}\}$$

Hence the method is $O(h^k)$ when $\epsilon_2 = O(1)$.

Proof. From Theorem 1.1 and the usual seminorm bounds, we get

$$\begin{aligned}\|(u - u_h, p - p_h)\|_\epsilon^2 \leq C(\beta_h, \alpha, \nu)\{\epsilon_1 H^{2k} |u|_{k+1} + \epsilon_2^{-1} H^{2k} |p|_k \\ + h^{2k}(|u|_{k+1} + |p|_k) + \epsilon_1 h^{2k} |u|_{k+1} + \epsilon_2^{-1} h^{2k} |p|_k\} \quad (16)\end{aligned}$$

Taking $H = 2h$ gives the result. □

We also consider the case of velocity-pressure elements (P_k, P_k) for $k \geq 2$. From [10], it is known that velocity-pressure spaces on a triangulation chosen to be the same degree polynomial (degree $k \geq 1$) spaces will satisfy the discrete inf-sup condition if the velocity space is enriched with the degree $(k+2)$ bubble functions (or, equivalently, linearly independent basis functions which vanish on the boundary) which provide $\frac{k(k+1)}{2}$ degrees of freedom. Since n uniform mesh refinements (dividing a triangle into 4 congruent triangles) generate $(2^n - 1)(2^{n-1} - 1)$ linearly independent basis functions which vanish on the element's boundary, generating T_h by enough uniform refinements of T_H (so that the degrees of freedom generated is at least as many as needed) will guarantee (X_h, Q_H) satisfies the discrete inf sup condition. More specifically, for a selected polynomial degree k, n must be chosen large enough to satisfy

$$(2^n - 1)(2^{n-1} - 1) \geq \frac{k(k+1)}{2}$$

Note that the degrees of freedom needed increases quadratically with the polynomial degree, and the degrees of freedom generated by refinements increases exponentially. Thus for reasonable k, the number n of refinements needed will be small.

Corollary 3.3. *(Higher Order Equal Order Elements) Suppose velocity-pressure elements are chosen to be (P_k, P_k) for $k \geq 2$, and T_h to be enough (as defined above) uniform refinements of T_H so that (X_h, Q_H) satisfies (12). Then by Theorem 1.1,*

$$\begin{aligned} \|(u - u_h, p - p_h)\|_\epsilon^2 \leq C(\beta_h, \alpha, \nu, |u|_{k+1}^2, |p|_{k+1}^2)\{\epsilon_1 H^{2k} \\ + \epsilon_2^{-1} H^{2k+2} + h^{2k} + h^{2k+2} + \epsilon_1 h_{2k} + \epsilon_2^{-1} h^{2k+2}\} \end{aligned} \quad (17)$$

Taking $H = 2^n h$, we see that the method is $O(h^k)$ (optimal) provided $\epsilon_1 \leq 2^{-2kn}$ and $\epsilon_2 \geq 2^{n(2k+2)} h^2$.

4 Proof of Theorem 1.1

We break the proof into several steps: development of the error equations, preliminary lemmas, existence of a solution, analysis of the linear terms in the error equations, and finally analysis of the nonlinear terms in the error equations. We begin the proof by subtracting (9)-(10) from (5)-(6) to obtain the error equations:

$$(u \cdot \nabla u - u_h \cdot \nabla u_h, v_h) + \nu(\nabla(u - u_h), \nabla v_h) - (p - p_h, \nabla \cdot v_h)$$
$$+ \varepsilon_1(\nabla(u - u_h)', (\nabla v_h)') = \varepsilon_1((\nabla u)', (\nabla v_h)') \qquad \forall v_h \in X_h \quad (18)$$

$$(\nabla \cdot (u - u_h), q_h) + \varepsilon_2^{-1}((p - p_h)', q_h') = \varepsilon_2^{-1}(p', q_h') \qquad \forall q_h \epsilon Q_h \qquad (19)$$

These equations will be used frequently throughout the rest of this report.

4.1 Preliminary bounds

To keep the proof as clean as possible, it will be helpful to first present the following bounds frequently used in the proof.

Lemma 4.1. *We have the following inequalities*

$$\frac{\nu}{2} \|\nabla u_h\|^2 + \epsilon_2^{-1} \|p_h'\|^2 + \epsilon_1 \|(\nabla u_h)'\|^2 \le \frac{1}{2\nu} \|f\|_*^2 \qquad (20)$$

$$\|\nabla u_h\| \le \nu^{-1} \|f\|_* \qquad (21)$$

$$\|\nabla u\| \le \nu^{-1} \|f\|_* \qquad (22)$$

Proof. For the first inequality, let $v_h = u_h$ and $q_h = p_h$ in (9)-(10) and add the equations. This gives

$$\nu \|\nabla u_h\|^2 + \varepsilon_2^{-1} \|p_h'\|^2 + \varepsilon_1 \|(\nabla u_h)'\|^2 = (f, u_h) \qquad (23)$$

Since $(f, u_h) \leq \|f\|_* \|\nabla u_h\| \leq \frac{1}{2\nu} \|f\|_*^2 + \frac{\nu}{2} \|\nabla u_h\|^2$, we have the first inequality. For the second, by (23) we have $\nu\|\nabla u_h\|^2 \leq (f, u_h)$, so
$\nu\|\nabla u_h\|^2 \leq \frac{1}{2\nu}\|f\|_*^2 + \frac{\nu}{2}\|\nabla u_h\|^2$.

For the third inequaltity, let $v = u$ and $q = p$ in (5)-(6) and add the equations. This gives $\nu \|\nabla u\|^2 = (f, u) \leq \frac{1}{2\nu} \|f\|_*^2 + \frac{\nu}{2} \|\nabla u\|^2$, from which the result follows. □

4.2 Existence of Solution

We now show that a solution to the method (9)-(10) exists, so that we may proceed to show convergence of the method.

Lemma 4.2. *If (X_h, Q_H) satisfy (12) and $0 < \epsilon_1, \epsilon_2, \nu$, then a solution to (9)-(10) exists.*

Proof. Define $T : X^* \rightarrow (X_h, Q_h)$ to be the solution operator of the linear problem: Given $f \in X^*$, find $(u_h, p_h) \in (X_h, Q_h)$ satisfying

$$\nu(\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) + \epsilon_1((\nabla u_h)', (\nabla v_h)') = (f, v_h) \quad \forall v_h \in X_h \quad (24)$$

$$(\nabla \cdot u_h, q_h) + \epsilon_2^{-1}(p_h', q_h') = 0 \quad \forall q_h \in Q_h \quad (25)$$

Since T is linear, we show T exists uniquely by showing $T(f) = (u_h, p_h) = 0$ when $f = 0$. Thus assume f=0. Let $v_h = u_h$ and $q_h = p_h$ in (24)-(25) and add the equations. This gives

$$\nu\|\nabla u_h\|^2 + \epsilon_1\|(\nabla u_h)'\|^2 + \epsilon_2^{-1}\|p_h'\|^2 = 0$$

Thus $p_h' = 0$, and since $u_h \in H_0^1$, $u_h = 0$. We still need p_H to be 0 if $T(f)$ is to be zero, since $p_h = p_H + p_h'$. Write $p_h = p_H + p_h'$ in (23), isolate the

$(p_H, \nabla \cdot v_h)$ term, apply Cauchy Schwarz to the right hand side, and divide both sides by $\|\nabla v_h\|$. This gives

$$\frac{(p_h, \nabla \cdot v_h)}{\|\nabla v_h\|} \le \nu \|\nabla u_h\| + \|p_h'\| + \epsilon_1 \|(\nabla u_h)'\| = 0 \quad \forall v_h \in X_h$$

Thus by (12), $\|p_H\| = 0$, and therefore we have that T exists uniquely. It is also clear that T is bounded and thus continuous.

Define the maps $N : (X_h, Q_h) \to X^*$ and $F : (X_h, Q_h) \to (X_h, Q_h)$ by $N(u,p) = f - u \cdot \nabla u$ and F by the composition of T and N, and consider the fixed point problem $F(u,p) = (u,p)$. Note that a solution to this fixed point problem is also a solution to (9)-(10). By Leray-Schauder, to show a fixed point exists, we need only show that all solutions to the fixed point problems

$$(u_\lambda, p_\lambda) = \lambda F((u_\lambda, p_\lambda)) \quad 0 \le \lambda \le 1 \tag{26}$$

are uniformly bounded independent of λ. Since

$$\lambda F((u_\lambda, p_\lambda)) = \lambda T(N((u_\lambda, p_\lambda))) = T(\lambda N((u_\lambda, p_\lambda))) = T(\lambda f - \lambda u_\lambda \cdot \nabla u_\lambda),$$

solutions to (26) are solutions of: Find $(u_\lambda, p_\lambda) \in (X_h, Q_h)$ satisfying

$$\lambda(u_\lambda \cdot \nabla u_\lambda, v_h) + \nu(\nabla u_\lambda, \nabla v_h) - (p_\lambda, \nabla \cdot v_h) + \epsilon_1((\nabla u_\lambda)', (\nabla v_h)') = \lambda(f, v_h) \qquad \forall v_h \in X_h \tag{27}$$

$$(\nabla \cdot u_\lambda, q_h) + \epsilon_2^{-1}(p_\lambda', q_h') = 0 \qquad \forall q_h \in Q_h \tag{28}$$

Choosing $v_h = u_\lambda, q_h = p_\lambda$ and adding (27)-(28) gives

$$\frac{\nu}{2}\|\nabla u_\lambda\|^2 + \epsilon_2^{-1}\|p_\lambda'\|^2 + \epsilon_1 \|(\nabla u_\lambda)'\|^2 \le \frac{1}{2\nu}\|f\|_*^2 \tag{29}$$

We have now bounded $\|p'_\lambda\|$ and $\|\nabla u_\lambda\|$ independent of λ, so to complete the proof we must still bound $\|(p_\lambda)_H\|$. We will proceed by using the discrete inf-sup condition, so we write $p_\lambda = p'_\lambda + (p_\lambda)_H$ in (27), isolate the $((p_\lambda)_H, \nabla \cdot v_h)$ term on the left hand side, apply Cauchy Schwarz and (13) on the right hand side, and divide both sides by $\|\nabla v_h\|$. Applying (12) gives

$$\beta_h \|(p_\lambda)_H\| \leq \|p'_\lambda\| + \lambda \|f\|_* + \epsilon_1 \|(\nabla u_\lambda)'\| + \nu \|\nabla u_\lambda\| + M \|\nabla u_\lambda\|^2$$

This bound coupled with (29) and the fact that $0 \leq \lambda \leq 1$ shows that $\|(p_\lambda)_H\| \leq C(\beta_h, \nu, M, \|f\|_*, \epsilon_1)$. Hence all solutions to (26) are bounded independent of λ, and thus a solution to (9)-(10) exists.

$\square$

Now that a solution to (9)-(10) is known to exist, we proceed to prove convergence of the method.

4.3 Error Analysis of the Linear Terms

This subsection defines a new projection $P : (X, Q) \to (X_h, Q_h)$ by $P(u, p) = (\widetilde{u}, \widetilde{p})$ to be the solution of the linear problem (30)-(31). Notice these equations are the full error equations without the nonlinear terms if (u, p) solves (5)-(6). This projection will later make the analysis of the error equations simpler and more compact. After we show the projection exists uniquely, we show it is bounded in $\|(\cdot, \cdot)\|_\epsilon$ and that

$$\|(u - \widetilde{u}, p - \widetilde{p})\|_\epsilon \leq \inf_{(v_h, q_h) \in (X_h, Q_h)} C(\beta_h, \nu) \|(u - v_h, p - q_h)\|_\epsilon \forall (v_h, q_h) \in (X_h, Q_h)$$

Definition 4.3. *Given* $(u,p) \in (X,Q), 0 < \epsilon_1, \epsilon_2, \nu \le 1$, *and suppose* (X_h, Q_H) *satisfies (12). Define the projection* $P(u,p) = (\widetilde{u}, \widetilde{p}) \in (X_h, Q_h)$ *to be the solution of the following linear system*

$$\nu(\nabla(u-\widetilde{u}), \nabla v_h) - (p-\widetilde{p}, \nabla \cdot v_h) + \varepsilon_1(\nabla(u-\widetilde{u})', (\nabla v_h)') = 0 \quad \forall v_h \in X_h \tag{30}$$

$$(\nabla \cdot (u-\widetilde{u}), q_h) + \varepsilon_2^{-1}((p-\widetilde{p})', q_h') = 0 \quad \forall q_h \in Q_h \tag{31}$$

To show that P exists uniquely, we show that the solution to the homogeneous problem has only the zero solution. Set $(u,p) = (0,0)$ and $(v_h, q_h) = (\widetilde{u}, \widetilde{p})$ in (30)-(31) and add the equations. This gives $\|\nabla \widetilde{u}\| = \|\widetilde{p}'\| = 0$. Again setting $(u,p) = (0,0)$, and applying (12) to (30) gives $\|\widetilde{p_H}\| = 0$. Hence if $(u,p) = (0,0)$, $\|\widetilde{p}\| \le \|\widetilde{p}'\| + \|\widetilde{p_H}\| = 0$ implies $\widetilde{p} = 0$, and since $\|\nabla \widetilde{u}\| = 0$ and $\widetilde{u} \in H_0^1$, we also have that $\widetilde{u} = 0$. Thus the linear system (30)-(31) gives a zero solution for zero data, so the solution to (30)-(31) exists and is unique, and therefore the projection P exists and is well-defined.

The goal of Lemmas 4.4 and 4.5 is to bound all of the terms in $\|P(u,p)\|_\epsilon$ by $C\|(u,p)\|_\epsilon$. Lemma 4.4 bounds the velocity and the fine scale pressure of the projection, i.e. three of the four terms in $\|P(u,p)\|_\epsilon$.

Lemma 4.4. *If* (X_h, Q_H) *satisfies (12) and* $0 < \epsilon_1, \epsilon_2, \nu \le 1$, *then* $P(u,p) = (\widetilde{u}, \widetilde{p})$ *satisfies the following inequality.*

$$\frac{\nu}{2}\|\nabla \widetilde{u}\|^2 + \varepsilon_1 \|(\nabla \widetilde{u})'\|^2 + \varepsilon_2^{-1}\|\widetilde{p}'\|^2 \le \nu \|\nabla u\|^2 + \varepsilon_1 \|(\nabla u)'\|^2 + \varepsilon_2^{-1}\|p'\|^2 + \frac{2}{\nu}\|p\|^2 + 2\|\nabla u\| \|\widetilde{p}\| \tag{32}$$

Proof. Separating terms and setting $v_h = \widetilde{u}$ and $q_h = \widetilde{p}$ in (30)-(31) yields

$$\nu(\nabla u, \nabla\widetilde{u}) - (p, \nabla\cdot\widetilde{u}) + \varepsilon_1((\nabla u)', (\nabla\widetilde{u})') = \nu\left\|\nabla\widetilde{u}\right\|^2 + \varepsilon_1\left\|(\nabla\widetilde{u})'\right\|^2 - (\widetilde{p}, \nabla\cdot\widetilde{u}) \tag{33}$$

$$(\nabla\cdot u, \widetilde{p}) + \varepsilon_2^{-1}(p', \widetilde{p}') = (\nabla\cdot\widetilde{u}, \widetilde{p}) + \varepsilon_2^{-1}\left\|\widetilde{p}'\right\|^2 \tag{34}$$

Adding the two equations and rearranging gives

$$\nu\left\|\nabla\widetilde{u}\right\|^2 + \varepsilon_1\left\|(\nabla\widetilde{u})'\right\|^2 + \varepsilon_2^{-1}\left\|\widetilde{p}'\right\|^2 = \nu(\nabla u, \nabla\widetilde{u}) - (p, \nabla\cdot\widetilde{u}) + \varepsilon_1((\nabla u)', (\nabla\widetilde{u})') + (\nabla\cdot u, \widetilde{p}) + \varepsilon_2^{-1}(p', \widetilde{p}') \tag{35}$$

Apply Cauchy-Schwarz to all terms on the right hand side.

$$\nu\left\|\nabla\widetilde{u}\right\|^2 + \varepsilon_1\left\|(\nabla\widetilde{u})'\right\|^2 + \varepsilon_2^{-1}\left\|\widetilde{p}'\right\|^2 = \nu\left\|\nabla u\right\|\left\|\nabla\widetilde{u}\right\| + \left\|p\right\|\left\|\nabla\widetilde{u}\right\| + \varepsilon_1\left\|(\nabla u)'\right\|\left\|(\nabla\widetilde{u})'\right\| + \left\|\nabla u\right\|\left\|\widetilde{p}\right\| + \varepsilon_2^{-1}\left\|p'\right\|\left\|\widetilde{p}'\right\| \tag{36}$$

Applying Young's inequality and reducing gives

$$\frac{\nu}{2}\left\|\nabla\widetilde{u}\right\|^2 + \varepsilon_1\left\|(\nabla\widetilde{u})'\right\|^2 + \varepsilon_2^{-1}\left\|\widetilde{p}'\right\|^2 \le \nu\left\|\nabla u\right\|^2 + \varepsilon_1\left\|(\nabla u)'\right\|^2 + \varepsilon_2^{-1}\left\|p'\right\|^2 + \frac{2}{\nu}\left\|p\right\|^2 + 2\left\|\nabla u\right\|\left\|\widetilde{p}\right\| \tag{37}$$

thus completing the proof of the Lemma. ☐

The last term we need to bound from $\|P(u,p)\|_\epsilon$ is the pressure of the projection, which we do now.

Lemma 4.5. *If* (X_h, Q_H) *satisfies (12),* $(\widetilde{u}, \widetilde{p}) := P(u,p))$*, and* $0 < \epsilon_1, \epsilon_2, \nu \le 1$*, then* $\|\widetilde{p}\|^2$ *satisfies the following inequality.*

$$\left\|\widetilde{p}\right\|^2 \le \left\|\widetilde{p_H}\right\|^2 + \left\|\widetilde{p}'\right\|^2 \le C(\beta_h)(\nu\left\|\nabla u\right\|^2 + \frac{1}{\nu}\left\|p\right\|^2 + \varepsilon_1\left\|(\nabla u)'\right\|^2 + \varepsilon_2^{-1}\left\|p'\right\|^2 + \left\|\nabla u\right\|\left\|\widetilde{p}\right\|) \tag{38}$$

Proof. Using Lemma 4.4, and the fact that $\varepsilon_2 < 1$, we immediately get an upper bound on $\|\widetilde{p}'\|^2$:

$$\|\widetilde{p}'\|^2 \;\leq\; \nu\,\|\nabla u\|^2 \;+\; \varepsilon_1\,\|(\nabla u)'\|^2 \;+\; \|p'\|^2 \;+\; \frac{2}{\nu}\,\|p\|^2 \;+\; 2\,\|\nabla u\|\,\|\widetilde{p}\| \tag{39}$$

To determine an upper bound for $\|\widetilde{p_H}\|^2$, we use (12). Write $\widetilde{p} = \widetilde{p}_H + \widetilde{p}'$, and isolate the $(\widetilde{p_H}, \nabla \cdot v_h)$ term in (30) to obtain

$$(\widetilde{p_H}, \nabla\cdot v_h) = -\nu(\nabla(u-\widetilde{u}), \nabla v_h) + (p, \nabla\cdot v_h) - \varepsilon_1(\nabla(u-\widetilde{u})', (\nabla v_h)') - (\widetilde{p}', \nabla\cdot v_h) \tag{40}$$

If we now apply Cauchy-Schwarz to the right hand side, note that $\|(\nabla v_h)'\| \leq \|\nabla v_h\|$ and $\|\nabla \cdot v_h\| \leq \|\nabla v_h\|$, and divide both sides by $\|\nabla v_h\|$, we get

$$\frac{(\widetilde{p_H}, \nabla \cdot v_h)}{\|\nabla v_h\|} \leq \nu\,\|\nabla u\| + \nu\,\|\nabla \widetilde{u}\| + \|p\| + \varepsilon_1\,\|(\nabla u)'\| + \varepsilon_1\,\|(\nabla \widetilde{u})'\| + \|\widetilde{p}'\| \tag{41}$$

Take the infemum over all v_h in X_h on both sides of the equation, and apply (12) to obtain

$$\beta_h\,\|\widetilde{p_H}\| \leq \nu\,\|\nabla u\| + \nu\,\|\nabla \widetilde{u}\| + \|p\| + \varepsilon_1\,\|(\nabla u)'\| + \varepsilon_1\,\|(\nabla \widetilde{u})'\| + \|\widetilde{p}'\| \tag{42}$$

Thus

$$\|\widetilde{p_H}\|^2 \leq C(\beta_h)(\nu^2\,\|\nabla u\|^2 + \nu^2\,\|\nabla \widetilde{u}\|^2 + \|p\|^2 + \varepsilon_1^{-2}\,\|(\nabla u)'\|^2 + \varepsilon_1^{-2}\,\|(\nabla \widetilde{u})'\|^2 + \|\widetilde{p}'\|^2) \tag{43}$$

Since $0 < \epsilon_1, \epsilon_2, \nu \leq 1$, this reduces to

$$\|\widetilde{p_H}\|^2 \leq C(\beta_h)(\nu\,\|\nabla u\|^2 + \|p\|^2 + \varepsilon_1\,\|(\nabla u)'\|^2 + (\nu\,\|\nabla \widetilde{u}\|^2 + \varepsilon_1\,\|(\nabla \widetilde{u})'\|^2 + \varepsilon_2^{-1}\,\|\widetilde{p}'\|^2)) \tag{44}$$

Apply Lemma 4.4 to the last 3 terms on the right hand side and reduce.

$$\|\widetilde{p_H}\|^2 \leq C(\beta_h) \left\{ \nu \|\nabla u\|^2 + \frac{2}{\nu} \|p\|^2 + \varepsilon_1 \|(\nabla u)'\|^2 + \varepsilon_2^{-1} \|p'\|^2 + 2 \|\nabla u\| \|\widetilde{p}\| \right\} \tag{45}$$

Adding the bounds (39) and (45) completes the proof. □

Proposition 4.6 now ties together Lemmas 4.4 and 4.5 to bound the projection P by a constant times its data, uniformly in ϵ_1 and ϵ_2.

Proposition 4.6. *If* (X_h, Q_H) *satisfies (12) and* $0 < \epsilon_1, \epsilon_2, \nu \leq 1$, *then the projection P satisfies* $\|P(u,p)\|_\epsilon^2 \leq C(\beta_h, \nu) \|(u,p)\|_\epsilon^2$.

Proof. From Lemma 4.4 and Lemma 4.5, we have that

$$\begin{aligned} \nu \|\nabla \widetilde{u}\|^2 + \varepsilon_1 \|(\nabla \widetilde{u})'\|^2 + \varepsilon_2^{-1} \|\widetilde{p}'\|^2 + \|\widetilde{p}\|^2 \leq C(\beta_h)\{\nu \|\nabla u\|^2 \\ + \varepsilon_1 \|(\nabla u)'\|^2 + \varepsilon_2^{-1} \|p'\|^2 + \nu^{-1} \|p\|^2 + \|\nabla u\| \|\widetilde{p}\|\} \end{aligned} \tag{46}$$

Applying Young's inequality to the $\|\nabla u\| \|\widetilde{p}\|$ term gives

$$\begin{aligned} \nu \|\nabla \widetilde{u}\|^2 + \varepsilon_1 \|(\nabla \widetilde{u})'\|^2 + \varepsilon_2^{-1} \|\widetilde{p}'\|^2 + \|\widetilde{p}\|^2 \leq \\ C(\beta_h) \left\{ \nu \|\nabla u\|^2 + \nu^{-1} \|p\|^2 + \varepsilon_1 \|(\nabla u)'\|^2 + \varepsilon_2^{-1} \|p'\|^2 \right\} \end{aligned} \tag{47}$$

Sufficiently increasing the constant C to account for the constant ν^{-1} on the right hand side completes the proof. □

Proposition 4.6 allows us to easily bound $\|(u - \widetilde{u}, p - \widetilde{p})\|_\epsilon$, which will be used in the analysis of the full error equations.

Proposition 4.7. *If* (X_h, Q_H) *satisfies (12) and* $0 < \epsilon_1, \epsilon_2, \nu \le 1$*, then the projection* P *satisfies*

$$\|(u,p)-P(u,p)\|_\epsilon^2 = \|(u-\widetilde{u}, p-\widetilde{p})\|_\epsilon^2 \le C(\beta_h,\nu) \inf_{(v_h,q_h)\in(X_h,Q_h)} \|(u-v_h, p-q_h)\|_\epsilon^2 \tag{48}$$

Proof. By the triangle inequality, for $(v_h, q_h) \in (X_h, Q_h)$ we have

$$\|(u-\widetilde{u}, p-\widetilde{p})\|_\epsilon^2 \le \|(u-v_h, p-q_h)\|_\epsilon^2 + \|(\widetilde{u}-v_h, \widetilde{p}-q_h)\|_\epsilon^2 \tag{49}$$

Now by the definition of P and Proposition 4.6,

$$\|(\widetilde{u}-v_h, \widetilde{p}-q_h)\|_\epsilon^2 = \|P(u-v_h, p-q_h)\|_\epsilon^2 \le C(\beta_h,\nu)\|(u-v_h, p-q_h)\|_\epsilon^2 \tag{50}$$

Inserting (50) into (49) gives the result. □

This new projection and associated bounds will greatly help to reduce the analysis needed to complete the proof of Theorem 1.1.

4.4 Completion of Proof for Theorem 1.1

We now complete the proof of the theorem by bounding the nonlinear terms in the error equations. With the previous analysis of the linear terms, we will be able to immediately reduce our original error equations, and with the aid of the new projection, and previous lemmas and propositions, we will be able to perform this analysis with a minimum of excessively long equations. Write $(u-u_h) = (u-\widetilde{u}) - (u_h - \widetilde{u})$ and $(p-p_h) = (p-\widetilde{p}) - (p_h - \widetilde{p})$ in the error equations (18)-(19), where $(\widetilde{u}, \widetilde{p}) = P(u, p)$.

$$
\begin{aligned}
&(u_h \cdot \nabla(u - \widetilde{u}), v_h) + ((u - \widetilde{u}) \cdot \nabla u, v_h) + \nu(\nabla(u - \widetilde{u}), \nabla v_h) \\
&\quad - ((p - \widetilde{p}), \nabla \cdot v_h) + \varepsilon_1(\nabla(u - \widetilde{u})', (\nabla v_h)') = \varepsilon_1((\nabla u)', (\nabla v_h)') \\
&\quad + (u_h \cdot \nabla(u_h - \widetilde{u}), v_h) + ((u_h - \widetilde{u}) \cdot \nabla u, v_h) + \nu(\nabla(u_h - \widetilde{u}), \nabla v_h) \\
&\quad - ((p_h - \widetilde{p}), \nabla \cdot v_h) + \varepsilon_1(\nabla(u_h - \widetilde{u})', (\nabla v_h)') \ \forall v_h \epsilon X_h \quad (51)
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla \cdot (u - \widetilde{u}), q_h) + \varepsilon_2^{-1}((p - \widetilde{p})', q_h') = \varepsilon_2^{-1}(p', q_h') \\
&\quad + (\nabla \cdot (u_h - \widetilde{u}), q_h) + \varepsilon_2^{-1}((p_h - \widetilde{p})', q_h') \forall q_h \in Q_h \quad (52)
\end{aligned}
$$

From the definition of the projection P, (51)-(52) reduces to

$$
\begin{aligned}
&(u_h \cdot \nabla(u - \widetilde{u}), v_h) + ((u - \widetilde{u}) \cdot \nabla u, v_h) = \varepsilon_1((\nabla u)', (\nabla v_h)') \\
&\quad + (u_h \cdot \nabla(u_h - \widetilde{u}), v_h) + ((u_h - \widetilde{u}) \cdot \nabla u, v_h) + \nu(\nabla(u_h - \widetilde{u}), \nabla v_h) - \\
&\quad ((p_h - \widetilde{p}), \nabla \cdot v_h) + \varepsilon_1(\nabla(u_h - \widetilde{u})', (\nabla v_h)') \quad \forall v_h \in X_h \quad (53)
\end{aligned}
$$

$$
0 = \varepsilon_2^{-1}(p', q_h') + (\nabla \cdot (u_h - \widetilde{u}), q_h) + \varepsilon_2^{-1}((p_h - \widetilde{p})', q_h') \quad \forall q_h \in Q_h \quad (54)
$$

Define $\phi_h = u_h - \widetilde{u}$, and $\eta = u - \widetilde{u}$.

$$
\begin{aligned}
&(u_h \cdot \nabla \eta, v_h) + (\eta \cdot \nabla u, v_h) = \varepsilon_1((\nabla u)', (\nabla v_h)') + (u_h \cdot \nabla \phi_h, v_h) + (\phi_h \cdot \nabla u, v_h) \\
&\quad + \nu(\nabla \phi_h, \nabla v_h) - (p_h - \widetilde{p}, \nabla \cdot v_h) + \varepsilon_1((\nabla \phi_h)', (\nabla v_h)') \quad \forall v_h \in X_h \quad (55)
\end{aligned}
$$

$$
\varepsilon_2^{-1}(p', q_h') + (\nabla \cdot \phi_h, q_h) + \varepsilon_2^{-1}((p_h - \widetilde{p})', q_h') = 0 \quad \forall q_h \in Q_h \quad (56)
$$

Set $q_h = p_h - \widetilde{p}$ and $v_h = \phi_h$ and add these equations.

$$\nu \|\nabla \phi_h\|^2 + \varepsilon_1 \|(\nabla \phi_h)'\|^2 + \varepsilon_2^{-1} \|(p_h - \widetilde{p})'\|^2 = (u_h \cdot \nabla \eta, \phi_h) + (\eta \cdot \nabla u, \phi_h) \\ - \varepsilon_2^{-1}(p', (p_h - \widetilde{p})') - \varepsilon_1((\nabla u)', (\nabla \phi_h)') - (\phi_h \cdot \nabla u, \phi_h) \quad (57)$$

Apply Lemma 2.2 to the trilinear forms and Cauchy Schwarz to the bilinear forms on the right hand side.

$$\nu \|\nabla \phi_h\|^2 + \varepsilon_1 \|(\nabla \phi_h)'\|^2 + \varepsilon_2^{-1} \|(p_h - \widetilde{p})'\|^2 \leq \\ M \|\nabla u_h\| \|\nabla \eta\| \|\nabla \phi_h\| + M \|\nabla u\| \|\nabla \eta\| \|\nabla \phi_h\| + \varepsilon_2^{-1} \|p'\| \|(p_h - \widetilde{p})'\| \\ + \varepsilon_1 \|(\nabla u)'\| \|(\nabla \phi_h)'\| + M \|\nabla u\| \|\nabla \phi_h\|^2 \quad (58)$$

Use Lemma 4.1 to bound $\|\nabla u\|$ and $\|\nabla u_h\|$.

$$\nu \|\nabla \phi_h\|^2 + \varepsilon_1 \|(\nabla \phi_h)'\|^2 + \varepsilon_2^{-1} \|(p_h - \widetilde{p})'\|^2 \leq 2M\nu^{-1} \|f\|_* \|\nabla \eta\| \|\nabla \phi_h\| \\ + \varepsilon_2^{-1} \|p'\| \|(p_h - \widetilde{p})'\| + \varepsilon_1 \|(\nabla u)'\| \|(\nabla \phi_h)'\| + M\nu^{-1} \|f\|_* \|\nabla \phi_h\|^2 \quad (59)$$

Apply Young's inequality on the right hand side and reduce to get

$$\nu \|\nabla \phi_h\|^2 + \frac{\varepsilon_1}{2} \|(\nabla \phi_h)'\|^2 + \frac{\varepsilon_2^{-1}}{2} \|(p_h - \widetilde{p})'\|^2 \leq 2M\nu^{-1} \|f\|_* \|\nabla \eta\| \|\nabla \phi_h\| \\ + \frac{\varepsilon_2^{-1}}{2} \|p'\|^2 + \frac{\varepsilon_1}{2} \|(\nabla u)'\|^2 + M\nu^{-1} \|f\|_* \|\nabla \phi_h\|^2 \quad (60)$$

Recalling $\alpha = M\nu^{-2} \|f\|_*$, (60) reduces to

$$2\nu(1-\alpha) \|\nabla \phi_h\|^2 + \varepsilon_1 \|(\nabla \phi_h)'\|^2 + \varepsilon_2^{-1} \|(p_h - \widetilde{p})'\|^2 \leq 4\nu\alpha \|\nabla \eta\| \|\nabla \phi_h\| \\ + \varepsilon_2^{-1} \|p'\|^2 + \varepsilon_1 \|(\nabla u)'\|^2 \quad (61)$$

We next bound $\|\nabla\eta\| \|\nabla\phi_h\|$ by $\|\nabla\eta\| \|\nabla\phi_h\| \leq \frac{(1-\alpha)}{4\nu\alpha} \|\nabla\phi_h\|^2 + \frac{\nu\alpha}{(1-\alpha)} \|\nabla\eta\|^2$ and apply it to (61).

$$\nu(1-\alpha) \|\nabla\phi_h\|^2 + \varepsilon_1 \|(\nabla\phi_h)'\|^2 + \varepsilon_2^{-1} \|(p_h - \widetilde{p})'\|^2 \leq 4\frac{\nu^2\alpha^2}{(1-\alpha)} \|\nabla\eta\|^2 + \varepsilon_2^{-1} \|p'\|^2 + \varepsilon_1 \|(\nabla u)'\|^2 \quad (62)$$

Thus,

$$\nu \|\nabla\phi_h\|^2 + \varepsilon_1 \|(\nabla\phi_h)'\|^2 + \varepsilon_2^{-1} \|(p_h - \widetilde{p})'\|^2 \leq 4\frac{\nu\alpha^2}{(1-\alpha)^2} \|\nabla\eta\|^2 + \frac{\varepsilon_2^{-1}}{(1-\alpha)} \|p'\|^2 + \frac{\varepsilon_1}{(1-\alpha)} \|(\nabla u)'\|^2 \quad (63)$$

We now seek a bound for $\|(p_h - \widetilde{p})_H\|^2$, and as before, we proceed by using the discrete inf-sup condition. Rearrange (55) to get

$$((p_h-\widetilde{p})_H, \nabla\cdot v_h) = -(u_h\cdot\nabla\eta, v_h) - (\eta\cdot\nabla u, v_h) + \varepsilon_1((\nabla u)', (\nabla v_h)') + (u_h\cdot\nabla\phi_h, v_h) + (\phi_h\cdot\nabla u, v_h) + \nu(\nabla\phi_h, \nabla v_h) - ((p_h-\widetilde{p})', \nabla\cdot v_h) + \varepsilon_1((\nabla\phi_h)', (\nabla v_h)') \ \forall v_h \in X_h \quad (64)$$

Using Cauchy-Schwarz, dividing both sides by $\|\nabla v_h\|$, and applying (12) gives

$$\frac{((p_h - \widetilde{p})_H, \nabla \cdot v_h)}{\|\nabla v_h\|} \leq M \|\nabla u_h\| \|\nabla\eta\| + M \|\nabla\eta\| \|\nabla u\| + \varepsilon_1 \|(\nabla u)'\| + M \|\nabla u_h\| \|\nabla\phi_h\| + M \|\nabla\phi_h\| \|\nabla u\| + \nu \|\nabla\phi_h\| + \|(p_h - \widetilde{p})'\| + \varepsilon_1 \|(\nabla\phi_h)'\| \quad \forall v_h \epsilon X_h \quad (65)$$

Use the assumed global uniqueness condition, Lemma 4.1, substitute in α, and as before, take infemum over all v_h in X_h on both sides and apply (12).

$$\beta_h \|(p_h - \widetilde{p})_H\| \leq 2\nu\alpha \|\nabla\eta\| + \varepsilon_1 \|(\nabla u)'\| + 2\nu\alpha \|\nabla\phi_h\| + \nu \|\nabla\phi_h\| + \|(p_h - \widetilde{p})'\| + \varepsilon_1 \|(\nabla\phi_h)'\| \quad (66)$$

Squaring both sides and using the assumption that $\varepsilon_1, \nu \leq 1$, we have

$$\|(p_h - \widetilde{p})_H\|^2 \leq C(\beta_h)\{\nu\alpha^2 \|\nabla\eta\|^2 + \varepsilon_1 \|(\nabla u)'\|^2 + \nu\alpha^2 \|\nabla\phi_h\|^2 + \nu \|\nabla\phi_h\|^2 + \|(p_h - \widetilde{p})'\|^2 + \varepsilon_1 \|(\nabla\phi_h)'\|^2\} \quad (67)$$

Thus we obtain a bound on $\|(p_h - \widetilde{p})\|^2$. By the triangle inequality,

$$\|(p_h - \widetilde{p})\|^2 \leq \|(p_h - \widetilde{p})_H\|^2 + \|(p_h - \widetilde{p})'\|^2 \quad (68)$$

Insert the bound (67) for $(p_h - \widetilde{p})_H$.

$$\|(p_h - \widetilde{p})\|^2 \leq C(\beta_h)\{\nu\alpha^2 \|\nabla\eta\|^2 + \varepsilon_1 \|(\nabla u)'\|^2 + \nu\alpha^2 \|\nabla\phi_h\|^2 + \nu \|\nabla\phi_h\|^2 + \|(p_h - \widetilde{p})'\|^2 + \varepsilon_1 \|(\nabla\phi_h)'\|^2\} + \|(p_h - \widetilde{p})'\|^2 \quad (69)$$

Reducing gives

$$\|(p_h - \widetilde{p})\|^2 \leq C(\beta_h)\{\nu\alpha^2 \|\nabla\eta\|^2 + \varepsilon_1 \|(\nabla u)'\|^2 + \nu\alpha^2 \|\nabla\phi_h\|^2 + \nu \|\nabla\phi_h\|^2 + \|(p_h - \widetilde{p})'\|^2 + \varepsilon_1 \|(\nabla\phi_h)'\|^2\} \quad (70)$$

Recall $0 < \epsilon_2 \leq 1$ and rearrange.

$$\|(p_h - \widetilde{p})\|^2 \leq C(\beta_h)\{\nu\alpha^2 \|\nabla\eta\|^2 + \varepsilon_1 \|(\nabla u)'\|^2 + \nu(\alpha^2 + 1) \|\nabla\phi_h\|^2 + \varepsilon_1 \|(\nabla\phi_h)'\|^2 + \varepsilon_2^{-1} \|(p_h - \widetilde{p})'\|^2\} \quad (71)$$

The last three terms are nearly identical to the left side of (63), so we multiply the last two terms by (α^2+1) and then factor it out of the last three terms:

$$\|(p_h-\widetilde{p})\|^2 \le C(\beta_h)\{\nu\alpha^2\|\nabla\eta\|^2+\varepsilon_1\|(\nabla u)'\|^2+(\alpha^2+1)\{\nu\|\nabla\phi_h\|^2+\varepsilon_1\|(\nabla\phi_h)'\|^2 + \varepsilon_2^{-1}\|(p_h-\widetilde{p})'\|^2\}\} \quad (72)$$

We now insert the bound (63) for those last three terms.

$$\|(p_h-\widetilde{p})\|^2 \le C(\beta_h)\{\nu\alpha^2\|\nabla\eta\|^2+\varepsilon_1\|(\nabla u)'\|^2+(\alpha^2+1)\{4\frac{\nu\alpha^2}{(1-\alpha)^2}\|\nabla\eta\|^2 + \frac{\varepsilon_2^{-1}}{(1-\alpha)}\|p'\|^2+\frac{\varepsilon_1}{(1-\alpha)}\|(\nabla u)'\|^2\}\} \quad (73)$$

or,

$$\|(p_h-\widetilde{p})\|^2 \le C(\beta_h)\{\nu\alpha^2(1+4\frac{(\alpha^2+1)}{(1-\alpha)^2})\|\nabla\eta\|^2+\varepsilon_1(1+\frac{(\alpha^2+1)}{(1-\alpha)})\|(\nabla u)'\|^2 + \frac{\varepsilon_2^{-1}(\alpha^2+1)}{(1-\alpha)}\|p'\|^2\} \quad (74)$$

Adding (63) and (74) gives

$$\begin{aligned}\nu\|\nabla\phi_h\|^2+\varepsilon_1\|(\nabla\phi_h)'\|^2+\varepsilon_2^{-1}\|(p_h-\widetilde{p})'\|^2+\|(p_h-\widetilde{p})\|^2 \\ \le C(\beta_h)\{\nu\alpha^2(1+4\frac{(\alpha^2+1)}{(1-\alpha)^2})\|\nabla\eta\|^2+\varepsilon_1(1+\frac{(\alpha^2+1)}{(1-\alpha)})\|(\nabla u)'\|^2 \\ + \frac{\varepsilon_2^{-1}(\alpha^2+1)}{(1-\alpha)}\|p'\|^2\} \quad (75)\end{aligned}$$

From the triangle inequality and the definition of $\|(\cdot,\cdot)\|_\epsilon^2$, we have

$$\begin{aligned}\|(u-u_h,p-p_h)\|_\epsilon^2 \le \nu\|\nabla\phi_h\|^2+\nu\|\nabla\eta\|^2+\varepsilon_1\|(\nabla\phi_h)'\|^2+\varepsilon_1\|(\nabla\eta)'\|^2 \\ +\varepsilon_2^{-1}\|(p_h-\widetilde{p})'\|^2+\varepsilon_2^{-1}\|(p-\widetilde{p})'\|^2+\|(p_h-\widetilde{p})\|^2+\|(p-\widetilde{p})\|^2 \quad (76)\end{aligned}$$

or,

$$\|(u-u_h,p-p_h)\|_\epsilon^2 \le \{\nu\|\nabla\eta\|^2+\varepsilon_1\|(\nabla\eta)'\|^2+\varepsilon_2^{-1}\|(p-\widetilde{p})'\|^2+\|(p-\widetilde{p})\|^2\}$$
$$+\{\nu\|\nabla\phi_h\|^2+\varepsilon_1\|(\nabla\phi_h)'\|^2+\varepsilon_2^{-1}\|(p_h-\widetilde{p})'\|^2+\|(p_h-\widetilde{p})\|^2\} \quad (77)$$

Applying the bound (75), we get

$$\|(u-u_h,p-p_h)\|_\epsilon^2 \le \{\nu\|\nabla\eta\|^2+\varepsilon_1\|(\nabla\eta)'\|^2+\varepsilon_2^{-1}\|(p-\widetilde{p})'\|^2+\|(p-\widetilde{p})\|^2\}$$
$$+C(\beta_h)\{\nu\alpha^2(1+4\frac{(\alpha^2+1)}{(1-\alpha)^2})\|\nabla\eta\|^2+\varepsilon_1(1+\frac{(\alpha^2+1)}{(1-\alpha)})\|(\nabla u)'\|^2$$
$$+\frac{\varepsilon_2^{-1}(\alpha^2+1)}{(1-\alpha)}\|p'\|^2\} \quad (78)$$

and reducing gives

$$\|(u-u_h,p-p_h)\|_\epsilon^2 \le C(\beta_h)\{\nu\alpha^2(1+4\frac{(\alpha^2+1)}{(1-\alpha)^2})\|\nabla\eta\|^2$$
$$+\varepsilon_1(1+\frac{(\alpha^2+1)}{(1-\alpha)})\|(\nabla u)'\|^2+\frac{\varepsilon_2^{-1}(\alpha^2+1)}{(1-\alpha)}\|p'\|^2+\varepsilon_1\|(\nabla\eta)'\|^2$$
$$+\varepsilon_2^{-1}\|(p-\widetilde{p})'\|^2+\|(p-\widetilde{p})\|^2\} \quad (79)$$

We rearrange terms to make the use of Proposition 4.7 more clear.

$$\|(u-u_h,p-p_h)\|_\epsilon^2 \le C(\beta_h)\{\varepsilon_1(1+\frac{(\alpha^2+1)}{(1-\alpha)})\|(\nabla u)'\|^2+\frac{\varepsilon_2^{-1}(\alpha^2+1)}{(1-\alpha)}\|p'\|^2$$
$$+\{\nu\alpha^2(1+4\frac{(\alpha^2+1)}{(1-\alpha)^2})\|\nabla\eta\|^2+\varepsilon_1\|(\nabla\eta)'\|^2+\varepsilon_2^{-1}\|(p-\widetilde{p})'\|^2+\|(p-\widetilde{p})\|^2\}\}$$
$$(80)$$

Apply Proposition 4.7.

$$\leq C(\beta_h)\{\varepsilon_1(1+\frac{(\alpha^2+1)}{(1-\alpha)})\|(\nabla u)'\|^2+\frac{\varepsilon_2^{-1}(\alpha^2+1)}{(1-\alpha)}\|p'\|^2$$
$$+\alpha^2(1+4\frac{(\alpha^2+1)}{(1-\alpha)^2})\inf_{v_h\in X_h}\inf_{\chi_h\in Q_h}\{\varepsilon_2^{-1}\|(p-\chi_h)'\|^2+\nu\|\nabla(u-v_h)\|^2$$
$$+\nu^{-1}\|(p-\chi_h)\|^2+\varepsilon_1\|\nabla(u-v_h)'\|^2\}\} \quad (81)$$

Adjust the constant to account for a factoring out of ν^{-1}, and the proof is complete.

5 Conclusions

In this paper we explored a multiscale discretization of the equilibrium Navier-Stokes equations arising from imposing finite dimensionality as a constraint. The discretization recovers both finest scale pressure regularization and sub-grid eddy viscosity models, and we showed the method is optimal for the linear-linear and linear-constant pairs of velocity-pressure elements. This report extended the framework of a model reduction via constraints idea from the linear Stokes problem [4] to the (nonlinear) equilibrium NSE.

6 References

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